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# Controllability of the 1D Schrödinger equation by the flatness approach

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**Abstract:** We derive in a straightforward way the exact controllability of the 1-D Schrödinger equation with a Dirichlet boundary control. We use the so-called *flatness approach*, which consists in parameterizing the solution and the control by the derivatives of a "flat output". This provides an explicit control input achieving the exact controllability in the energy space. As an application, we derive an explicit pair of control inputs achieving the exact steering to zero for a simply-supported beam.

*Keywords:* Partial differential equations, Schrödinger equation, beam equation, boundary control, exact controllability, path planning, flatness.

## 1. INTRODUCTION

The exact controllability of the linear Schrödinger equation (or of the plate equation) was investigated in Lions (1988), Machtyngier (1994), Komornik (1994) with the multiplier method, in Haraux (1989), Jaffard (1990), Komornik and Loreti (2005) with nonharmonic Fourier analysis, in Lebeau (1992) with microlocal analysis, and in Chen et al. (1991); Liu (1997) with frequency domain tests. The exact controllability was extended to the semilinear Schrödinger equation in Rosier and Zhang (2009b,a, 2010), and Laurent (2010a,b) by means of Strichartz estimates and Bourgain analysis.

All the above results rely on some observability inequalities for the adjoint system. A direct approach which does not involve the adjoint system was proposed in Littman and Markus (1988); Rosier (2002); Littman and Taylor (2007). The result in Rosier (2002) used a fundamental solution of the Schrödinger equation with compact support in time, and provided controls that are Gevrey.

In this paper, we derive in a straightforward way the null (or, equivalently, exact) controllability of the (linear) Schrödinger equation on the interval $(0,1)$ with a Dirichlet control at $x=1$. More precisely, for any final time $T>0$ and any $\theta_0 \in L^2(0,1)$, we provide an explicit and regular control such that the state reached at time $T$ is exactly zero. We use the so-called *flatness* approach Fliess et al. (1995), which consists in parameterizing the solution $\theta$ and the control $u$ by the derivatives of a "flat output" $y$; this notion was initially introduced for finite-dimensional (nonlinear) systems, and later extended to PDEs, Laroche et al. (2000); Lynch and Rudolph (2002); Meurer et al. (2008); Meurer (2013). In Martin et al. (2013a,b,c), we proved that the null controllability of the heat equation could be derived with the flatness approach, and that this approach provided very efficient numerical schemes by taking partial sums in the series. The design of the control in Martin et al. (2013a) was done in two steps. In the first one, a null control was used to reach an intermediate state Gevrey of order 1/2, while a flatness-based control was used in the second step to drive this (regular) intermediate state to 0. Clearly, this strategy cannot be used for the Schrödinger equation, as an application of a null control in the first step would yield a state which may be only $L^2$. Instead, using some idea in Littman and Taylor (2007); Rosier and Zhang (2009a), we apply here a *nonull* control in the first step, using a smoothing effect for the Schrödinger equation on $\mathbb{R}$, to reach an intermediate state which is again Gevrey of order 1/2. The second step is then carried out as in Martin et al. (2013a).

The paper is outlined as follows. In Section 2 we consider the control problem for the Schrödinger equation in dimension $N=1$. In Proposition 1 we investigate an ill-posed problem with Cauchy data in a Gevrey class and prove its (global) well-posedness. Theorem 2 then establishes the null controllability in small time for any initial data in $L^2$. Section 3 is devoted to the exact controllability of a simply-supported beam. In Theorem 6 we derive a null controllability result by the flatness approach for the beam equation with two boundary controls. Some additional computations needed to compute numerically the control for the Schrödinger equation are provided in Section 4.

