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Optimal LQ-Type Switched Control Design for a Class of Linear Systems with Piecewise Constant Inputs

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Abstract: This paper is devoted to a specific LQ-type optimal control problem (OCP) in the presence of additional control constraints. We consider control processes governed by linear differential equations with a priori known control switchings. The piecewise constant structure of the admissible controls under consideration is motivated by variety of concrete engineering applications and moreover, can be interpreted as a result of a quantization procedure applied to the original dynamics. We propose an new implementable numeric algorithm that makes it possible to calculate a consistent approximating solution to the initial constrained LQ-type OCPs. The contribution discusses some theoretic aspects of the obtained computational scheme and also contains a numerical example.

Keywords: switched linear systems, piecewise constant controls, specific LQ-type optimal control design, numerical methods.

1. INTRODUCTION

The control design techniques based on advanced optimization techniques are nowadays a mature methodology for the practical synthesis of several types of modern controllers associated with switched and hybrid dynamic systems (see e.g., [2-3, 9-10, 14-15, 17, 19-20, 24, 30, 37-38],). Recently, the problem of effective numerical methods for the constrained LQ based systems optimization has attracted a lot of attention, thus both theoretical results and applications were developed, (see e.g., [4-5, 7, 16, 18-22, 25]). Note that handling constraints in practical system design is an important issue in most, if not all, real world applications. It is readily appreciated that the implementable control systems have a corresponding set of constraints; for example, inputs always have maximum and minimum values and states are usually required to lie within certain ranges. Moreover, it is generally true that optimal levels of performance are associated with operating on, or near, constraint boundaries (see [28, 30]). Thus, a control engineer really can not ignore constraints without incurring a performance penalty.

The aim of our contribution is to elaborate a consistent computational algorithm for an LQ-type OCP in the presence of piecewise constant control inputs. The given restrictive structure of the admissible control function under consideration is motivated by some important engineering applications (see [22, 25, 36]) as well as by the possible quantization procedure applied to the original dynamics (see e.g., [12, 26]). Note that quadratic optimal control of piecewise linear systems was addressed earlier in [8, 32]. The treatment there was based on the backward solutions of Riccati differential equations, and the optimum had to be recomputed for each new final state. Computation of nonlinear gain using the Hamilton-Jacobi-Bellman equation and the convex optimization techniques has also been done in [32]. On the other hand, the above-mentioned optimization approaches to linear constrained systems are not sufficiently advanced to the LQ-type OCPs governed by linear systems with piecewise constant controls. In our paper we propose a new numerical method based on a specific relaxation scheme in combination with the projection scheme. And, it should be noted already at this point that a computational algorithm we propose can be effectively used in concrete control synthesis procedures associated with some important classes of switched systems.

Recall that the general hybrid and switched systems constitute formal models where two types of dynamics are present, continuous and discrete event dynamic behavior (see e.g. [11, 27]). In order to understand how these systems can be operated efficiently, both aspects of the dynamics have to be taken into account during the optimization phase. The non-stationary linear systems we study in this paper include a particular family of switched systems with the time-driven location transitions. We refer to [6, 11, 27, 35, 39] for the basic concepts and some technical details.

The remainder of our paper is organized as follows: Section 2 contains the problem statement and necessary preliminary facts and concepts. Section 3 deals with a specific relaxation scheme of the initial constrained LQ-type OCP. Moreover, we also propose a projected gradient method for the concrete numerical treatment of the studied OCPs. In this section we also discuss a controlability result for the class of dynamic processes under consideration. Section 4 is devoted to numerical aspects of the elaborated computational algorithm and contains two illustrative examples Section 5 summarizes the paper.

2. PROBLEM FORMULATION AND SOME BASIC FACTS

Consider the following linear non-stationary system with a switched control structure

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t),\ t \in [t_0, t_f], \\ x(t_0) &= x_0, \end{aligned} \tag{1}$$

where $A(\cdot) \in \mathbb{L}^{\infty}[t_0,t_f;\mathbb{R}^{n\times n}]$, $B(\cdot) \in \mathbb{L}^{\infty}[t_0,t_f;\mathbb{R}^{n\times m}]$. Here $\mathbb{L}^{\infty}[t_0,t_f;\mathbb{R}^{n\times n}]$ and $\mathbb{L}^{\infty}[t_0,t_f;\mathbb{R}^{n\times m}]$ are the standard Lebesgue spaces of the essentially bounded matrix-functions defined on a bounded time interval $[t_0,t_f]$.Similarly to the classic LQR (the Linear Quadratic Regulator) theory it is desired to minimize the following quadratic cost functional associated with (1)

$$\begin{aligned} J(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} (\langle Q(t)x(t), x(t)\rangle + \\ \langle R(t)u(t), u(t)\rangle)dt + \frac{1}{2}\langle Gx(t_f), x(t_f)\rangle, \end{aligned} \tag{2}$$

where $G \in \mathbb{R}^{n\times n}$ and the matrix-functions $Q(\cdot)$ and $R(\cdot)$ are assumed to be integrable. Following the conventional LQR theory we next introduce the standard regularity/positivity hypothesis

$$G \geq 0, \quad Q(t) \geq 0,\ R(t) \geq \delta I,\ \delta > 0\ \ \forall t \in [t_0,t_f].$$

