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Linear-Quadratic Control of Discrete-Time Stochastic Systems with Indefinite Weight Matrices and Mean-Field Terms

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Abstract: In this paper, the linear-quadratic optimal control problem is considered for discrete-time stochastic systems with indefinite weight matrices in the cost function and mean-field terms in both the cost function and system dynamics. A set of generalized difference Riccati equations (GDREs) is introduced in terms of algebraic equality constraints and matrix pseudo-inverse. It is shown that the solvability of the GDRE is not only sufficient but also necessary for the well-posedness of the indefinite mean-field linear-quadratic optimal control problem and the existence of optimal feedback as well as open-loop controls.

Keywords: Indefinite linear-quadratic control, mean-field theory, stochastic system.

1. INTRODUCTION

In this paper, we consider the linear-quadratic (LQ) optimal control problem of mean-field type in a finite horizon. Precisely, the dynamic model of the system is:

$$\begin{cases} x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k + \bar{B}_k \mathbb{E}u_k) \\ \qquad\quad + (C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k u_k + \bar{D}_k \mathbb{E}u_k) w_k, \\ x_l = \zeta, \quad k \in \mathbb{N}_l, \end{cases} \tag{1}$$

where $A_k, \bar{A}_k, C_k, \bar{C}_k \in \mathbb{R}^{n\times n}$, and $B_k, \bar{B}_k, D, \bar{D}_k \in \mathbb{R}^{n\times m}$ are given deterministic matrices; $\mathbb{N}_l$ denote the set $\{l, l+1, \cdots, N-1\}$ for a given positive integrer N and $l \in \{0, 1, \cdots, N-1\}$. In the sequel, $\{l, l+1, \cdots, N\}$ will be denoted by $\bar{\mathbb{N}}_l$, and when $l = 0$, $\mathbb{N}_l$ and $\bar{\mathbb{N}}_l$ will be simply denoted by $\mathbb{N}$ and $\bar{\mathbb{N}}$, respectively. In (1), $\{x_k, k \in \bar{\mathbb{N}}\}$, $\{u_k, k \in \mathbb{N}\}$ and $\{w_k, k \in \mathbb{N}\}$ are the state, control and disturbance process, respectively; $\{w_k\}$ is assumed to be a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, P)$, and

$$\mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0, \quad \mathbb{E}[(w_{k+1})^2|\mathcal{F}_k] = 1, \tag{2}$$

with $\mathcal{F}_k$ being the σ-algebra generated by $\{x_0, w_l, l = 0, 1, \cdots, k\}$. The initial value ζ is measurable with respect to $\mathcal{F}_{l-1}$, and is square integrable. The cost functional associated with (1) is

$$\begin{aligned} J(l,\zeta,u) = \mathbb{E}\Big[\sum_{k=l}^{N-1}\Big(& x_k^T Q_k x_k + (\mathbb{E}x_k)^T \bar{Q}_k \mathbb{E}x_k + u_k^T R_k u_k \\ &+ (\mathbb{E}u_k)^T \bar{R}_k \mathbb{E}u_k\Big)\Big] \\ &+ \mathbb{E}\left(x_N^T G_N x_N\right) + (\mathbb{E}x_N)^T \bar{G}_N \mathbb{E}x_N, \end{aligned} \tag{3}$$

where $Q_k, \bar{Q}_k, R_k, \bar{R}_k, k \in \mathbb{N}_l, G_N, \bar{G}_N$ are deterministic symmetric matrices with appropriate dimensions. We introduce the following admissible control set

$$\left\{ u = (u_0, u_1, \cdots, u_{N-1}) \;\middle|\; \begin{array}{l} u_k \text{ is } \mathcal{F}_{k-1}\text{-measurable}, \\ \sum_{k=l}^{N-1} \mathbb{E}|u_k|^2 < \infty. \end{array} \right\}$$

denoted by $\mathcal{U}_{ad}$. The optimal control problem considered in this paper is stated as follows:

Problem (MF-LQ). *For any given initial pair (l, ζ) with ζ being $\mathcal{F}_{l-1}$-measurable and square-integrable, find a $u^* \in \mathcal{U}_{ad}$ such that*

$$J(\zeta, u^*) = \inf_{u\in\mathcal{U}_{ad}} J(l, \zeta, u). \tag{4}$$

We then call u^* an optimal control for Problem (MF-LQ).

