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Overcoming the Dissipation Condition in Passivity-based Control for a class of mechanical systems

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Abstract: A well-known problem in the application of the Interconnection and Damping Assignment technique for the stabilization of underactuated mechanical systems is dissipation in unactuated coordinates, since it may impede the definiteness requirements for the closed-loop system. Recently, the expansion of the closed-loop Hamiltonian function by a cross term between coordinates and momenta has been explored showing promising results. However, the large number of free parameters is an issue for the tuning of the closed-loop system, and the solution of the matching partial differential equations (PDEs) remains a difficult task. In this work, we aim at giving the closed-loop augmented Hamiltonian more structure in order to simplify the controller parametrization. The result is desired behavior at the equilibrium avoiding the solution of PDEs. Simulations and experiments demonstrate the applicability of the method.

1. INTRODUCTION

Total energy shaping control techniques like Interconnection and Damping Assignment Passivity-Based control (IDA-PBC) and the method of Controlled Lagrangians (CL) have become popular during the last years. These methods shape the energy of the system but preserve its physical structure and have thus become attractive for the stabilization of (underactuated [2]) mechanical systems (see Acosta et al. [2005], Ortega and Spong [2000], Bloch et al. [2000]). The controller design can be summarized in two simple steps: *energy shaping* of the conservative system in order to assign a local minimum of the closed-loop energy at the desired equilibrium and *damping injection* to asymptotically stabilize the equilibrium. The role of damping for the stability of mechanical systems is ambiguous, though: it either has a stabilizing or destabilizing effect. Krechetnikov and Marsden [2007] describe with several examples the phenomenon of dissipation-induced instabilities - for systems stabilized by gyroscopic forces. Physical dissipation is mostly neglected in IDA-PBC or CL for the sake of simplicity and mathematical elegance. Yet, it plays a crucial role in the applicability of energy shaping control techniques to real physical systems: as shown in Gómez-Estern and van der Schaft [2004], physical dissipation in unactuated degrees of freedom can impede the implementation of an IDA-PBC controller. The so-called *dissipation condition* determines if required definiteness properties for the closed-loop system can be fulfilled in the presence of dissipation or not. For a related analysis of the effect of physical damping from the CL point of view see for example Woolsey et al. [2004].

To overcome the dissipation condition, the desired Hamiltonian function can be augmented by a cross term between coordinates and momenta as reported in Kotyczka and Delgado L. [2012]. This leads to a more general representation of the closed-loop port-Hamiltonian (pH) system and a considerable amount of free parameters. The tuning of the desired system is no longer intuitive: it is not possible to achieve a *physically motivated* choice of the design parameters, since the approach breaks the physical structure of the system. Transparency with respect to achievable dynamics can however be provided by *local linear dynamics assignment* presented in Kotyczka [2011]. The resulting nonlinear controller guarantees desired local behavior and provides an estimate of the domain of attraction based on standard IDA-PBC arguments.

In the most common version of IDA-PBC for underactuated mechanical systems, the structure of the interconnection and damping matrices is fixed, and all assignable energy functions are characterized by the solution of a set of PDEs (see e.g. Ortega and Spong [2000]). Here, we show a systematic way to compute the controller without solving any PDEs: we fix the desired Hamiltonian and parametrize the closed-loop system based on the solution of one Lyapunov and some algebraic equations. In Acosta and Astolfi [2009], another approach is pursued to obviate the solution of PDEs for general input-affine systems by designing an approximating integral together with a dynamic extension to replace the PDEs with algebraic inequalities.

The remaining of the paper is organized as follows. Section 2 recalls the main idea of IDA-PBC for underactuated mechanical systems and introduces the dissipation condition. In Section 3 the issues of the dissipation condition are exemplarily explained on a linear mechanical system. Therefrom, an intuitive solution can be derived for the nonlinear system in case the dissipation condition is not satisfied. The main results of this note are presented in

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[2] A mechanical system is said to be underactuated, if not all degrees of freedom can be controlled directly.

Section 4 and Section 5. The controller design procedure, simulations and experimental results are finally shown in Section 6 to demonstrate its practicability.

Remark 1. In this paper we only consider the IDA-PBC framework. The CL case can be tackled in a similar manner, since both formulations are equivalent as shown in Blankenstein et al. [2002] and Chang et al. [2002].