It is well known that the classic LQ optimal control strategy $u^{opt}(\cdot)$ does not incorporate any additional (state or control) restrictions into the resulting design procedure. Let us recall here the explicit formula for $u^{opt}(\cdot)$ (see e.g., (10; 24))

$$u^{opt}(t) = -R^{-1}(t)\left[B^T(t)P(t)\right]x^{opt}(t), \tag{3}$$

where $P(\cdot)$ is the matrix-function, namely, the solution to the classic differential matrix Riccati equation associated with the LQ problem (1)-(2). In the above-mentioned conventional case (1)-(2) the optimization problem is formally studied in the full space $\mathbb{L}^2[t_0,t_f;\mathbb{R}^m]$ of square integrable control functions. In contrast to the classic case, we consider system (1) in combination with the specific piecewise constant admissible inputs $u(\cdot)$ of the following type (see Fig.1).

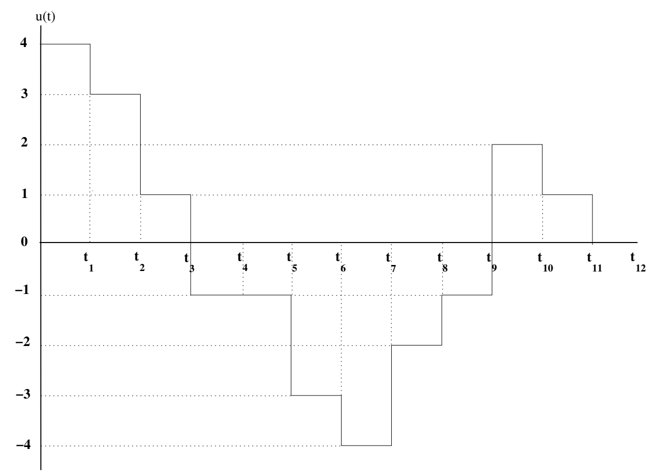

Fig. 1. The admissible switched-type control inputs $u(\cdot)$.

Resulting from the admissibility assumption the main minimization problem for the linear system (1) can be interpreted as a restricted LQ optimization problem. For example, the control signal $u(\cdot)$ showed in Fig. 1 can only take a value (level) within the finite set $\mathcal{Q} = \{-4,-3,-2,-1,0,1,2,3,4\}$ during the time interval $[t_{i-1},t_i), i = 1,\ldots,12$. In addition the control signal here is only allowed to change its value at the times $t_0,t_1,\ldots,t_f$ being fixed between these times.

Let us now specify formally the set of admissible piecewise constant control functions for system (1) in a general case. For each component $u^{(k)}(\cdot)$ of the feasible control input $u(\cdot)$ we introduce the following finite set of feasible (bounded) value levels:

$$\mathcal{Q}^k := \{q_j^{(k)} \in \mathbb{R},\ j = 1,\ldots,M_k\},\ M_k \in \mathbb{N}, \quad k = 1,\ldots,m.$$

The combinatorial character of the examined control functions associated with the initial system (1) can be illustrated by a simple example.

Example 1. Suppose $u(t) \in \mathbb{R}^2$ and

$$\mathcal{Q}^1 = \{0,1,2\},\ \mathcal{Q}^2 = \{0,-1\}.$$

Furthermore, the set of switching times for each control component is assumed to be given by

$$\mathcal{T}^1 = \{0,0.5,1\},\ \mathcal{T}^2 = \{0,0.33,0.66,1\}.$$

Resulting from the above definitions, the set $\mathcal{S}$ in (4) can now be written as $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, where:

$$\begin{aligned} &\mathcal{S}_1 = \{v : [t_0,t_f] \to \mathbb{R} \mid v(t) = I_{[0,0.5)}(t)q_{j_1}^{(1)} + \\ &I_{[0.5,1)}(t)q_{j_2}^{(1)},\ q_{j_i}^{(1)} \in \mathcal{Q}^1\}; \\ &\mathcal{S}_2 = \{w : [t_0,t_f] \to \mathbb{R} \mid w(t) = I_{[0,0.33)}(t)q_{j_1}^{(2)} + \\ &I_{[0.33,0.66)}(t)q_{j_2}^{(2)} + I_{[0.66,1)}(t)q_{j_3}^{(2)}, q_{j_i}^{(2)} \in \mathcal{Q}^2\} \end{aligned}$$

In that concrete case we evidently have: $M_1 = 3$, $M_2 = 2$, $N_1 = 2$ and $N_2 = 3$. The cardinality of the control set $\mathcal{S}$ is given by $|\mathcal{S}| = 3^2 \cdot 2^3 = 72$. In other words, we have 72 admissible control inputs, among which we must find the one that minimizes the quadratic performance criterion.