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The above formulation is similar to that of Elliott et al. (2013). But in this paper, the weight matrices $R_k, R_k+\bar{R}_k$, $k \in \mathbb{N}$ in (3) are allowed to be negative in some sense. In other words, here what we are concerned with is an indefinite version of Elliott et al. (2013), which is termed as indefinite mean-field stochastic LQ control problem. The classical indefinite LQ control was first studied by Chen et al. (1998), and then, has been attracting more and more researchers from the control community (Ait Rami et al. (2000, 2001); Chen and Zhou (2000); Yao et al. (2004). This kind of control theory turns out to be useful in solving the well-known mean-variance portfolio selection problems. In Zhou and Li (2000); Lim and Zhou (2002), the analytical results on multi-period mean-variance model comparable to those in the single-period model are firstly achieved. The new feature of (1) and (3) is that both the system dynamics and the cost functions are influenced by mean-field terms. Thus, the problem is essentially a combination of mean-field theory and LQ optimization problem. Mean-field theory was developed to study the collective behaviors resulting from individuals' mutual interactions in various physical and sociological dynamic systems, according to which the interactions among agents can be modeled by a mean-field term. When the number of individuals goes to infinity, the mean-field term will approach the expectation. An exact derivation of this can be achieved by the classical McKean-Vlasov argument. For details, readers may see, for example, Bemsoussan et al. (2011); Mckean (1966); Sznitman (1989) and the references therein. Just as Yong (2013) points out that, in order to make the state process and the control process insensitive to the random disturbances, it is an efficient way to include the variations $\mathrm{var}(x_k)$ and $\mathrm{var}(u_k)$ into the cost functional. This motivates the inclusion of $\mathbb{E}x_k$ and $\mathbb{E}u_k$ into the cost functional (3). An example of this case is the well known mean-variance portfolio selection problems (Li and Ng (2000); Zhou and Li (2000)), in which the risk is quantified by using variance.

Large population stochastic dynamical games (LPSDGs) or mean-field games Huang et al. (2003, 2007); Lasry and Lions (2007) relate to the topic of this paper, which come from the study of multi-agent systems. In Huang et al. (2003, 2007), the authors propose a method of state aggregation and the Nash certainty equivalence principle to construct decentralized ε-Nash equilibria to decrease the computational complexity. Further, Huang et al. (2010) studies LQG mixed games with continuum-parameterized minor players; Li and Zhang (2008) propose ε-Nash equilibria for stochastic cost functionals with both non-coupled and coupled dynamics; Wang and Zhang (2012) discusses the case when each agent's dynamics has random coefficients. Independently, Lasry and Lions (Lasry and Lions (2007)) introduce similar problems from the viewpoint of mean-field theory directly. Compared to mean-field games, the centralized optimal control problem will be studied in this paper.

In this paper, we introduce a set of new type of difference Riccati equations—called the generalized difference Riccati equations (GDREs), which involves matrix pseudo-inverse and algebraic constraints, and turns out to be quite suitable for studying Problem (MF-LQ). This is because that the solvability of this set of equations is not only sufficient but also necessary for the well-posedness of the LQ problem and the attainability of the corresponding optimal controls. Moreover, all the corresponding optimal controls can be derived via the solutions of the GDREs. The obtained results extends those in Elliott et al. (2013).

The remainder of this paper is organized as follows. In Section 2, the optimal control within the set of all the mean-field-type linear feedback controls is searched. This is referred as the closed-loop formulation, where two coupled GDREs are introduced. In Section 3, the solvability of GDREs is shown to be necessary and sufficient for the well-posedness of Problem (MF-LQ) and the existence of an optimal control. Section 4 presents some extension of Problem (MF-LQ). Finally, in Section 5, we give some concluding remarks.

2. CLOSED-LOOP FORMULATION

In this section, we shall search the optimal control within the set of all the mean-field-type linear feedback controls. This is referred as the closed-loop formulation of Problem (MF-LQ). It is shown in this section that if Problem (MF-LQ) is solved by a linear feedback control, then a set of GDREs is existent and solvable. This will be undertaken via matrix minimum principle (Athans (1968)).

Define now the value function of Problem (MF-LQ)

$$V(l,\zeta) = \inf_{u\in\mathcal{U}_{ad}} J(l,\zeta,u).$$

Since the weighting matrices $Q_k, \bar{Q}_k, R_k, \bar{R}_k, k \in \mathbb{N}, G, \bar{G}$ are possibly negative define, Problem (MF-LQ) may be ill-posed. In the following, several notions about this are given.

Definition 2.1. (i) Problem (MF-LQ) is said to be finite or well-posed at (l,ζ) if

$$V(l,\zeta) > -\infty.$$

Problem (MF-LQ) is said to be finite or well-posed if it is finite or well-posed at any (l,ζ).

(ii) Problem (MF-LQ) is said to be (uniquely) solvable or attainable at (l,ζ) if there exists a (unique) $u^* \in \mathcal{U}_{ad}$ such that (4) holds at (l,ζ). Problem (MF-LQ) is said to be (uniquely) solvable or attainable if it is solvable at any (l,ζ).

We recall the pseudo-inverse of a matrix. By Penrose (1955), for a given matrix $M \in \mathbb{R}^{n\times m}$, there exists a unique matrix in $\mathbb{R}^{m\times n}$ denoted by $M^\dagger$ such that

$$\begin{cases} MM^\dagger M = M, \quad M^\dagger MM^\dagger = M^\dagger, \\ (MM^\dagger)^T = MM^\dagger, \quad (M^\dagger M)^T = M^\dagger M. \end{cases} \tag{5}$$

This $M^\dagger$ is called the Moore-Penrose inverse of M. If M is symmetric, by (Ait Rami et al. (2001)), $M^\dagger = (M^\dagger)^T$, $MM^\dagger = M^\dagger M$, and $M \geq 0$ if and only if $M^\dagger \geq 0$.