Notation: If obvious from the context, arguments are dropped for simplicity. The index 0 as in x_0 denotes an initial state; in the case of a matrix (e.g. W_0), it denotes $W(q)$ evaluated at the equilibrium. The column vector of partial derivatives with respect to x is represented as ∇_x.

2. IDA-PBC FOR UNDERACTUATED MECHANICAL SYSTEMS

Let us first briefly introduce the common IDA-PBC approach for underactuated mechanical systems as presented in Ortega and Spong [2000]. We consider Hamiltonian systems of the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & -R(q) \end{bmatrix} \begin{bmatrix} \nabla_q H(q,p) \\ \nabla_p H(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u, \tag{1}$$

where $q \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ are the generalized coordinates and momenta, respectively, $u \in \mathbb{R}^m$ is the input, and $G(q) \in \mathbb{R}^{n\times m}$ the input matrix with $\mathrm{rank}(G(q)) = m < n$. The dissipation matrix $R(q) = R^T(q) \geq 0$ is assumed to satisfy $G_\perp R = 0$ [3]. The Hamiltonian

$$H(q,p) = \frac{1}{2} p^T M^{-1}(q) p + V(q) \tag{2}$$

corresponds to the total energy with inertia matrix $M(q) > 0$ and potential energy $V(q)$. The state feedback

$$\begin{aligned} u = G^\dagger \left(\nabla_q H + RM^{-1}p - M_d M^{-1} \nabla_q H_d\right) \\ + G^\dagger \left((J_2 - R_2) M_d^{-1} p\right) \end{aligned} \tag{3}$$

with $G^\dagger = (G^T G)^{-1} G^T$, transforms (1) into a pH system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2 - R_2 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \tag{4}$$

with new (shaped) energy

$$H_d(q,p) = \frac{1}{2} p^T M_d^{-1}(q) p + V_d(q) \tag{5}$$

if the projected matching equations

$$\begin{aligned} G_\perp \left(M_d M^{-1} \nabla_q (p^T M_d^{-1} p) - 2 J_{21} M_d^{-1} p\right) \\ - G_\perp \nabla_q (p^T M^{-1} p) = 0 \end{aligned} \tag{6}$$

$$G_\perp \left(\nabla_q V - M_d M^{-1} \nabla_q V_d\right) = 0 \tag{7}$$

are satisfied. Here, $G_\perp \in \mathbb{R}^{(n-m)\times n}$ is a full rank left annihilator satisfying $G_\perp G = 0$, and $J_2 = J_{20}(q) + J_{21}(q,p)$ is a skew-symmetric matrix with J_{21} linear in p [4]. The inertia matrix PDE (6) and the potential energy PDE (7) correspond to the terms that are quadratic in p or independent from p, respectively. If further

$$q^* = \arg\min V_d(q), \ \ M_d(q) > 0 \ \text{ and } \ R_2(q) \geq 0 \tag{8}$$

in a neighborhood of q^*, then the equilibrium $(q^*, 0)$ is (locally) stable with Lyapunov function $H_d(q,p)$. Asymptotic stability can be shown by invoking LaSalle's invariance principle. See Gómez-Estern and van der Schaft [2004], and Ortega and Spong [2000] for proofs and details.

[3] no dissipation in unactuated coordinates

[4] Note that J_{20} is chosen such that $G_\perp J_{20} = 0$ and $R_2 = G K_d G^T$ for a positive semidefinite matrix $K_d(q) \in \mathbb{R}^{m\times m}$, since $G_\perp R = 0$.

2.1 Dissipation Condition

In contrast to above, we consider the case $G_\perp R \neq 0$. Thus, we additionally get a new set of algebraic matching equations (corresponding to the terms that are linear in p)

$$G_\perp \left(RM^{-1}p + (J_{20} - R_2) M_d^{-1} p\right) = 0. \tag{9}$$

Gómez-Estern and van der Schaft [2004] derive from (9) the dissipation condition

$$G_\perp \left(RM^{-1}M_d + M_d M^{-1} R\right) G_\perp^T \geq 0, \tag{10}$$

and show, that it is a necessary and sufficient condition for the existence of a passive closed-loop system with positive definite storage function H_d. It is yet known, that in the presence of physical damping in unactuated degrees of freedom for many mechanical systems - such as the Acrobot system, the Furuta and the inverted pendulum among others - it is not possible to find a solution of the matching equations (6), (7), and (9) which satisfies the definiteness requirements (8) (see for example Kotyczka and Delgado L. [2012], Gómez-Estern and van der Schaft [2004], or Woolsey et al. [2004] for the CL point of view).