Let us introduce some definitions and notations. We say a function $y \in C^\infty([0,T])$ is *Gevrey of order* $s \geq 0$ *on* $[0,T]$ if there exist some positive constants $M, R$ such that

$$|y^{(p)}(t)| \leq M \frac{(p!)^s}{R^p} \qquad \forall t \in [0,T],\ \forall p \geq 0; \qquad (1)$$

If $y$ is complex-valued the definition applies for the real and complex parts. More generally, for any compact set $K \subset \mathbb{R}^N$ $(N \geq 1)$, and any function $y: x=(x_1,x_2,...,x_N) \in K \mapsto y(x) \in \mathbb{R}$ which is of class $C^\infty$ on $K$ (i.e. $y$ is the restriction to $K$ of a function of class $C^\infty$ on some open neighborhood $\Omega$ of $K$), we shall say that $y$ *is Gevrey of order* $s_1$ *in* $x_1$, $s_2$ *in* $x_2$, ..., $s_N$ *in* $x_N$ *on* $K$, where $s_i \geq 0$ for $1 \leq i \leq N$, if there are some positive constants $M, R_1, ..., R_N$ such that

$$|\partial_{x_1}^{p_1}\partial_{x_2}^{p_2}\cdots\partial_{x_N}^{p_N}y(x)| \leq M\frac{\prod_{i=1}^N (p_i!)^{s_i}}{\prod_{i=1}^N R_i^{p_i}}, \quad \forall x \in K,\ \forall p \in \mathbb{N}^N. \tag{2}$$

## 2. CONTROLLABILITY OF THE SCHRÖDINGER EQUATION

We are concerned with the (null) controllability of the system

$$i\theta_t + \theta_{xx} = 0, \qquad (t,x) \in (0,T)\times(0,1), \tag{3}$$
$$\theta(t,0) = 0, \qquad t \in (0,T), \tag{4}$$
$$\theta(t,1) = u(t), \qquad t \in (0,T), \tag{5}$$
$$\theta(0,x) = \theta_0(x), \quad x \in (0,1), \tag{6}$$

where $\theta_0 \in L^2(0,1)$ is given; $\theta, \theta_0$ and $u$ are complex-valued functions.

Let

$$y(t) = \theta_x(t,0), \qquad t \in [0,T]. \tag{7}$$

We claim that (3)-(4) is "flat" with $y$ as flat output, which means that the map $\theta \to y$ is a bijection between appropriate spaces of smooth functions. Indeed, let us seek a formal solution $(\theta, u)$ of (3)-(5) and (7) in the form

$$\theta(t,x) = \sum_{k\geq 0}\frac{x^k}{k!}a_k(t), \quad u(t) = \sum_{k\geq 0}\frac{a_k(t)}{k!},$$

where the $a_k$'s are functions yet to define. Plugging the formal solution into (3) yields

$$\sum_{i\geq 0}\frac{x^k}{k!}[a_{k+2} + ia_k'] = 0.$$

Thus

$$a_{k+2} = -ia_k', \quad \forall k \geq 0.$$

On the other hand, we infer from (4) and (7) that $a_0(t) = 0$ and $a_1(t) = y(t)$. It follows that for all $k \geq 0$

$$a_{2k} = 0, \quad a_{2k+1} = (-i)^k y^{(k)}.$$

Eventually,

$$\theta(t,x) = \sum_{k\geq 0}\frac{x^{2k+1}}{(2k+1)!}(-i)^k y^{(k)}(t) \tag{8}$$

$$u(t) = \sum_{k\geq 0}\frac{(-i)^k y^{(k)}(t)}{(2k+1)!}. \tag{9}$$

In particular $\theta$ is uniquely defined in terms of $y$.

Conversely, if $y \in C^\infty([0,T])$ and the first formal series in (8) is convergent in $C^2([0,T]\times[0,1])$, then it is easily seen that (3)-(5) and (7) hold. Our first result shows that the above computations are fully justified when $y$ is Gevrey of order $s \in [0,2)$.

*Proposition 1.* Let $s \in [0,2)$, $-\infty < t_1 < t_2 < \infty$, and $y \in C^\infty([t_1,t_2])$ satisfying for some constants $M, R > 0$

$$|y^{(k)}(t)| \leq M\frac{k!^s}{R^k}, \qquad \forall k \geq 0,\ \forall t \in [t_1,t_2]. \tag{10}$$

Then the function $\theta$ defined in (8) is Gevrey of order $s$ in $t$ and $s/2$ in $x$ on $[t_1,t_2]\times[0,1]$.