Let us start from a generic linear feedback control

$$u_k = L_k x_k + \bar{L}_k \mathbb{E}x_k, \quad L_k, \bar{L}_k \in \mathbb{R}^{m\times n}, \quad k \in \mathbb{N}. \tag{6}$$

Under (6), the closed-loop system (1) becomes

$$\begin{cases} x_{k+1} \\ = (A_k + B_k L_k)x_k + [B_k \bar{L}_k + \bar{A}_k + \bar{B}_k (L_k + \bar{L}_k)]\mathbb{E}x_k \\ \quad + (C_k + D_k L_k)x_k w_k \\ \quad + \left[[D_k \bar{L}_k + \bar{C}_k + \bar{D}_k (L_k + \bar{L}_k)]\mathbb{E}x_k \right] w_k, \\ x_l = \zeta, \end{cases} \tag{7}$$

and the cost functional (3) is

$$\begin{aligned} & J(l,\zeta,u) \\ & = \sum_{k=l}^{N-1} \mathbb{E}\left[x_k^T Q_k x_k + (\mathbb{E}x_k)^T \bar{Q}_k \mathbb{E}x_k \right. \\ & \qquad + (L_k x_k + \bar{L}_k \mathbb{E}x_k)^T R_k (L_k x_k + \bar{L}_k \mathbb{E}x_k) \\ & \qquad \left. +((L_k + \bar{L}_k)\mathbb{E}x_k)^T \bar{R}_k (L_k + \bar{L}_k)\mathbb{E}x_k \right] \\ & \quad + \mathbb{E}\left(x_N^T G_N x_N \right) + (\mathbb{E}x_N)^T \bar{G}_N \mathbb{E}x_N \\ & = \sum_{k=l}^{N-1} \left\{ Tr\left[\left(Q_k + L_k^T R_k L_k \right) \mathbb{E}\left(x_k x_k^T \right) \right] \right. \\ & \qquad \left. + Tr\left[\bar{\Phi}_k \left(\mathbb{E}x_k (\mathbb{E}x_k)^T \right) \right] \right\} \\ & \quad + Tr\left[G_N \mathbb{E}\left(x_N x_N^T \right) \right] + Tr\left[\bar{G}_N \left(\mathbb{E}x_N (\mathbb{E}x_N)^T \right) \right], \end{aligned} \tag{8}$$

where

$$\begin{aligned} \bar{\Phi}_k &= \bar{Q}_k + (L_k + \bar{L}_k)^T \bar{R}_k (L_k + \bar{L}_k) \\ &\quad + L_k^T R_k \bar{L}_k + \bar{L}_k^T R_k L_k + \bar{L}_k^T R_k \bar{L}_k. \end{aligned}$$

From the form (6) of the control, we can view $\{(L_k, \bar{L}_k), k \in \mathbb{N}\}$ as the new control input. Also (8) reminds us that $\mathbb{E}x_k(\mathbb{E}x_k)^T$, $\mathbb{E}\left(x_k x_k^T\right)$ can be regarded as the new system states. Write $X_l = \mathbb{E}(\zeta\zeta^T)$, $X_k = \mathbb{E}\left(x_k x_k^T\right)$, $\bar{X}_l = \mathbb{E}\zeta\left(\mathbb{E}\zeta\right)^T$ and $\bar{X}_k = \mathbb{E}x_k(\mathbb{E}x_k)^T$. Then, by (7) we have

$$\begin{aligned} & X_{k+1} \\ & = (A_k + B_k L_k) X_k (A_k + B_k L_k)^T \\ & \quad + (A_k + B_k L_k) \bar{X}_k \left[\bar{A}_k + B_k \bar{L}_k + \bar{B}_k (L_k + \bar{L}_k) \right]^T \\ & \quad + \left[\bar{A}_k + B_k \bar{L}_k + \bar{B}_k (L_k + \bar{L}_k) \right] \bar{X}_k (A_k + B_k L_k)^T \\ & \quad + \left[\bar{A}_k + B_k \bar{L}_k + \bar{B}_k (L_k + \bar{L}_k) \right] \bar{X}_k \\ & \qquad \cdot \left[\bar{A}_k + B_k \bar{L}_k + \bar{B}_k (L_k + \bar{L}_k) \right]^T \\ & \quad + (C_k + D_k L_k) X_k (C_k + D_k L_k)^T \\ & \quad + (C_k + D_k L_k) \bar{X}_k \left[\bar{C}_k + D_k \bar{L}_k + \bar{D}_k (L_l + \bar{L}_k) \right]^T \\ & \quad + \left[\bar{C}_k + D_k \bar{L}_k + \bar{D}_k (L_k + \bar{L}_k) \right] \bar{X}_k (C_k + D_k L_k)^T \\ & \quad + \left[\bar{C}_k + D_k \bar{L}_k + \bar{D}_k (L_k + \bar{L}_k) \right] \bar{X}_k \\ & \qquad \cdot \left[\bar{C}_k + D_k \bar{L}_k + \bar{D}_k (L_k + \bar{L}_k) \right]^T \\ & \equiv \mathcal{X}_k(L_k, \bar{L}_k), \end{aligned} \tag{9}$$