3. MOTIVATING EXAMPLE

The idea of the present note is easily motivated by looking at a linear mechanical system. A substantial analysis of IDA-PBC for linear time-invariant (LTI) systems can be found in Prajna et al. [2002] and Ortega and Liu [2012]. The CL case is treated in Zenkov [2002]. We consider LTI mechanical systems represented in Hamiltonian form

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & -R \end{bmatrix} \begin{bmatrix} \nabla_q H(q,p) \\ \nabla_p H(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u = Ax + Bu, \tag{11}$$

where the constant system matrices can be written as:

$$A = \begin{bmatrix} 0 & M^{-1} \\ -Q & -RM^{-1} \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ G \end{bmatrix}. \tag{12}$$

The Hamiltonian in (11) takes the quadratic form

$$H(q,p) = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} q^T Q q. \tag{13}$$

As in the previous section, the goal is to transform (11) by state feedback into a new LTI mechanical system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2 - R_2 \end{bmatrix} \begin{bmatrix} Q_d & 0 \\ 0 & M_d^{-1} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = A_d x \tag{14}$$

with shaped energy function

$$H_d(q,p) = \frac{1}{2} p^T M_d^{-1} p + \frac{1}{2} q^T Q_d q. \tag{15}$$

The applicability of IDA-PBC to LTI systems is equivalent to the solvability of the LMIs (Proposition 3.1 in Ortega and Liu [2012] and Proposition 7 in Prajna et al. [2002])

$$P > 0 \tag{16}$$

$$\mathrm{sym}\{B_\perp A P^{-1} B_\perp^T\} \leq 0, \tag{17}$$

where $B_\perp$ is a full rank left annihilator of B ($B_\perp B = 0$). The matrix P is the Hessian of the desired quadratic energy function which in this case has a predefined block-diagonal structure arising from (15):

$$P = \begin{bmatrix} Q_d & 0 \\ 0 & M_d^{-1} \end{bmatrix}. \tag{18}$$

The matching of (11) and (14) is satisfied if there exists a solution of the set of LMIs (16) and (17) restricted to

(18) (see e.g. Zenkov [2002] and Kotyczka and Delgado L. [2012]). The LMI (17) can also be written as

$$G_\perp(M_d M^{-1} - Q Q_d^{-1}) = 0 \tag{19}$$

$$-G_\perp(R M^{-1} M_d + M_d M^{-1} R) G_\perp^T \leq 0, \tag{20}$$

which represent the matching of the potential energy (7), and the dissipation condition (10), respectively. LTI Systems trivially satisfy the kinetic energy matching equation. For the inverted pendulum, there exist no $M_d, Q_d > 0$ which solve (19) and (20) in the case $G_\perp R \neq 0$.

The central question which is addressed in this note is whether or not it is possible to transform a damped mechanical system into a closed-loop pH system by static feedback if the dissipation condition is not satisfied. Ortega and Liu [2012] showed that IDA-PBC is equivalent to stabilizability. The (damped) inverted pendulum is controllable and thus stabilizable. By allowing off diagonal entries in the matrix P - representing a cross term between coordinates and momenta in the energy function (15) - the set of LMIs (16) and (17) can be easily solved.

Remark 2. In Zenkov [2002], the stabilization of (conservative) linear mechanical systems using only position feedback in the CL framework is discussed. Therein, the closed-loop Hamiltonian is initially assumed to have a non-block-diagonal Hessian, which corresponds to the structure of the closed-loop Hamiltonian (21) in the present note.