In general, all the sets $\mathcal{Q}^k$ are different (contains different levels) and have various numbers of elements. In addition, each $\mathcal{Q}^k$ possesses a strict order property $q_1^{(k)} < q_2^{(k)} < \ldots < q_{M_k}^{(k)}$. We now introduce the set of switching times associated with an admissible control function

$$\mathcal{T}^k := \{t_i^{(k)} \in \mathbb{R}_+, i = 1,\ldots,N_k\},\ N_k \in \mathbb{N}, \quad k = 1,\ldots,m$$

The sets $\mathcal{T}^k$ are assumed to be defined for each control component $u^{(k)}(\cdot)$, $k = 1,\ldots,m$, where $\mathbb{R}_+$ denotes a nonnegative semiaxis. Let $t_0 < t_1^{(k)} < \ldots < t_{N_k}^{(k)}$. For the final time instants for each $\mathcal{T}^k$ we put here $t_{N_1}^{(1)} = \ldots = t_{N_m}^{(m)} = t_f$. Using the notation of the level sets $\mathcal{Q}^k$ and the fixed switching times $\mathcal{T}^k$ introduced above, the set of admissible controls $\mathcal{S}$ can now be easily specified by the Cartesian product

$$\mathcal{S} := \mathcal{S}_1 \times \ldots \times \mathcal{S}_m, \tag{4}$$

where each set $\mathcal{S}_k$, $k = 1,...,m$ is defined as follows

$$\begin{aligned} &\mathcal{S}_k := \{v : [t_0,t_f] \to \mathbb{R} \mid v(t) = \sum_{i=1}^{N_k} I_{[t_{i-1}^{(k)},t_i^{(k)})}(t)q_{j_i}^{(k)}; \\ &q_{j_i}^{(k)} \in \mathcal{Q}^k;\ j_i \in \mathbb{Z}[1,M_k];\ t_i^{(k)} \in \mathcal{T}^k\}. \end{aligned}$$

By $\mathbb{Z}[1,M_k]$ we denote here the set of all integers into the interval $[1,M_k]$ and $I_{[t_{i-1}^k,t_i^k)}(t)$ is the characteristic function of the interval $[t_{i-1}^k, t_i^k)$. Evidently, the set of admissible control inputs $\mathscr{S}$ can be qualitatively interpreted as the set of all the possible functions $u : [t_0, t_f] \to \mathbb{R}^m$, such that each component $u^{(k)}(\cdot)$ of $u(\cdot)$ attains a constant level value $q_{j_i}^{(k)} \in \mathscr{Q}^k$ for t from $[t_{i-1}^k, t_i^k)$. Moreover, the component level changes occur only at the prescribed times $t_i^k \in \mathscr{T}^k$, $i = 1, ..., N_k$. In general, the cardinality of the set $\mathscr{S}$ for the admissible control inputs $u(\cdot)$, $u(t) \in \mathbb{R}^m$ can be expressed as follows

$$|\mathscr{S}| = \prod_{l=1}^{m} M_l^{N_l}. \tag{5}$$

Motivating from various engineering applications, we now can formulate the following specific constrained LQ-type OCP

$$\begin{aligned} &\text{minimize } J(u(\cdot)) \\ &\text{subject to } u(\cdot) \in \mathscr{S}, \end{aligned} \tag{6}$$

where $J(\cdot)$ is the costs functional defined in (2). Note that $\mathscr{S}$ constitutes a nonempty subset of the space $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$. However, the classically LQ-optimal control input $u^{opt}(\cdot)$ in (3) does not belong to the introduced specific set $\mathscr{S}$. Due to the highly restrictive condition $u(\cdot) \in \mathscr{S}$, the main optimization problem (6) can not be generally solved by a direct application of the classic Pontryagin Maximum Principle. A possible application of a suitable hybrid version of the conventional Maximum principle from [2-3, 14-15, 35, 39]) is also complicated by a non-standard structure of the simple switchings under consideration. Let us additionally note that the value of an exponentially growing cardinality $|\mathscr{S}|$ exacerbates crucially a possible application of some combinatorial and various state/control discretization based numerical algorithms for OCPs (see e.g., [1, 13, 28,-31, 37-38, 40] and the references therein).

The aim of this contribution is to propose a relative simple implementable computational procedure for a consistent and implementable numerical treatment of the constrained OCP (6). We use a basic relaxation technique associated with the main OCP (6) in combination with a gradient based algorithm for this purpose. We first obtain an optimal solution of the convex-type relaxed OCP. Next we use it in a constructive solution procedure for the original problem (5).

3. THE GRADIENT-BASED APPROACH TO THE RELAXED OPTIMAL CONTROL PROBLEM

In this section we propose a constructive computational scheme for the constrained LQ-type OCP (6) formulated above. This scheme we propose incorporates a specifically relaxed OCPs associated with the initial problem (6). Let us first recall a simple auxiliary result from the classic convex analysis (see [23, 29, 33]): it is a well known fact that the composition of two convex functionals is not necessarily convex. In the following we will need a basic result providing conditions that ensure convexity of the composition (see e.g., [23, 33]).