and

$$\begin{aligned} \bar{X}_{k+1} &= \left[(A_k + \bar{A}_k) + (B_k + \bar{B}_k)(L_k + \bar{L}_k) \right] \bar{X}_k \\ &\quad \cdot \left[(A_k + \bar{A}_k) + (B_k + \bar{B}_k)(L_k + \bar{L}_k) \right]^T \\ &\equiv \bar{\mathcal{X}}_k(L_k, \bar{L}_k). \end{aligned} \tag{10}$$

Using $\bar{X}$ and $\bar{X}$, $J(l,\zeta,u)$ with u defined in (6) can be represented as

$$\begin{aligned} & J(l,\zeta,u) \\ & = \sum_{k=l}^{N-1} \left\{ Tr\left[\left(Q_k + L_k^T R_k L_k \right) X_k \right] + Tr\left(\bar{\Phi}_k \bar{X}_k \right) \right\} \\ & \quad + Tr\left(G_N X_N \right) + Tr\left(\bar{G}_N \bar{X}_k \right) \\ & \equiv \mathcal{J}(X_l, \bar{X}_l, \mathcal{L}), \end{aligned} \tag{11}$$

where $\mathcal{L} \equiv \{L_k, \bar{L}_k, k \in \mathbb{N}\}$. Therefore, Problem (MF-LQ) is equivalent to the following problem:

$$\text{Problem (MDO)}: \quad \begin{cases} \min\limits_{L_k, \bar{L}_k \in R^{m \times n}, k \in \mathbb{N}_l} \mathcal{J}(X_l, \bar{X}_l, \mathcal{L}) \\ \text{subject to (9)(10).} \end{cases} \tag{12}$$

Clearly, this is a matrix dynamic optimization problem. The formulation from (9) to (12) has appeared in Elliott et al. (2013), where the definite Mean-field LQ problem is studied.

A natural way to deal with Problem (MDO) is to resort to the matrix minimum principle. For details about this principle, readers may refer to Athans (1968). Following the framework above, we have the following results.

Theorem 2.1. If Problem (MF-LQ) is attained by a linear feedback control of the form

$$u_k = L_k x_k + \bar{L}_k \mathbb{E}x_k, \quad L_k, \bar{L}_k \in \mathbb{R}^{m \times n}, \quad k \in \mathbb{N}$$

with $L_k, \bar{L}_k, k \in \mathbb{N}$ being constant deterministic matrices, then the following GDREs have solution $\{(P_k, T_k), k \in \bar{\mathbb{N}}\}$

$$\begin{cases} P_k = Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k - H_k^T W_k^\dagger H_k, \\ T_k = Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^T P_{k+1} (C_k + \bar{C}_k) \\ \qquad + (A_k + \bar{A}_k)^T T_{k+1} (A_k + \bar{A}_k) - \bar{H}_k^T \bar{W}_k^\dagger \bar{H}_k, \\ P_N = G_N, \quad T_N = G_N + \bar{G}_N, \\ W_k, \bar{W}_k \geq 0, \quad W_k W_k^\dagger H_k - H_k = 0, \\ \bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0, \\ k \in \mathbb{N}, \end{cases} \tag{13}$$

with

$$\begin{cases} W_k = R_k + B_k^T P_{k+1} B_k + D_k^T P_{k+1} D_k, \\ H_k = B_k^T P_{k+1} A_k + D_k^T P_{k+1} C_k, \\ \bar{W}_k = R_k + \bar{R}_k + (B_k + \bar{B}_k)^T T_{k+1} (B_k + \bar{B}_k) \\ \qquad + (D_k + \bar{D}_k)^T P_{k+1} (D_k + \bar{D}_k), \\ \bar{H}_k = (B_k + \bar{B}_k)^T T_{k+1} (A_k + \bar{A}_k) \\ \qquad + (D_k + \bar{D}_k)^T P_{k+1} (C_k + \bar{C}_k), \\ k \in \mathbb{N}. \end{cases} \tag{14}$$

Proof. The proof is omitted here due to the space limitation. □

3. NECESSITY AND SUFFICIENCY OF THE GDRES

Though the GDREs (13) is derived for the closed-loop formulation, we shall show in this section that the solvability of the GDREs (13) is not only sufficient but also necessary to the well-posedness of Problem (MF-LQ).