4. MAIN RESULT

In Kotyczka and Delgado L. [2012] the violation of the dissipation condition in applying the IDA procedure for mechanical systems has been overcome by assuming a closed-loop Hamiltonian, augmented by a mixed term of coordinates and momenta $p^T n(q)$. The approach was motivated by the existence of a stabilizing linear state feedback. The Lyapunov function related to the stabilization problem is shown to possess a non-block-diagonal solution. However, a series of difficulties arises in the application of the proposed augmented IDA-PBC approach: The solution of additional PDEs for the new functions $n_i(q)$ is required, the estimate of the region of attraction is poor, and the controller becomes confusingly complicated. The present work aims at solving some of these issues. It turns out that the choice $p^T n(q) = -p^T K(q) \nabla_q V_d(q)$ for a regular matrix $K(q)$ simplifies the augmented IDA-PBC approach. Consider the Hamiltonian [5]

$$H_d(q,p) = \frac{1}{2} p^T {M_d}^{-1} p + V_d(q) - p^T K \nabla_q V_d(q) \tag{21}$$

for the generalized target pH system

$$\mathcal{F} \nabla H_d = \begin{bmatrix} W(q) & X(q) \\ Y(q) & Z(q) + J_{21}(q,p) \end{bmatrix} \begin{bmatrix} \nabla_q H_d(q,p) \\ \nabla_p H_d(q,p) \end{bmatrix}, \tag{22}$$

with $J_{21}(q,p) = -J_{21}(q,p)^T$ linear in p. Note that the closed-loop interconnection and damping matrices in $\mathcal{F}(q,p) = \mathcal{J}(q,p) - \mathcal{R}(q)$ are of a more general form as in the classic approach. The goal is to find a static state feedback which renders (1) the modified pH system (22). For a given closed-loop Hamiltonian (21), $W(q)$ and $X(q)$ can be explicitly calculated: The matching of the first rows of (1) and (22) leads to - splitting the equation in terms independent and linear in p:

[5] The matrices K and M_d are chosen to be constant for simplicity.

$$W(\nabla_q^2 V_d K^T) - X M_d^{-1} + M^{-1} = 0 \tag{23}$$

$$(W - XK) \nabla_q V_d = 0. \tag{24}$$

With V_d, M_d and K fixed, (23) and (24) are satisfied by

$$X = M^{-1} \left(M_d^{-1} - K \nabla_q^2 V_d K^T \right)^{-1}, \tag{25}$$

$$W = XK = M^{-1} \left(M_d^{-1} - K \nabla_q^2 V_d K^T \right)^{-1} K. \tag{26}$$

Furthermore, the unactuated part of the second rows of (1) and (22) must match. A sufficient condition is the solution of the *new* matching equations (splitting the equation in different dependencies on p - quadratic, linear and independent):

$$G_\perp \left(Y \nabla_q (p^T M_d^{-1} p) + 2 J_{21} M_d^{-1} p \right) + G_\perp \nabla_q (p^T M^{-1} p) = 0 \tag{27}$$

$$G_\perp \left(R M^{-1} + Z M_d^{-1} - Y \nabla_q^2 V_d K^T \right) p - G_\perp J_{21} K \nabla_q V_d = 0 \tag{28}$$

$$G_\perp \left(\nabla_q V + (Y - ZK) \nabla_q V_d \right) = 0. \tag{29}$$

Assumption 1. The inertia matrix M does not depend on unactuated coordinates, i.e. $G_\perp \nabla_q (p^T M^{-1} p) = 0$.

Proposition 1. Given the solution $L(q)$ of

$$G_\perp \left(\nabla_q V + (L + R) M^{-1} M_d K \nabla_q V_d \right) = 0 \tag{30}$$

for a given mechanical system (1) with inertia matrix M, potential energy V and dissipation matrix R, and for fixed V_d, M_d and K, the state feedback

$$u = G^\dagger \left(\nabla_q H + R M^{-1} p + Y \nabla_q H_d + Z \nabla_p H_d \right) \tag{31}$$

with $G^\dagger = (G^T G)^{-1} G^T$, $Y = LW$, and

$$Z = \left(Y \nabla_q^2 V_d K^T - R M^{-1} \right) M_d + G v^T,$$

transforms (1) into the closed-loop system (22).