**Proof.** We want to prove the formal series

$$\partial_t^m\partial_x^n\theta(t,x) = \sum_{2k+1\geq n}\frac{x^{2k+1-n}}{(2k+1-n)!}(-i)^k y^{(k+m)}(t) \tag{11}$$

is uniformly convergent on $[t_1,t_2]\times[0,1]$ with an estimate of its sum of the form

$$|\partial_t^m\partial_x^n\theta(t,x)| \leq C\frac{m!^s}{R_1^m}\frac{n!^{\frac{s}{2}}}{R_2^n}. \tag{12}$$

By (10), we have for all $(t,x) \in [t_1,t_2]\times[0,1]$

$$\begin{aligned}
&\left|\frac{x^{2k+1-n}}{(2k+1-n)!}y^{(k+m)}(t)\right| \\
&\leq \frac{M}{R^{k+m}}\frac{(k+m)!^s}{(2k+1-n)!} \\
&\leq \frac{M}{R^{k+m}}\frac{(2^{k+m}k!\,m!)^s}{(2k+1-n)!} \\
&\leq \frac{M}{R_1^{k+m}}\frac{(2^{-2k}\sqrt{\pi k}\,(2k)!)^{\frac{s}{2}}}{(2k+1-n)!}m!^s \\
&\leq \frac{M}{R_1^{k+m}}\frac{(2^{-2k-1}\sqrt{\pi k}\,(2k+1)!)^{\frac{s}{2}}}{(k+\frac{1}{2})^{\frac{s}{2}}(2k+1-n)!}m!^s \\
&\leq M\frac{(\pi k)^{\frac{s}{4}}}{R_1^k(k+\frac{1}{2})^{\frac{s}{2}}(2k+1-n)!^{1-\frac{s}{2}}}n!^{\frac{s}{2}}\frac{m!^s}{R_1^m},
\end{aligned}$$

where we have set $R_1 = 2^{-s}R$ and used twice $(p+q)! \leq 2^{p+q}p!q!$, and

$$(2k)! \sim \frac{2^{2k}}{\sqrt{\pi k}}k!^2 \tag{13}$$

which follows at once from Stirling's formula. Since $\sum_{2k+1\geq n}\frac{(\pi k)^{\frac{s}{4}}}{R_1^k(k+\frac{1}{2})^{\frac{s}{2}}(2k+1-n)!^{1-\frac{s}{2}}} < \infty$, we infer the uniform convergence of the series in (11) for all $m,n \geq 0$. This shows that $\theta \in C^\infty([t_1,t_2]\times[0,1])$. On the other hand, picking any $R_2 \in (0,\sqrt{R_1})$, since

$$\begin{aligned}
&M\sum_{2k+1\geq n}\frac{(\pi k)^{\frac{s}{4}}}{R_1^k(k+\frac{1}{2})^{\frac{s}{2}}(2k+1-n)!^{1-\frac{s}{2}}} \\
&\leq KR_1^{-\frac{n}{2}}\sum_{j\geq 0}\frac{j^{\frac{s}{4}}+n^{\frac{s}{4}}}{R_1^{\frac{j}{2}}j!^{1-\frac{s}{2}}} \\
&\leq CR_2^{-n},
\end{aligned}$$

for some constants $K, C > 0$ independent of $n$, we conclude that

$$|\partial_t^m\partial_x^n\theta(t,x)| \leq C\frac{n!^{\frac{s}{2}}}{R_2^n}\frac{m!^s}{R_1^m},$$

which proves that $\theta$ is Gevrey of order $s$ in $t$ and $s/2$ in $x$, as desired. □

Let $\theta_0 \in L^2(0,1)$ be given. Define $v_0 \in L^2(\mathbb{R})$ as

$$v_0(x) = \begin{cases}\theta_0(x) & \text{if } x \in (0,1),\\ -\theta_0(-x) & \text{if } x \in (-1,0),\\ 0 & \text{if } x \in (-\infty,-1)\cup(1,+\infty).\end{cases} \tag{14}$$