3.1 Sufficiency

By taking expectation on both sides of System (1) we have

$$\begin{cases} \mathbb{E}x_{k+1} = (A_k + \bar{A}_k)\mathbb{E}x_k + (B_k + \bar{B}_k)\mathbb{E}u_k, \\ \mathbb{E}x_l = \mathbb{E}\zeta. \end{cases} \quad (15)$$

Subtracting (15) from (1) leads to

$$\begin{cases} x_{k+1} - \mathbb{E}x_{k+1} \\ = [A_k(x_k - \mathbb{E}x_k) + B_k(u_k - \mathbb{E}u_k)] \\ \quad + [C_k(x_k - \mathbb{E}x_k) + (C_k + \bar{C}_k)\mathbb{E}x_k \\ \quad + D_k(u_k - \mathbb{E}u_k) + (D_k + \bar{D}_k)\mathbb{E}u_k]w_k, \\ x_l - \mathbb{E}x_l = \zeta - \mathbb{E}\zeta. \end{cases} \quad (16)$$

Theorem 3.1. If the GDREs (13) admit solution $(P_\cdot, T_\cdot)$, then Problem (MF-LQ) is well-posed and solvable. Further,

$$u_k = -\bar{W}_k^\dagger \bar{H}_k \mathbb{E}x_k - W_k^\dagger H_k (x_k - \mathbb{E}x_k), \quad k \in \mathbb{N}, \quad (17)$$

is an optimal control, and the corresponding value function is

$$V(\zeta) = \mathbb{E}\big[(x_l - \mathbb{E}x_l)^T P_l (x_l - \mathbb{E}x_l) \big] + (\mathbb{E}x_l)^T T_l \mathbb{E}x_l. \quad (18)$$

Proof. The proof follows easily from the method of completing the square. □

Corollary 3.1. If $Q_k, Q_k + \bar{Q}, G_N, G_N + \bar{G}_N \geq 0, R_k, R + \bar{R}_k > 0$ in Theorem 3.1, then there is a unique optimal control

$$u_k = -\bar{W}_k^{-1} \bar{H}_k \mathbb{E}x_k - W_k^{-1} H_k (x_k - \mathbb{E}x_k), \quad k \in \bar{\mathbb{N}}_l.$$

Proof. Under the given condition, we can get that $W_k, \bar{W}_k > 0$. Thus, the GDREs (13) is solvable. This completes the proof. □

3.2 Necessity of the GDREs

Consider the following convex set of coupled symmetric matrices on $\mathbb{N}$.

$$\mathcal{M} = \left\{ P_k, T_k \,\middle|\, \begin{array}{l} \left[\begin{array}{c|c} \bar{J}_k & \bar{H}_k^T \\ \hline \bar{H}_k & \bar{W}_k \end{array}\right] \geq 0, \left[\begin{array}{c|c} J_k & H_k^T \\ \hline H_k & W_k \end{array}\right] \geq 0, \\ k \in \mathbb{N}, P_N \leq G_N, T_N \leq G_N + \bar{G}_N \end{array} \right\}, \quad (19)$$

where

$$\begin{cases} \bar{J}_k = Q_k + \bar{Q}_k + (C_k + \bar{C}_k)^T P_{k+1}(C_k + \bar{C}_k) \\ \qquad + (A_k + \bar{A}_k)^T T_{k+1}(A_k + \bar{A}_k) - T_k, \\ \bar{H}_k = (B_k + \bar{B}_k)^T T_{k+1}(A_k + \bar{A}_k) \\ \qquad + (D_k + \bar{D}_k)^T P_{k+1}(C_k + \bar{C}_k) \\ \bar{W}_k = R_k + \bar{R}_k + (D_k + \bar{D}_k)^T P_{k+1}(D_k + \bar{D}_k) \\ \qquad + (B_k + \bar{B}_k)^T T_{k+1}(B_k + \bar{B}_k), \\ J_k = Q_k + A_k^T P_{k+1} A_k + C_k^T P_{k+1} C_k - P_k, \\ H_k = B_k^T P_{k+1} A_k + D_k^T P_{k+1} C_k, \\ W_k = R_k + B_k^T P_{k+1} B_k + D_k^T P_{k+1} D_k. \end{cases} \quad (20)$$

Theorem 3.2. If $\mathcal{M}$ is nonempty, Problem (MF-LQ) is well-posed.

Proof. Similar to Theorem 3.1, we have

$$\begin{aligned} &J(l,\zeta,u) \\ &= \sum_{k=l}^{N-1} \left\{ \begin{bmatrix} \mathbb{E}x_k \\ \mathbb{E}u_k \end{bmatrix}^T \left[\begin{array}{c|c} \bar{J}_k & \bar{H}_k^T \\ \hline \bar{H}_k & \bar{W}_k \end{array}\right] \begin{bmatrix} \mathbb{E}x_k \\ \mathbb{E}u_k \end{bmatrix} \right. \\ &\quad \left. + \mathbb{E}\left(\begin{bmatrix} x_k - \mathbb{E}x_k \\ u_k - \mathbb{E}u_k \end{bmatrix}^T \left[\begin{array}{c|c} J_k & H_k^T \\ \hline H_k & W_k \end{array}\right] \begin{bmatrix} x_k - \mathbb{E}x_k \\ u_k - \mathbb{E}u_k \end{bmatrix} \right) \right\} \\ &\quad + \mathbb{E}\left[(x_N - \mathbb{E}x_N)^T G_N (x_N - \mathbb{E}x_N) \right] \\ &\quad + (\mathbb{E}x_N)^T (G_N + \bar{G}_N)\mathbb{E}x_N \\ &\quad + \mathbb{E}\big[(x_l - \mathbb{E}x_l)^T P_l (x_l - \mathbb{E}x_l) \big] + (\mathbb{E}x_l)^T T_l \mathbb{E}x_l. \end{aligned}$$