Proof. Let $J_{21} = 0$ and M_d be constant, such that (27) is satisfied. The matching of the actuated part of (1) and (22) yields the control law (31). The solution of (28) requires

$$Z = \left(Y \nabla_q^2 V_d K^T - R M^{-1} \right) M_d + G v^T \tag{32}$$

for an arbitrary vector $v = v(q) \in \mathbb{R}^n$. With Z as in (32), equation (29) takes the form

$$G_\perp (\nabla_q V + \Sigma \nabla_q V_d) = 0, \tag{33}$$

where $\Sigma = Y - \left(Y \nabla_q^2 V_d K^T - R M^{-1} \right) M_d K$, or after some simple calculations

$$\begin{aligned} \Sigma &= \left(Y K^{-1} \left(M_d^{-1} - K \nabla_q^2 V_d K^T \right) + R M^{-1} \right) M_d K \\ &= \left(Y W^{-1} + R \right) M^{-1} M_d K. \end{aligned} \tag{34}$$

according to (26). Defining $L = Y W^{-1}$ and rewriting equation (33) with (34) finishes the proof.

5. STABILITY AND CONTROLLER DESIGN PROCEDURE

One of the fundamental difficulties of IDA-PBC is the parametrization of the closed-loop system: For the desired equilibrium to be stable, $(q^*, 0)$ needs to be an isolated minimum of the desired Hamiltonian, and $\mathcal{R}(q) \geq 0$ is required around q^*. The vast amount of degrees of freedom of the presented approach makes it hard to choose an appropriate parametrization. Stability and desired local behavior for the closed-loop system can be guaranteed, however, by matching the linearized closed-loop pH system with a desired LTI system.

Assumption 2. $(q^*, 0)$ is an admissible equilibrium, i.e. $G^{\perp}\nabla_q V(q)|_{q^*} = 0$ holds.

Let $\Delta\dot{x} = A\Delta x + Bu$ be the linearized mechanical system around the equilibrium $x^* = (q^{*T}, 0^T)^T$, $(\Delta x = x - x^* = (q - q^*, p))$ and let the pair (A, B) be controllable. Design a linear state feedback $u = -D\Delta x$ such that the state matrix $A_d = A - BD$ is Hurwitz. Then there exists a unique positive definite matrix P, which solves the Lyapunov equation

$$A_d P^{-1} + P^{-1} A_d^T = -2R_0 \tag{35}$$

for any $R_0 > 0$ (see e.g. Boyd et al. [1994]).

Proposition 2. Let the linear feedback $u = -D\Delta x$ for the damped underactuated mechanical system (1) result in a closed-loop system locally approximated by the state matrix A_d. Take a positive definite matrix $R_0 > 0$ and compute the solution

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0 \tag{36}$$

of (35). Set

$$Q_d = P_{11}, \;\; M_d = P_{22}^{-1}, \;\; K = -P_{12}^T P_{11}^{-1} \tag{37}$$

$$\text{and} \;\; \begin{bmatrix} W_0 & X_0 \\ Y_0 & Z_0 \end{bmatrix} = A_d P^{-1}. \tag{38}$$

Fix $V_d(q)$ with $\nabla_q V_d|_{q^*} = 0$ and $\nabla_q^2 V_d|_{q^*} = Q_d$ and suppose one can find a solution $L(q)$ of (30), satisfying $L(q^*) = Y(q^*)W(q^*)^{-1} = Y_0 W_0^{-1}$. Then, the nonlinear control law (31) asymptotically stabilizes the equilibrium $(q^*, 0)$. From the largest bounded level set of $H_d(q, p)$ where $\mathcal{R}(q) > 0$ holds, an estimate of the domain of attraction can be derived. The closed-loop system is locally approximated by

$$\Delta\dot{x} = A_d \Delta x. \tag{39}$$

Proof. The control law (31) transforms (1) into a port-Hamiltonian system (22) under the conditions of the proposition and with W, X and Z according to (25), (26) and (32), respectively. From the parameter choice in (37) and the structure of the closed-loop energy (21),

$$\frac{\partial^2 H_d}{\partial x^2}|_{x^*} = P > 0 \tag{40}$$

can be deduced, i.e. positive definiteness of H_d in an open neighborhood of $(q^*, 0)$ is guaranteed. The dissipation matrix at the equilibrium is

$$\mathcal{R}(q^*) = -\frac{1}{2}(A_d P^{-1} + P^{-1} A_d) = R_0 > 0. \tag{41}$$

Since the elements of $\mathcal{R}(q)$ are continuous functions in q, strong dissipativity in an open neighborhood of $(q^*, 0)$ is guaranteed. An estimate of the region of attraction follows from usual Lyapunov arguments. The linearization of the closed-loop pH system around $(q^*, 0)$ yields directly (39).