Lemma 1. Let $g^1 : \mathscr{W} \to \mathbb{R}$ be a convex functional determined on a convex set $\mathscr{W} \subseteq \mathbb{R}^p$ and $g^2 : \mathscr{V} \to \mathscr{W}$ be an affine mapping defined on a convex subset $\mathscr{V}$ of a real Hilbert space H. Then the composed functional $g : \mathscr{V} \to \mathbb{R}$, $g(\cdot) := g^1(g^2(\cdot))$ is convex.

Let now $x^u(\cdot)$ be a solution to the initial value problem (1) generated by an admissible control $u(\cdot) \in \mathscr{S}$. Evidently, every component of $x^u(\cdot)$ is an affine function (functional) of $u(\cdot)$

$$x(t,u) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau. \tag{7}$$

Here $\Phi(\cdot, \tau)$ is the fundamental solution matrix associated with (1). Let us note that set of admissible controls $\mathscr{S}$ constitutes a non-convex set. This fact is due to the originally combinatorial structure of $\mathscr{S}$ determined in (4). Motivated from this fact let us consider the convex hull $\text{conv}(\mathscr{S})$ associated with $\mathscr{S}$

$$\text{conv}(\mathscr{S}) := \{v(\cdot) \mid v(t) = \sum_{s=1}^{|\mathscr{S}|} \lambda_s u_s(t),\ \sum_{s=1}^{|\mathscr{S}|} \lambda_s = 1,$$
$$\lambda_s \geq 0,\ u_s(\cdot) \in \mathscr{S},\ s = 1, \dots, |\mathscr{S}|\}.$$

From the definition of $\mathscr{S}$ we conclude that the convex set $\text{conv}(\mathscr{S})$ is closed and bounded. Using (4), we also can give the alternative characterization of $\text{conv}(\mathscr{S})$

$$\text{conv}(\mathscr{S}) = \text{conv}(\mathscr{S}_1) \times \ldots \times \text{conv}(\mathscr{S}_m),$$

where $\text{conv}(\mathscr{S}_k)$ is a convex hull of $\mathscr{S}_k$ $k = 1, ..., m$. Since $\text{conv}(\mathscr{Q}^k) \equiv [q_1^{(k)}, q_{M_k}^{(k)}]$, we have

$$\text{conv}(\mathscr{S}_k) := \{v(\cdot) \mid v(t) = \sum_{i=1}^{N_k} I_{[t_{i-1}^{(k)}, t_i^{(k)})}(t) q_{j_i}^{(k)};$$
$$q_{j_i}^{(k)} \in [q_1^{(k)}, q_{M_k}^{(k)}];\ j_i \in \mathbb{Z}[1, M_k];\ t_i^{(k)} \in \mathscr{T}^k\}.$$

Roughly speaking $\text{conv}(\mathscr{S})$ contains all the piecewise constant functions $u(\cdot)$ such that the constant value $u^{(k)}(t)$ belongs to the interval $[q_1^{(k)}, q_{M_k}^{(k)}]$ for all $t \in [t_{i-1}^{(k)}, t_i^{(k)})$. Let us note that in contrast to the initial set $\mathscr{S}$, the corresponding convex hull $conv(\mathscr{S})$ is an infinite dimensional space. Using the above convex construction, we can formulate the following auxiliary OCP

$$\begin{aligned} &\text{minimize } J(u(\cdot)) \\ &\text{subject to } u(\cdot) \in \text{conv}(\mathscr{S}). \end{aligned} \tag{8}$$

The problem (8) formulated above is in fact a relaxation of the initial OCP (6). We will study this problem and use it for a constructive numerical treatment of (6). Let us firstly formulate the following key property of the auxiliary OCP (8).

Theorem 2. The cost functional $J : \text{conv}(\mathscr{S}) \to \mathbb{R}$

$$J(u(\cdot)) = \frac{1}{2} \int_{t_0}^{t_f} [\langle Q(t)x^u(t), x^u(t)\rangle +$$
$$\langle R(t)u(t), u(t)\rangle] dt + \frac{1}{2}\langle Gx^u(t_f), x(t_f)\rangle$$

is convex and the auxiliary OCP (8) constitutes a convex optimization problem in the Hilbert space $\mathbb{L}^2[t_0, t_f; \mathbb{R}^m]$.

Theorem 2 is in fact an immediate consequence of Lemma 1. As we can see (8) is a convex relaxation of the initial OCP (6). The proved convexity of OCP (8) makes it possible to apply the powerful numerical convex programming approaches to this auxiliary optimization problem. In this paper, we use a variant of the projected gradient method for a concrete numerical treatment of (8). Note that under the basic assumptions introduced in Section 2 the following mapping $x^u(t) : \mathbb{L}^2[t_0, t_f; \mathbb{R}^m] \to \mathbb{R}^n$ is Fréchet differentiable for every $t \in [t_0, t_f]$ (see [17, 23]). Therefore, the quadratic costs functional $J(\cdot)$ in (8) is also Fréchet differentiable. We refer to [23, 29] for the corresponding differentiability concept. Assume $u^*(\cdot) \in \text{conv}(\mathscr{S})$ is an optimal solution of (8). The existence of an optimal input $u^*(\cdot)$ is guaranteed in the convex problem (8) (see e.g., [33]). By $x^*(\cdot)$ we next denote the corresponding optimal trajectory (solution) of (1) generated by $u^*(\cdot)$. The projected gradient method for problem (8) can now be expressed as follows:

$$u_{l+1}(\cdot) = \mathscr{P}_{\text{conv}(\mathscr{S})}[u_l(\cdot) - \alpha_l \nabla J(u_l(\cdot))] \quad (9)$$

where $\mathscr{P}_{\text{conv}(\mathscr{S})}$ is the operator of projection on to convex set $\text{conv}(\mathscr{S})$ and $\{\alpha_l\}$ is a sequence of step sizes. The conventional projection operator $\mathscr{P}_{\text{conv}(\mathscr{S})}$ is defined as usual:

$$\mathscr{P}_{\text{conv}(\mathscr{S})}(u(\cdot)) := \text{Arg} \min_{v(\cdot)\in \text{conv}(\mathscr{S})} \left(||v(\cdot) - u(\cdot)||_{\mathbb{L}^2[t_0,t_f;\mathbb{R}^m]} \right)$$

Recall that the projected gradient iterations (9) generate a minimizing sequence for the convex optimization problem (8). Some useful mathematically exact convergence theorems for iterations (9) can be found in [29, 33]. We also refer to [1, 3, 6] for the related results. In the context of OCP (8) and method (9) the basic convergence result can be reformulated as follows.

Theorem 3. Assume that all the hypotheses of Section 2 are satisfied. Consider a sequence of control functions generated by (9). Then there exists an admissible initial data $(u^0(\cdot), x^0(\cdot))$ and a sequence of the step-sizes $\{\alpha_l\}$ such that $\{u_l(\cdot)\}$ is a minimizing sequence for (8), i.e., $\lim_{l\to\infty} J(u_l(\cdot) = J(u^*(\cdot))$.

The proposed gradient-type method (9) provides a possible basis for the computational approach to (8). Using an obtained optimal solution $u^*(\cdot) \in \text{conv}(\mathscr{S})$ we next need to determine a suitable approximation for a solution to the original OCP (1). In the next section we propose a constructive numerical procedure for this purpose.

The study of OCPs with piecewise constant controls also involves a question of the general interest. Consider the initial dynamic system (1) on the given set of admissible controls $\mathscr{S}$ and reformulate the classical controllability question associated with the specific control set of piecewise constant inputs: system (1) on $\mathscr{S}$ is said to be controllable if for any initial state $x(t_0)$ and any final state $x(t_f)$, there exist an admissible function $u(\cdot) \in \mathscr{S}$ that transfers $x(t_0)$ to $x(t_f)$ in finite time. It is necessary to stress, that there are some (expectable) examples of non-controllable linear system involving the piecewise constant controls. In connection with this observation we can formulate a new controllability criterion for the simplified case of constant system/control matrices $A(t) \equiv A$, $B(t) \equiv B$ and unified switching times $N_k \equiv N$ $\mathscr{T}^k \equiv \mathscr{T}$ for all $k = 1,...,m$.

Theorem 4. The stationary linear system (1) is controllable for $u(\cdot) \in \mathscr{S}$ and $N_k \equiv N$, $\mathscr{T}^k \equiv \mathscr{T}$, $k = 1,...,m$ if and only if the following matrix

$$W(N) := \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} e^{-A\tau} d\tau BB^T \int_{t_{i-1}}^{t_i} e^{-A^T\tau} d\tau \right],\ t_i \in \mathscr{T}.$$

is nonsingular.

Theorem 4 can be proved by application of some standard arguments from the lineat control theory (see e.g., [13]). Note that the obtained result makes it possible to establish the existence of an optimal solution to the restricted OCP of the type (6) with additional terminal constraint $x(t_f) = x_f$, where $x_f \in \mathbb{R}^n$ is a prescribed final state.

4. NUMERICAL TREATMENT OF THE INITIAL OPTIMAL CONTROL PROBLEM

Theorem 2 and the classic gradient-type iterations (9) provide a theoretic basis for a simple computational approach to the initial OCP (6). Recall that in contrast to the relaxed optimization problem (8) the original OCP (6) does not possesses any convexity property. Define the formal Hamiltonian associated with problems (6) and (8)

$$H(t,x,u,p) = \langle p, A(t)x + B(t)u\rangle - \frac{1}{2}\left(\langle Q(t)xt, x\rangle + \langle R(t)u,u\rangle\right).$$

where $p \in \mathbb{R}^n$ is the adjoint variable. By $\hat{u}(\cdot) \in \mathscr{S}$ we now denote an optimal solution to the initial OCP (6). Using the explicit representation of the gradient $\nabla J(u_l(\cdot))$ in OCPs with ordinary differential equations (see e.g., [1, 3, 6, 28-29, 37, 40]), we can propose a conceptual computational scheme for the numerical treatment of the initial problem (6).