Therefore,

$$\begin{aligned} J(l,\zeta,u) &\geq \mathbb{E}\big[(x_l - \mathbb{E}x_l)^T P_l (x_l - \mathbb{E}x_l) \big] + (\mathbb{E}x_l)^T T_l \mathbb{E}x_l \\ &= \mathbb{E}\big[(\zeta - \mathbb{E}\zeta)^T P_l (\zeta - \mathbb{E}\zeta) \big] + (\mathbb{E}\zeta)^T T_l \mathbb{E}\zeta > \infty. \end{aligned}$$

Hence, Problem (MF-LQ) is well-posed. □

We now state the main result of this section.

Theorem 3.3. The following statements are equivalent

(i) Problem (MF-LQ) is well-posed;

(ii) Problem (MF-LQ) is attainable;

(iii) $\mathcal{M}$ (19) is nonempty;

(iv) the GDREs (13) are solvable.

Moreover, when any of the above statements is satisfied, Problem (MF-LQ) is attainable by

$$u_k = -\bar{W}_k^\dagger \bar{H}_k \mathbb{E}x_k - W_k^\dagger H_k (x_k - \mathbb{E}x_k), \quad k \in \mathbb{N}. \quad (21)$$

Proof. The proof is omitted here due to the space limitation. □

4. EXTENSIONS

This section extends results of the previous sections to the case with uncertainties from multi-channels. In this case, the system equation is

$$\begin{cases} x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k + \bar{B}_k \mathbb{E}u_k) \\ \qquad + \sum_{i=1}^{p} (C_{i,k} x_k + \bar{C}_{i,k} \mathbb{E}x_k + D_{i,k} u_k + \bar{D}_{i,k} \mathbb{E}u_k) w_{i,k} \\ \qquad + w_k, \\ x_l = \zeta, \quad k \in \mathbb{N}_l, \end{cases} \quad (22)$$

where $w_{i,k} \in \mathbb{R}$, $w_k \in \mathbb{R}^n$. In (22), the noises $w^i = \{w_{i,k}, k \in \mathbb{N}\}, i = 1, 2, \cdots, p$, satisfy the following conditions:

$$\begin{cases} \mathbb{E}[w_{i,k+1}|\mathcal{F}_k] = 0, \ \mathbb{E}[(w_{i,k+1})^2|\mathcal{F}_k] = \rho_{k+1}^{ii}, \\ \mathbb{E}[w_{i,k+1} w_{i,k+1}|\mathcal{F}_k] = \rho_{k+1}^{ij}, \quad i,j = 1, \cdots, p, i \neq j, \\ \mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0, \ \mathbb{E}[w_{k+1} w_{k+1}^T|\mathcal{F}_k] = V_{k+1}, \\ \mathbb{E}[w_{i,k+1} w_{k+1}|\mathcal{F}_k] = v_{k+1}^i, i = 1, 2, \cdots, p. \end{cases} \quad (23)$$

Taking expectations on both sides of (22), we have

$$\begin{cases} \mathbb{E}x_{k+1} = (A_k + \bar{A}_k)\mathbb{E}x_k + (B_k + \bar{B}_k)\mathbb{E}u_k, \\ \mathbb{E}x_l = \mathbb{E}\zeta. \end{cases} \quad (24)$$

Therefore,

$$\begin{cases} x_{k+1} - \mathbb{E}x_{k+1} \\ = [A_k(x_k - \mathbb{E}x_k) + B_k(u_k - \mathbb{E}u_k)] \\ \quad + \sum_{i=1}^{p} [C_{i,k}(x_k - \mathbb{E}x_k) + (C_{i,k} + \bar{C}_{i,k})\mathbb{E}x_k \\ \quad + D_{i,k}(u_k - \mathbb{E}u_k) + (D_{i,k} + \bar{D}_{i,k})\mathbb{E}u_k] w_{i,k} + w_k \\ \equiv [A_k(x_k - \mathbb{E}x_k) + B_k(u_k - \mathbb{E}u_k)] \\ \quad + \sum_{i=1}^{p} \mathcal{A}_{i,k} w_{i,k} + w_k, \\ x_l - \mathbb{E}x_l = \zeta - \mathbb{E}\zeta. \end{cases} \tag{25}$$