Six steps summarize the controller design procedure

Step 1: Linearize the mechanical system around the desired equilibrium $(q^*, 0)$ and design a linear state feedback $u = -D\Delta x$, such that the closed-loop dynamics are given by

$$\Delta\dot{x} = A_d\,\Delta x, \qquad \Delta x = (q - q^*, p) \in \mathbb{R}^{2n} \tag{42}$$

with A_d Hurwitz. Desired local performance properties can be formulated in terms of the eigenvalues of A_d or an LQR design to determine D.

Step 2: Fix $R_0 = R_0^T > 0$ and calculate the solution P^{-1} of the Lyapunov equation

$$A_d P^{-1} + P^{-1} A_d^T = -2R_0. \tag{43}$$

The matrix P is the Hessian of the desired Hamiltonian function at the equilibrium:

$$\nabla^2 H_d(q, p)|_{(q^*,0)} = \begin{bmatrix} Q_d & -Q_d K^T \\ -KQ_d & M_d^{-1} \end{bmatrix} = P. \tag{44}$$

Further, calculate the interconnection and damping matrices at the equilibrium:

$$\begin{bmatrix} W_0 & X_0 \\ Y_0 & Z_0 \end{bmatrix} = A_d P^{-1}. \tag{45}$$

Step 3: Fix the potential energy of the closed-loop system $V_d(q)$, such that

$$\nabla_q^2 V_d(q)|_{q^*} = Q_d, \;\; \nabla_q V_d(q)|_{q^*} = 0$$

Step 4: Solve

$$G_{\perp}\left(\nabla_q V + (L + R)\, M^{-1} M_d K \nabla_q V_d\right) = 0$$

for an arbitrary matrix L satisfying $L(q^*) = Y_0 W_0^{-1}$

Step 5: Calculate

$$Y = LW = LM^{-1}\left(K^{-1}M_d^{-1} - \nabla_q^2 V_d K^T\right)^{-1}$$

and

$$Z = \left(Y\nabla_q^2 V_d K^T - RM^{-1}\right) M_d + Gv^T$$

with an arbitrary vector $v = v(q)$ such that $Z(q^*) = Z_0$.

Step 6: Compute the control law (31)

The approach is quite systematic and therefore easy to implement. Only Step 4 should be done with care: Some elements of L might need to be smartly fixed in order to get a suitable solution. In Step 5, the vector v can be further used to inject nonlinear damping (see e.g. example below). Furthermore, the choice of R_0 has some implications regarding the estimate of the region of attraction. How to optimally choose R_0 is still an open question.

6. EXAMPLE - INVERTED PENDULUM ON A CART

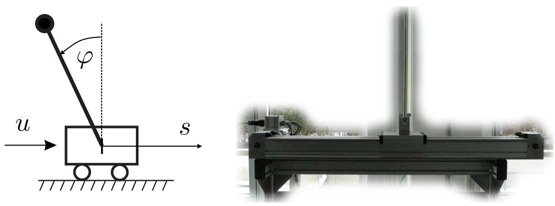

Fig. 1. The inverted pendulum on a cart. Scheme (left) and test rig (right).

To illustrate the approach, consider the pendulum on a cart depicted in Figure 1. This is a classical example of an underactuated mechanical system: it has one single input u and two degrees of freedom corresponding to the horizontal motion of the cart and the rotation of the pendulum, represented in local coordinates by $q = (s, \varphi)^T$. The equations of motion after a partial feedback linearization (PFL) (see Spong [1994]) are given by (1) with

$$M = I_{2\times 2}, \quad V(\varphi) = ag\cos\varphi,$$
$$G = \begin{bmatrix} 1 \\ a\cos\varphi \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & r_{\varphi} \end{bmatrix}, \tag{46}$$

where $I_{2\times 2}$ denotes the identity matrix (i.e. $p = \dot{q}$), g is the gravity constant and r_φ the normalized viscous damping coefficient for the unactuated coordinate (the damping in the actuated coordinate is assumed to be compensated). The parameter a is given by

$$a = \frac{m_p l_p}{m_p l_p^2 + \theta_p}, \tag{47}$$

where l_p is the distance from the rotation axis to the center of gravity of the pendulum, and m_p and θ_p are the pendulum mass and moment of inertia, respectively. The system parameters of the test rig are shown in Table 1 [6].