Let $v = v(t,x)$ denote the solution of the Cauchy problem

$$iv_t + v_{xx} = 0, \quad (t,x) \in \mathbb{R}^2, \tag{15}$$
$$v(0,x) = v_0(x), \quad x \in \mathbb{R}. \tag{16}$$

The following properties are well known, see e.g. Cazenave (2003); Linares and Ponce (2009); Rosier and Zhang (2009a):

$$v(t,-x) = -v(t,x) \quad \text{for a.e. } x\in\mathbb{R}, \text{ for all } t\in\mathbb{R}, \tag{17}$$
$$v\in C^\infty(\mathbb{R}_t\setminus\{0\}\times\mathbb{R}_x), \tag{18}$$
$$\left(\int_{-\infty}^{\infty} ||v||^4_{L^\infty(\mathbb{R})}dt\right)^{\frac14} \le c||v_0||_{L^2(\mathbb{R})}, \tag{19}$$
$$\sup_x \int_{-\infty}^{\infty} |D_x^{\frac12} v|^2 dt \le c||v_0||^2_{L^2(\mathbb{R})}. \tag{20}$$

(18) rests on the fact that $v_0$ is *compactly supported.* To justify (18), it is sufficient to introduce the operator

$$Pu := (x+2it\partial_x)u = 2ite^{i\frac{|x|^2}{4t}}\partial_x\big(e^{-i\frac{|x|^2}{4t}}u\big)$$

and to notice that it commutes with the Schrödinger operator $L = i\partial_t + \partial_{xx}$. The same is true for $P^k$ and $L$ for all $k\in\mathbb{N}$, and hence

$$||(P^kv)(t,.)||_{L^2(\mathbb{R})} = ||(P^kv)(0,.)||_{L^2(\mathbb{R})} = ||x^kv_0||_{L^2(\mathbb{R})} < \infty,$$

where we used the conservation of the $L^2(\mathbb{R})-$norm for the solutions of (15)-(16). This yields $v\in L^\infty((t_1,t_2),H^k_{loc}(\mathbb{R}))$ for all $0<t_1<t_2$ and $k\in\mathbb{N}$, and hence (18) by using (15) inductively. In particular, it follows from (18) and (19) that for all $\tau>0$,

$$v(.,1)\in C^\infty((0,\tau])\cap L^4(0,\tau). \tag{21}$$

With Proposition 1 at hand, we can derive a null controllability result obtained in a constructive way by the flatness approach.

*Theorem 2.* Let $\theta_0\in L^2(0,1)$ and $T>0$ be given. Let $v$ denote the solution of (15)-(16), where $v_0$ is as in (14). Pick any $\tau\in(2T/3,T)$ and any $s\in(1,2)$. Then there exists a function $y:[\tau,T]\to\mathbb{R}$ Gevrey of order $s$ on $[\tau,T]$ such that, setting

$$u(t) = \begin{cases} v(t,1) & \text{if } \ 0<t\le\tau,\\ \displaystyle\sum_{k\ge1}\frac{(-i)^k y^{(k)}(t)}{(2k+1)!} & \text{if } \ \tau<t\le T,\end{cases} \tag{22}$$

the solution $\theta$ of (3)-(6) satisfies $\theta(T,.)=0$. Furthermore, the control function $u$ is in $L^4(0,T)$ and it is Gevrey of order $s$ in $t$ on $[\varepsilon,T]$ for all $\varepsilon\in(0,T)$, $\theta\in C([0,T],L^2(0,1))\cap C^\infty((0,T]\times[0,1])$, and $\theta$ is Gevrey of order $s$ in $t$ and $s/2$ in $x$ on $[\varepsilon,T]\times[0,1]$ for all $\varepsilon\in(0,T)$.

**Proof.** The idea is to apply first a control in the time interval $[0,\tau]$ to smooth out the state function, and next to use the above flatness approach to steer the (more regular) state function to 0 in the time interval $[\tau,T]$.