To use the method of completing the square, we need some calculations. Precisely, we have

$$\begin{aligned} &\mathbb{E}\left[(x_{k+1} - \mathbb{E}x_{k+1})^T P_{k+1}(x_{k+1} - \mathbb{E}x_{k+1})\right] \\ &= \mathbb{E}\big[(x_k - \mathbb{E}x_k)^T A_k^T P_{k+1} A_k (x_k - \mathbb{E}x_k) \\ &\quad + (x_k - \mathbb{E}x_k)^T \big(A_k^T P_{k+1} B_k + B_k^T P_{k+1} A_k\big)(u_k - \mathbb{E}u_k)\big] \\ &\quad + \mathbb{E}\big[(u_k - \mathbb{E}u_k)^T B_k^T P_{k+1} B_k (u_k - \mathbb{E}u_k) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} \big(\mathcal{A}_{i,k}^T P_{k+1} \mathcal{A}_{j,k} w_{i,k} w_{j,k}\big)\big] \\ &\quad + \mathbb{E}\big[\sum_{i=1}^{p} \big(\mathcal{A}_{i,k}^T P_{k+1} w_k w_{i,k} + w_k^T P_{k+1} \mathcal{A}_{i,k} w_{i,k}\big)\big] \\ &\quad + \mathbb{E}\big[w_k^T P_{k+1} w_k\big]. \end{aligned}$$

By properties of the noises, we can have an expression of $\mathbb{E}\left[(x_{k+1} - \mathbb{E}x_{k+1})^T P_{k+1}(x_{k+1} - \mathbb{E}x_{k+1})\right]$, with which we then can get the following:

$$\begin{aligned} &J(l, \zeta, u) + \mathbb{E}\left[(x_N - \mathbb{E}x_N)^T P_N (x_N - \mathbb{E}x_N)\right] \\ &\quad - \mathbb{E}\left[(x_l - \mathbb{E}x_l)^T P_l (x_l - \mathbb{E}x_l)\right] \\ &\quad + (\mathbb{E}x_N)^T T_N \mathbb{E}x_N - (\mathbb{E}x_l)^T T_l \mathbb{E}x_l \\ &\quad + 2(\mathbb{E}x_N)^T \phi_N - 2(\mathbb{E}x_l)^T \phi_l \\ &= \sum_{k=l}^{N} \mathbb{E}\Big\{(x_k - \mathbb{E}x_k)^T \big(Q_k + A_k^T P_{k+1} A_k \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} \big(C_{i,k}^T P_{k+1} C_{j,k} \rho_k^{ij}\big) - P_k\big)(x_k - \mathbb{E}x_k) \\ &\quad + (x_k - \mathbb{E}x_k)^T \big(A_k^T P_{k+1} B_k + B_k^T P_{k+1} A_k \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} \rho_k^{ij} \big(C_{i,k}^T P_{k+1} D_{j,k} + C_{j,k} P_{k+1} D_{i,k}\big)\big)(u_k - \mathbb{E}u_k) \\ &\quad + (u_k - \mathbb{E}u_k)^T \big[R_k + B_k^T P_{k+1} B_k \\ &\quad + \sum_{i=1}^{p}\sum_{i=1}^{p} \big(\rho_k^{ij} D_{i,k}^T P_{k+1} D_{j,k}\big)\big](u_k - \mathbb{E}u_k) \\ &\quad + (\mathbb{E}x_k)^T \big[Q_k + \bar{Q}_k + (A_k + \bar{A}_k)^T T_{k+1}(A_k + \bar{A}_k) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} \rho_k^{ij} (C_{i,k} + \bar{C}_{i,k})^T P_{k+1}(C_{j,k} + \bar{C}_{j,k}) - T_k\big]\mathbb{E}x_k \\ &\quad + 2(\mathbb{E}x_k)^T \big[(A_k + \bar{A}_k)^T T_{k+1}(B_k + \bar{B}_k) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} \rho_k^{ij} (C_{i,k} + \bar{C}_{i,k})^T P_{k+1}(D_{j,k} + \bar{D}_{j,k})\big]\mathbb{E}u_k \\ &\quad + (\mathbb{E}u_k)^T \big[R_k + \bar{R}_k + (B_k + \bar{B}_k)^T T_{k+1}(B_k + \bar{B}_k) \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} \rho_k^{ij} (D_{i,k} + \bar{D}_{i,k})^T P_{k+1}(D_{j,k} + \bar{D}_{j,k})\big]\mathbb{E}u_k \\ &\quad + 2(\mathbb{E}x_k)^T \big[\sum_{i=1}^{p} (C_{i,k} + \bar{C}_{i,k})^T P_{k+1} v_k^i \\ &\quad + (A_k + \bar{A}_k)\phi_{k+1} - \phi_k\big] \\ &\quad + 2(\mathbb{E}u_k)^T \big[\sum_{i=1}^{p} (D_{i,k} + \bar{D}_{i,k})^T P_{k+1} v_k^i \\ &\quad + (B_k + \bar{B}_k)^T \phi_{k+1}\big] + Tr[P_{k+1} V_k]\Big\} \\ &\quad + \mathbb{E}\left[(x_N - \mathbb{E}x_N)^T G_N (x_N - \mathbb{E}x_N)\right] \\ &\quad + (\mathbb{E}x_N)^T (G_N + \bar{G}_N)\mathbb{E}x_N. \end{aligned} \tag{26}$$