Conceptual Algorithm 1. (0) Set the initial condition of the iterative scheme $u_{(0)}(\cdot)$, calculate the corresponding trajectory $x_{(l)}(\cdot)$ of (1) and put the iterations register $l := 0$.

(1) Calculate $\nabla J(u_{(l)})(\cdot)$ in accordance with the gradient formalism from [1,3,6,40]

(2) Calculate the projection of $u_{(l)}(\cdot) - \alpha_{(l)} \nabla J(u_{(l)}(\cdot))$ on the convex restriction set $\text{conv}(\mathscr{S})$ and set

$$\bar{u}_{(l+1)}(\cdot) := \mathscr{P}_{\text{conv}(\mathscr{S})}(\bar{u}_{(l)}(\cdot)).$$

(3) Evaluate the $(l+1)$ iteration of the control function given by components $u^{(k)}_{(l+1)}(t) = \sum_{i=1}^{N_k} I_{[t^{(k)}_{i-1}, t^{(k)}_i)}(t)\bar{q}^{(k)}_{i,n}$ for all $k = 1,\dots,m$, where:

$$\bar{q}^{(k)}_{i,l} := \begin{cases} q^{(k)}_1, & \bar{\bar{q}}^{(k)}_{i,l} < q^{(k)}_1 \\ \bar{\bar{q}}^{(k)}_{i,l}, & q^{(k)}_1 \le \bar{\bar{q}}^{(k)}_{i,l} \le q^{(k)}_{M_k} \\ q^{(k)}_{M_k}, & q^{(k)}_{M_k} \le \bar{\bar{q}}^{(k)}_{i,l} \end{cases},\ i = 1,\dots,N_k.$$

and $q^{(k)}_j \in \mathscr{Q}^k$, $\forall j = 1,\dots,M_k$,

$$\bar{\bar{q}}^{(k)}_{i,l} := \frac{1}{\Delta_i}\int_{t_{i-1}}^{t_i} \bar{u}^{(k)}_{(l)}(t)dt,\ \Delta_i := t_i - t_{i-1}.$$

(4) Calculate the difference $|J(u_{(l+1)}(\cdot)) - J(u_{(l)}(\cdot))|$. If it is less than a prescribed accuracy $\varepsilon > 0$, then put $u^*(\cdot) \equiv u_{(l+1)}(\cdot)$ (an approximating optimal solution to (8)). Else, update the iteration register $l = l+1$ and go to Step (1).

(5) Using the evaluated function $u^*(\cdot)$, the approximating optimal control $\hat{u}(\cdot) \in \mathscr{S}$ is finally calculated componentwise

$$\hat{u}^{(k)}(\cdot) = \sum_{i=1}^{N_k} I_{[t^{(k)}_{i-1}, t^{(k)}_i)}(t)\hat{q}^{(k)}_i \quad \forall k = 1,\dots,m.$$

where $\hat{q}^k_i := \text{Arg}\min_{v\in\mathscr{Q}^k} |v - \bar{q}^{(k)}_{i,l+1}|$. Solve (1) with the obtained control input $\hat{u}(\cdot) \in \mathscr{S}$ and obtain the approximating optimal trajectory $\hat{x}(\cdot)$. Stop.

We now illustrate the effectiveness of the proposed Conceptual Algorithm and consider two illustrative examples.

Example 2. Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) + u(t) \end{bmatrix},\ t \in [0,5],$$

for $x(0) = (1,-1)^T$ associated with

$$J(u(\cdot)) = \frac{1}{2}\int_0^5 \left(x_1^2(t) + 10x_2^2(t) + u^2(t)\right)dt,$$

Let $\mathscr{Q} = \{0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, \dots, 5\}$ be the given finite set of constant control values. The classic LQ optimal control $u^{opt}(\cdot)$ can be here easily calculated. Applying Conceptual Algorithm I, we compute $\hat{u}(\cdot)$ (see Fig. 2) and the corresponding optimal trajectory (see Fig. 3).

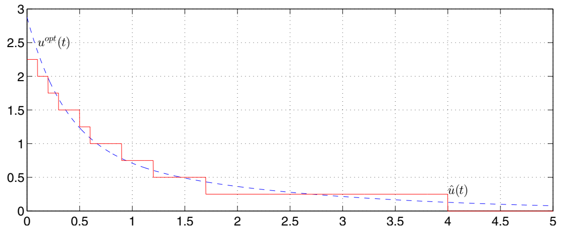

Fig. 2. Control inputs $u^{opt}(t)$ and $\hat{u}(t)$

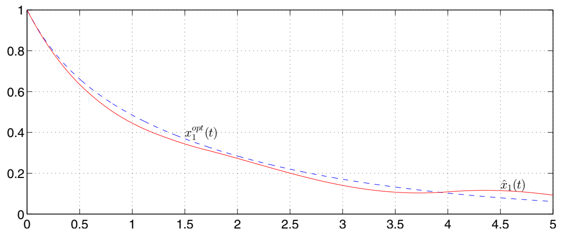

Fig. 3. Components of the optimal trajectories $x_1^{opt}(t)$ and $\hat{x}_1(t)$

The calculated cost in problem (6) for this example is equal to $J(\hat{u}(\cdot)) = 7.5362$.