STEP 1. FREE EVOLUTION.

We set $u(t)=v(t,1)$ for $t\in[0,\tau]$. By (17)-(18) $v$ is smooth and odd for all $t>0$ hence $v(t,0)=0$. Therefore, $\theta(t,x)=v(t,x)$ for $(t,x)\in[0,\tau]\times[0,1]$. Introduce the usual fundamental solution of the Schrödinger equation

$$E(t,x) = \frac{1}{(4\pi it)^{\frac12}}e^{i\frac{x^2}{4t}}.$$

Then

$$v(t,x) = (E(t,.)*v_0)(x) = \int_{-1}^1 E(t,x-y)v_0(y)dy. \tag{23}$$

*Lemma 3.* The function $v(t,x)$ is Gevrey of order 1 in $t$ and $1/2$ in $x$ on $[t_1,t_2]\times[-L,L]$ for all $0<t_1<t_2$ and $L>0$. Furthermore, we can write

$$v(\tau,x) = \sum_{k\ge0} y_k(-i)^k\frac{x^{2k+1}}{(2k+1)!}, \quad x\in\mathbb{R} \tag{24}$$

with

$$|y_k| \le C(\tau)||\theta_0||_{L^1(0,1)}\Big(\frac{2}{\tau}\Big)^k k! \tag{25}$$

where $C(\tau)$ is a continuous function on $(0,+\infty)$.

*Proof of Lemma 3:* Pick any $k\in\mathbb{N}$. Then

$$\partial_x^k v(t,x) = \int_{-1}^1(\partial_x^kE)(t,x-y)v_0(y)\,dy$$
$$= \frac{1}{(4\pi it)^{\frac12}}\int_{-1}^1 \partial_x[e^{i\frac{x^2}{4t}}](t,x-y)v_0(y)dy. \tag{26}$$

Let us first check that $e^{i\frac{x^2}{4t}}$ is Gevrey of order $1/2$ in $x$. Clearly

$$e^{i\frac{x^2}{4t}} = \sum_{k\ge0}\Big(\frac{i}{4t}\Big)^k\frac{x^{2k}}{k!} = \sum_{l\ge0}a_l\frac{x^l}{l!},$$

where

$$a_l = \begin{cases}\Big(\dfrac{i}{4t}\Big)^k\dfrac{(2k)!}{k!} & \text{if } l=2k,\\ 0 & \text{if } l=2k+1.\end{cases}$$

Then for $t_1\le t\le t_2$ and $x\in[-L,L]$,

$$|a_{2k}| \le \frac{c}{(4t)^k}\frac{(2k)!}{(2k)!^{\frac12}2^{-k}(\pi k)^{\frac14}}$$
$$\le c'\frac{(2k)!^{\frac12}}{(2k)^{\frac14}(\sqrt{2t})^{2k}}, \tag{27}$$

where we used again (13), and where $c$ and $c'$ denote some universal constants. The following lemma is needed.

*Lemma 4.* Let $s\in(0,1)$, and let $(a_k)_{k\ge0}$ be a sequence such that

$$|a_k|\le C\frac{k!^s}{R^kk^\alpha} \qquad \forall k\ge0$$

for some constants $C>0$, $R>0$ and $\alpha\ge0$. Then the function

$$f(x) = \sum_{k\ge0}a_k\frac{x^k}{k!}$$

is Gevrey of order $s$ on $[-L,L]$ for all $L>0$.