Now, we introduce a set of GDREs

$$\begin{cases} P_k = Q_k + A_k^T P_{k+1} A_k + \sum_{i=1}^{p}\sum_{j=1}^{p} \big(C_{i,k}^T P_{k+1} C_{j,k} \rho_k^{ij}\big) \\ \qquad - H_k^T W_k^\dagger H_k, \\ T_k = Q_k + \bar{Q}_k + (A_k + \bar{A}_k)^T T_{k+1}(A_k + \bar{A}_k) \\ \qquad + \sum_{i=1}^{p}\sum_{j=1}^{p} \rho_k^{ij} (C_{i,k} + \bar{C}_{i,k})^T P_{k+1}(C_{j,k} + \bar{C}_{j,k}) \\ \qquad - \bar{H}_k^T \bar{W}_k^\dagger \bar{H}_k, \\ P_N = G_N, \quad T_N = G_N + \bar{G}_N, \\ W_k, \bar{W}_k \geq 0, \quad W_k W_k^\dagger H_k - H_k = 0, \\ \bar{W}_k \bar{W}_k^\dagger \bar{H}_k - \bar{H}_k = 0, \\ k \in \mathbb{N}, \end{cases} \tag{27}$$

where

$$\begin{cases} W_k = R_k + B_k^T P_{k+1} B_k + \sum_{i=1}^{p}\sum_{i=1}^{p} \big(\rho_k^{ij} D_{i,k}^T P_{k+1} D_{j,k}\big), \\ H_k = B_k^T P_{k+1} A_k + \sum_{i=1}^{p}\sum_{j=1}^{p} \big(\rho_k^{ij} D_{i,k}^T P_{k+1} C_{j,k}\big), \\ \bar{W}_k = R_k + \bar{R}_k + (B_k + \bar{B}_k)^T T_{k+1}(B_k + \bar{B}_k) \\ \qquad + \sum_{i=1}^{p}\sum_{j=1}^{p} \rho_k^{ij} (D_{i,k} + \bar{D}_{i,k})^T P_{k+1}(D_{j,k} + \bar{D}_{j,k}), \\ \bar{H}_k = (B_k + \bar{B}_k)^T T_{k+1}(A_k + \bar{A}_k) \\ \qquad + \sum_{i=1}^{p}\sum_{j=1}^{p} \rho_k^{ij} (D_{j,k} + \bar{D}_{j,k})^T P_{k+1}(C_{i,k} + \bar{C}_{i,k}), \\ k \in \mathbb{N}. \end{cases} \tag{28}$$

Let

$$\begin{cases} \phi_k = (A_k + \bar{A}_k)\phi_{k+1} + \sum_{i=1}^{p} (C_{i,k} + \bar{C}_{i,k})^T P_{k+1} v_k^i \\ \qquad - \bar{H}_k^T \bar{W}_k^\dagger \psi_k, \\ \bar{W}_k \bar{W}_k^\dagger \psi_k - \psi_k = 0, \\ \phi_N = 0, \end{cases} \tag{29}$$

with

$$\psi_k = \sum_{i=1}^{p} (D_{i,k} + \bar{D}_{i,k})^T P_{k+1} v_k^i + (B_k + \bar{B}_k)^T \phi_{k+1}. \quad (30)$$

Then, we have the following theorem which provides the optimal solution to LQ problem for general system (22) with performance functional (3).

Theorem 4.1. Suppose that the GDREs (27) and equation (29) are solvable. Then Problem (MF-LQ) with system (22) is attained by an optimal control in the form

$$u_k = -\bar{W}_k^\dagger (\bar{H}_k \mathbb{E}x_k + \psi_k) - W_k^\dagger H_k \left(x_k - \mathbb{E}x_k \right), \quad (31)$$

where the matrices are defined in (27)-(30). Further, the optimal cost value is given by

$$\begin{aligned} V(l,\zeta) = & \sum_{k=l}^{N-1} \Big[Tr[P_{k+1} V_k - \psi_k^T \bar{W}_k^\dagger \psi_k \Big] \\ & + \mathbb{E}\big[\left(x_l - \mathbb{E}x_l \right)^T P_l \left(x_l - \mathbb{E}x_l \right) \big] \\ & + \left(\mathbb{E}x_l \right)^T T_l \mathbb{E}x_l + 2\phi_l^T \mathbb{E}x_l. \end{aligned}$$

Proof. The proof is omitted here due to the space limitation. □

5. CONCLUSION

It is shown in this paper that the solvability of the GDRE is not only sufficient but also necessary for the well-posedness of the indefinite mean-field LQ optimal control problem and the existence of optimal feedback as well as open-loop controls. For future research, we may study the definite and indefinite mean-field LQ control with regime switching (or Markov jump parameter).

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