Example 3. We now consider (1) for the case $n = 3$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -0.875x_2(t) - 20x_3(t) \\ -50x_3(t) + 50u(t) \end{bmatrix},$$
$$x(0) = [\,1 \;\; 0 \;\; -1\,]^T,$$

where $t \in [0,1]$. The quadratic cost functional under consideration has the following form,

$$J(u(\cdot)) = \frac{1}{2}\int_0^1 \left(3x_1^2(t) + x_2^2(t) + 2x_3^2(t) + u^2(t)\right) dt,$$

Assume $\mathcal{Q} = \{-5, -4.5, -4, -3.5, \ldots, 3.5, 4, 4.5, 5\}$. We now apply Conceptual Algorithm I and calculate $\hat{u}(\cdot)$ and the corresponding trajectory in that case. The calculated cost associated with the initial OCP (6) for this example is equal to $J(\hat{u}(\cdot)) = 2.0237$. The corresponding computational results are presented on Fig. 4 and Fig. 5.

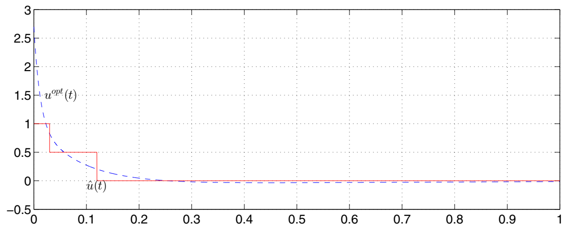

Fig. 4. Control inputs $u^{opt}(t)$ and $\hat{u}(t)$

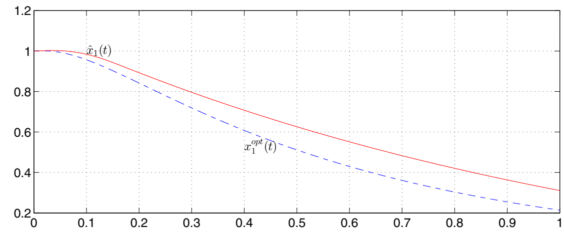

Fig. 5. Components of the optimal trajectories $x_1^{opt}(t)$ and $\hat{x}_1(t)$.

5. CONCLUSION

In this contribution, we proposed a new numerical approach to a non-specific LQ type OCP. This computational scheme is based on a convex relaxation procedure for the initial problem in combination with the usual gradient techniques. We firstly reformulate the original OCP in a relaxed form and establish the corresponding convexity properties. Next we use the obtained relaxation in a constructive solution treatment of the initial OCP. The convex structure of the auxiliary problem makes it possible to take into consideration diverse powerful algorithms of the classic convex programming. We also study the general controllability question associated with the constrained linear dynamic systems under consideration.

Finally, note that the theoretical and computational approaches presented in this paper can be applied to some alternative classes of constrained LQ-type OCPs with switched structure. Let us also note that the proposed numerical algorithm can also constitute a constructive part of some general numerical techniques based on discretizations and linear approximations of the nonlinear OCPs.

REFERENCES

[1] V. Azhmyakov, J. Raisch, Convex control systems and convex optimal control problems with constraints, *IEEE Transactions on Automatic Control*, vol. 53, 2008, pp. 993 – 998.

[2] V. Azhmyakov, S.A. Attia and J. Raisch, On the Maximum Principle for the impulsive hybrid systems, *Lecture Notes in Computer Science*, vol. 4981, Springer, Berlin, 2008, pp. 30 – 42.

[3] V. Azhmyakov, V. Boltyanski, A. Poznyak, Optimal control of impulsive hybrid systems, *Nonlinear Analysis: Hybrid Systems*, vol. 2, 2008, pp. 1089 – 1097.

[4] V. Azhmyakov, R. Galvan-Guerra and M. Egerstedt, Hybrid LQ-optimization using Dynamic Programming, in: *Proceedings of the 2009 American Control Conference*, St. Louis, USA, 2009, pp. 3617 – 3623.

[5] V. Azhmyakov, R. Galvan-Guerra and M. Egerstedt, On the LQ-based optimization technique for impulsive hybrid control systems, in: *Proceedings of the 2010 American Control Conference*, Baltimor, USA, pp. 129 – 135, 2010.

[6] V. Azhmyakov, M. Basin and J. Raisch, A Proximal point based approach to optimal control of affine switched systems, *Discrete Event Dynamic Systems*, vol. 22, no. 1 (2012), pp 61 - 81.

[7] M.V. Basin, A. Ferrara, and D. Calderon-Alvarez, Sliding mode regulator solution to optimal control problem, in: *Proceedings of the 47th Conference on Decision and Control*, Cancun, Mexico, 2008, pp. 2184 – 2189.

[8] S. P. Banks and S. A. Khathur, Structure and control of piecewise-linear systems, *International Journal of Control*, vol. 50, 1989, pp. 667 – 686.

[9] J. Betts, *Practical Methodsfor Optimal Control Problems Using Nonlinear Programming*, SIAM, Philadelphia, USA, 2001.