Parameter	Value
a	3.9
g	9.81
r_φ	$3.34 \cdot 10^{-4}$

Table 1. System parameters

6.1 Controller Design

The linearized system around the unstable equilibrium is controllable. It is thus possible to design a linear state feedback, such that the equation of motion of the closed-loop system is locally approximated by $\dot{x} = A_d x$, where $x = (\Delta s, \Delta\varphi, \dot{s}, \dot{\varphi})^T$ and A_d is Hurwitz. The assumptions above are valid for the inverted pendulum after a PFL: The inertia matrix does not depend on unactuated coordinates, and the desired equilibrium is admissible. We can therefore apply the six steps presented in Section 5 to design a nonlinear stabilizing controller for the system.

The steps 1-3 are implemented using a LQR controller (eigenvalues at $-163.6, -1.03, -6.2, -6.1$) and with

$$R_0 = \operatorname{diag}\{1, 1, \frac{1}{3}, \frac{1}{3}\} \quad \text{and} \quad V_d = \frac{1}{2} q^T Q_d q. \tag{48}$$

Fix the elements of the first row of L to the constant values $L(q)_{11} = L(q^*)_{11}$ and $L(q)_{12} = L(q^*)_{12}$ to calculate the solution of (30) (Step 4). Equation (30) results in

$$\begin{aligned} 0 = (c_1\, L_{2,1} + c_2 \cos\varphi + c_3\, L_{2,2} + c_4)\, s + \\ c_5 \varphi \cos\varphi + (c_6\, L_{2,2} + c_7\, L_{2,1} + c_8)\varphi + c_9 \sin\varphi \end{aligned} \tag{49}$$

for some constants c_i. The remaining two elements $L_{2,1}$ and $L_{2,2}$ are chosen depending only on the angle φ. The matrices Y and Z are calculated in Step 5 with

$$v^T = v_0^T + K_{di}(\varphi) G^T, \tag{50}$$

where the constant vector v_0 is chosen such that $Z(q^*) = Z_0$ and $K_{di} = 200 \left(\frac{1}{\cos\varphi} - 1\right)$: It turns out, that adding more damping for larger values of φ improves the transient behavior. Step 6 is straightforward.

6.2 Simulations

Figure 2 shows the response of the system controlled with the LQR controller of the previous section, with an IDA-PBC controller as found in Acosta et al. [2005] for the undamped system as an exponent of the classical passivity-based control approach for mechanical systems [7], and with the augmented IDA-PBC controller presented in this paper. Near the desired equilibrium, the systems

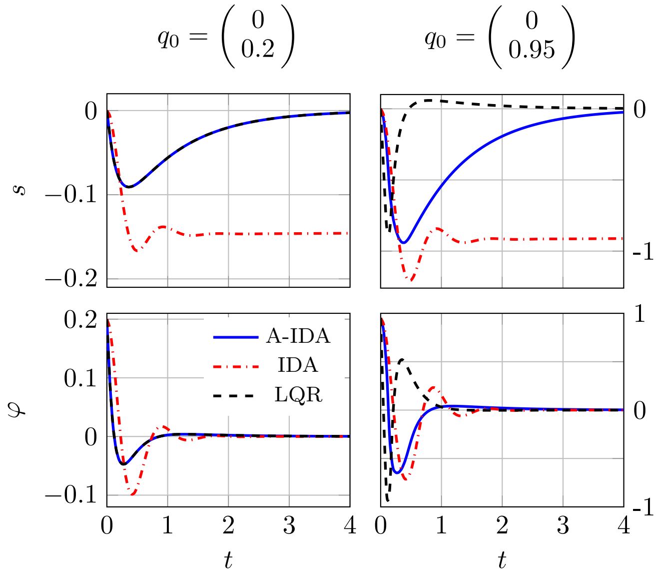

Fig. 2. Response of the cart position s (top) and angle of the pendulum φ (bottom) for two different initial angles $\varphi_0 = 0.2$ rad (left) and $\varphi_0 = 0.95$ rad (right) and $s_0 = \dot{s}_0 = \dot{\varphi}_0 = 0$.