*Proof of Lemma 4.* Note first that $f$ is well defined and analytic on $\mathbb{R}$, for

$$|a_k|\frac{L^k}{k!} \le C\sum_{k\ge0}\frac{L^k}{R^kk^\alpha k!^{1-s}} < \infty.$$

Taking the derivatives of the terms in the series, we infer that $f^{(m)}(x) = \sum_{k\ge0}a_{k+m}\frac{x^k}{k!}$, and hence for any $x\in[-L,L]$

$$|f^{(m)}(x)| \le C\sum_{k\ge0}\frac{L^k(k+m)!^s}{R^{k+m}(k+m)^\alpha k!}$$
$$\le C\frac{2^{sm}}{R^mm^\alpha}m!^s\sum_{k\ge0}\frac{(2^sL)^k}{R^kk!^{1-s}}, \tag{28}$$

where we used the estimate

$$(k+m)! \le 2^{k+m}k!\,m!. \tag{29}$$

The proof of Lemma 4 is complete. □

It follows from (27) and Lemma 4 that $e^{i\frac{x^2}{4t}}$ is Gevrey of order $1/2$ in $x$. More precisely, using (27) and (28), we infer that for $x\in[-L,L]$ and $k\in\mathbb{N}$

$$|\partial_x^k\big(e^{i\frac{x^2}{4t}}\big)| \le C(t,L)\frac{k!^{\frac12}}{k^{\frac14}(\sqrt{t})^k}. \tag{30}$$

Using (26), this yields for $x \in [-L, L]$ and $t_1 \leq t \leq t_2$

$$|\partial_x^k v(t,x)| \leq C(t_1,t_2,L) \int_{-1}^{1} \frac{k!^{\frac{1}{2}}}{k^{\frac{1}{4}}(\sqrt{t})^k} |v_0(y)| dy$$
$$\leq C \frac{k!^{\frac{1}{2}}}{k^{\frac{1}{4}}(\sqrt{t})^k} ||\theta_0||_{L^1(0,1)}.$$

Combined with (13) and (15), this gives for $(t,x) \in [t_1,t_2] \times [-L,L]$ and $(k,l) \in \mathbb{N}^2$

$$|\partial_x^k \partial_t^l v(t,x)| = |\partial_x^{k+2l} v(t,x)|$$
$$\leq C \frac{(k+2l)!^{\frac{1}{2}}}{(k+2l)^{\frac{1}{4}}(\sqrt{t})^{k+2l}} ||\theta_0||_{L^1(0,1)}$$
$$\leq C \frac{k!^{\frac{1}{2}}(2l)!^{\frac{1}{2}} 2^{k+2l}}{(k+2l)^{\frac{1}{4}}(\sqrt{t})^{k+2l}} ||\theta_0||_{L^1(0,1)}$$
$$\leq C \frac{k!^{\frac{1}{2}}}{R_1^k} \frac{l!}{R_2^l} ||\theta_0||_{L^1(0,1)}$$

where $C, R_1, R_2$ are some constants that depend on $t_1, t_2$, and $L$.

Thus, $v$ is Gevrey of order 1 in $t$ and 1/2 in $x$. In particular, $v(\tau, .)$ is an analytic function on $\mathbb{R}$. Being odd, it can be written as

$$v(\tau,x) = \sum_{k\geq 0} y_k (-i)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R} \qquad (31)$$

with

$$|y_k| = |\partial_x^{2k+1} v(\tau,0)|$$
$$\leq C(\tau) ||\theta_0||_{L^1(0,1)} \frac{(2k+1)!^{\frac{1}{2}}}{(2k+1)^{\frac{1}{4}}(\sqrt{\tau})^{2k+1}}$$
$$\leq C(\tau) ||\theta_0||_{L^1(0,1)} (\frac{2}{\tau})^k k!,$$

where (13) was used again. The proof of Lemma 3 is complete. □

Since $v \in C(\mathbb{R}, L^2(\mathbb{R}))$, $\theta \in C([0,\tau], L^2(0,1))$.

Step 2. Construction of the control on $[\tau, T]$

We need the following

*Lemma 5.* Let $\tau \in (\frac{2}{3}T, T)$ and $1 < s < 2$ be numbers, and let $(y_k)_{k\geq 0}$ be as in (25). Then there exists a function $y : [\tau, T] \to \mathbb{R}$ which is Gevrey of order $s$ on $[\tau, T]$ and such that

$$y^{(k)}(\tau) = y_k \qquad \forall k \geq 0, \qquad (32)$$
$$y^{(k)}(T) = 0 \qquad \forall k \geq 0, \qquad (33)$$
$$|y^{(k)}(t)| \leq C ||\theta_0||_{L^1(0,1)} \frac{(k!)^s}{R^k}, \ \forall k \geq 0, \ \forall t \in [\tau, T] \qquad (34)$$

for some constants $C = C(\tau, T, s) > 0$ and $R = R(\tau, T, s) > 0$.