[10] V. Boltyanski, A. Poznyak, *The Robust Maximum Principle*, Birkhaeuser, New York, USA, 2012.

[11] M.S. Branicky, V.S. Borkar and S.K. Mitter, A unifed framework for hybrid control: model and optimal control theory, *IEEE Transactions on Automatic Control*, vol. 43, 1998, pp. 31 – 45.

[12] R.W. Brockett and D. Liberzon, Quantized feedback stabilization of linear systems, *IEEE Transactions on Automatic Control*, vol. 45, 2000, pp. 1279 – 1289.

[13] A. E. Bryson and Y-C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*, Hemisphere Publishing Corp., New York, USA, 1975.

[14] C. Cassandras, D.L. Pepyne and Y. Wardi, Optimal control of class of hybrid systems, *IEEE Transactions on Automatic Control*, vol. 46, 2001, pp.398 – 415.

[15] X.C. Ding, Y. Wardi and M. Egerstedt, On-line optimization of switched-mode dynamical systems, *IEEE Transactions on Automatic Control*, vol. 54, 2009, pp. 2266 – 2271.

[16] M. Egerstedt, Y. Wardi, and H. Axelsson, Transition-time optimization for switched-mode dynamical systems, *IEEE Transactions on Automatic Control*, vol. 51, 2006, pp. 110 – 115.

[17] H.O. Fattorini, *Infinite-Dimensional Optimization and Control Theory*, Cambridge University Press, Cambridge, UK, 1999.

[18] R. Galvan-Guerra, V. Azhmyakov, M. Egerstedt, Optimization of the multiagent systems with increasing state dimensions: hybrid LQ approach, in: *Proceedings of the 2011 American Control Conference*, San Francisco, USA, 2011, pp. 881 - 887.

[19] M. Garavello and B. Piccoli, Hybrid necessary principle, *SIAM Journal on Control and Optimization*, vol. 43, 2005, pp. 1867 – 1887.

[20] P.E. Gill, W. Murray and M.H. Wright, *Practical Optimization*, Academic Press, New York, USA, 1981.

[21] R. Goebel and M. Subbotin, Continuous time linear quadratic regulator with control constraints via convex duality, *IEEE Trans. Automatic Control*, AC-52, pp. 886 – 892, 2007.

[22] G.C. Goodwin, M.M. Seron and J.A. Dona, *Constrained Control and Estimation*, Springer, London, UK, 2005.

[23] A.D. Ioffe and V.M. Tichomirov, *Theory of Extremal Problems*, North Holland, Amsterdam, The Netherlands 1979.

[24] D. Kirk, *Optimal Control Theory*, Dover, New York, USA, 1998.

[25] A. Kojimaa and M. Morari, LQ control for constrained continuous-time systems, *Automatica*, vol. 40, 2004, pp. 1143 – 1155.

[26] D. Liberzon, Hybrid feedback stabilization of systems with quantized signals, *Automatica*, vol. 39, 2003, pp. 1543 – 1554.

[27] J. Lygeros, *Lecture Notes on Hyrid Systems*, Cambridge University Press, Cambridge, UK, 2003.

[28] E. Polak, *Optimization*, Springer-Verlag, New York, USA, 1997.

[29] B.T. Polyak, *Introduction to Optimization, Optimization Software*, Inc. Publ. Division, New York, USA, 1987.

[30] A. Poznyak, *Advanced Mathematical Tools for Automatic Control Engineers*. Elsevier, Amsterdam, The Netherlands, 2008.

[31] Pytlak, R., *Numerical Methods for Optimal Control Problems with State Constraints*, Springer-Verlag, Berlin, Germany, 1999.

[32] A. Rantzer and M. Johansson, Piecewise linear quadratic optimal control, *IEEE Transactions on Automatic Control*, vol. 45, 2000, pp. 629 – 637.

[33] R.T. Rockafellar, *Convex Analysis*. Princeton University Press, Princeton, USA, 1970.

[34] W.J. Rugh. *Linear Sistem Theory*, Prentice Hall, USA, 1996.

[35] M.S. Shaikh and P.E. Caines, On the hybrid optimal control problem: theory and algorithms, *IEEE Transactions on Automatic Control*, vol. 52, 2007, pp. 1587 – 1603.

[36] S.A. Stanton, B.G. Marchand, Finite set control transcription for optimal control applications, *Journal of Spacecraft and Rockets*, vol. 47, 2010, pp. 457 – 471.

[37] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer, New York, USA, 2002.

[38] J. L. Speyer and D.H. Jacobson, *Primer on Optimal Control Theorey*, SIAM, Philadelphia, USA, 2010.

[39] F. Taringoo and P. Caines, The sensitivity of hybrid systems optimal cost functions with respect to switching manifoold parameters, in: *Lecture Notes in Computer Science*, vol. 5469, Springer, Berlin, Germany, 2009, pp. 475 – 479.

[40] K.L. Teo, C.J. Goh and K.H. Wong, *A Unifed Computational Approach to Optimal Control Problems*, Wiley, New York, USA, 1991.