controlled with the augmented IDA-PBC and with the LQR controller behave equal. The first one shows, however, a slightly smoother response for larger initial angles. Figure 3 shows level sets of the augmented closed-loop Hamiltonian H_d in the (s, φ)-plane, where $R_d > 0$. The region bounded by the largest level set is an estimate of the domain of attraction of the equilibrium point: Since the energy function is of quadratic form and the dissipation solely depends on the angle φ, the 4-dimensional sublevel set of H_d completely contained in the region where $R_d > 0$ is an estimate of the domain of attraction. Table 2 shows

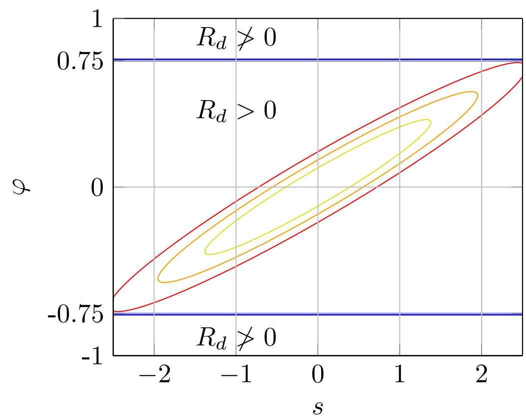

Fig. 3. Level sets of H_d in the plane $p = 0$ and level set where the smallest eigenvalue of R_d equals 0.

the maximal stabilizable initial angle φ_0 when starting at rest for the different controller types. These values have been determined by simulation [8].

[6] The system has been non-dimensionalized

[7] The IDA-PBC controller has been parametrized with the aim of a large estimate of the region of attraction. Other parametrizations (and approaches) show different responses, the oscillations and the relative slow convergence of the cart's position, however, remain. See e.g. Woolsey et al. [2004] for the CL case.

[8] In Woolsey et al. [2004] the equilibrium is asymptotically stable for $|\varphi_0| < \frac{\pi}{2}$

Stabilizable initial angles in radians	
Classic IDA	$-1.45 < \varphi_0 < 1.45$
Augmented IDA	$-1.20 < \varphi_0 < 1.20$
LQR	$-0.96 < \varphi_0 < 0.96$

Table 2. Simulative estimate of the stabilizable initial angles φ_0 for $(s_0 = 0, \varphi_0, \dot{s}_0 = 0, \dot{\varphi}_0 = 0)$

Remark 3. The fact that the dissipation condition is not satisfied does not imply instability of the closed-loop system. It is in fact possible to prove stability for the system controlled by the classical IDA-PBC approach in the presence of physical damping by a spectral stability analysis associated with the linearized dynamics as shown in Woolsey et al. [2004]. However, the analysis is cumbersome and one loses the Lyapunov function, and therewith the proof of the region of attraction. On the other hand, Woolsey et al. [2004] confirm that physical damping degrades the local performance of the energy shaping controller, whereas a (well-tuned) linear static state feedback ensures good local performance eliminating undesired oscillations.

6.3 Experimental results

Figure 4 shows the behavior of the augmented IDA-PBC controlled test rig. The same controller parametrization from the simulation has been also used for the experiments, which has been chosen rather "slow" to clearly visualize the results. The desired position of the cart changes from −0.1m to 0.1m at 0.85s. As shown in the plot, the cart smoothly reaches the desired position as expected, keeping the pendulum close to its desired equilibrium and showing a similar transient to that of the simulations.

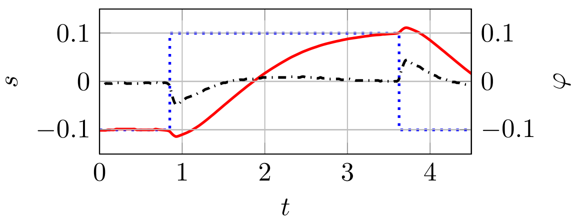

Fig. 4. Position control of the inverted pendulum: the dash-dotted line shows the behavior of the pendulum's angle φ and the solid line represents the position of the cart s

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7. CONCLUSION

This note presents an IDA-PBC controller design approach for underactuated mechanical systems based on a more general closed-loop Hamiltonian function which a) is easy to parametrize b) does not get affected by physical damping, since dissipation is considered in the controller design and c) does not require the solution of any PDE. A framework of 6 steps has been presented for the controller design for a class of mechanical systems. Simulations and experimental results confirm the applicability of the method.

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