*Proof of Lemma 5.* Let

$$\bar{y}(t) = \sum_{k\geq 0} y_k \frac{(t-\tau)^k}{k!}, \qquad t \in [\tau, T]. \qquad (35)$$

From $\tau > 2T/3$, we infer that $2(T-\tau)/\tau < 1$, and that for $t \in [\tau, T]$

$$\sum_{k\geq 0} |y_k \frac{(t-\tau)^k}{k!}| \leq C(\tau) ||\theta_0||_{L^1(0,1)} \sum_{k\geq 0} \left| \frac{2(T-\tau)}{\tau} \right|^k < \infty.$$

Thus $\bar{y}$ is analytic on $[\tau, T]$, and hence Gevrey of order $s$ on $[\tau, T]$, with

$$\bar{y}^{(k)}(\tau) = y_k, \qquad k \geq 0.$$

Introduce the "step function"

$$\phi_s(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ \dfrac{e^{-(1-t)^{-\kappa}}}{e^{-(1-t)^{-\kappa}} + e^{-t^{-\kappa}}} & \text{if } t \in (0,1), \\ 0 & \text{if } t \geq 1, \end{cases}$$

where $\kappa = (s-1)^{-1}$. Then $\phi_s$ is Gevrey of order $s$ on $[0,1]$ (in fact on $\mathbb{R}$) with $\phi_s(0) = 1$, $phi_s(1) = 0$ and for all $i \geq 1$ $\phi_s^{(i)}(0) = \phi_s^{(i)}(1) = 0$.

The desired function $y$ is given by

$$y(t) = \phi_s \Big( \frac{t-\tau}{T-\tau} \Big) \bar{y}(t).$$

Indeed, since products of Gevrey functions of order $s$ are Gevrey functions of order $s$, see e.g. Rudin (1987); Yamanaka (1989), we infer that the function $y : [\tau, T] \to \mathbb{R}$ is Gevrey of order $s$ on $[\tau, T]$. Furthermore, (32) and (33) hold. Let us now prove (34). Pick any $\rho > 1$ such that $2\rho(T-\tau)/\tau < 1$. Then for any $z \in \mathbb{C}$ with $|z| \leq \rho(T-\tau)$, we have

$$\sum_{k\geq 0} \left| y_k \frac{z^k}{k!} \right| \leq M := C(\tau) ||\theta_0||_{L^1(0,1)} \sum_{k\geq 0} \left| \frac{2\rho(T-\tau)}{\tau} \right|^k < \infty.$$

It follows from Cauchy's formula that for $|z| \leq T - \tau$ and $m \geq 0$,

$$\left| \partial_z^m \sum_{k\geq 0} y_k \frac{z^k}{k!} \right| \leq M \frac{m!}{(\rho-1)^m (T-\tau)^m}$$

Thus, for $\tau \leq t \leq T$ and $m \geq 0$,

$$|\bar{y}^{(m)}(t)| \leq M \frac{m!}{(\rho-1)^m (T-\tau)^m}. \qquad (36)$$

Then (34) follows from (36) and (Rudin, 1987, Theorem 19.7). The proof of Lemma 5 is complete. □

Let, for $(t,x) \in [\tau, T] \times [0,1]$,

$$\theta(t,x) = \sum_{k\geq 0} \frac{x^{2k+1}}{(2k+1)!} (-i)^k y^{(k)}(t), \qquad (37)$$
$$u(t) = \sum_{k\geq 0} \frac{(-i)^k y^{(k)}(t)}{(2k+1)!}. \qquad (38)$$

By Proposition 1 and Lemma 5, the function $\theta$ is well-defined and Gevrey of order $s$ in $t$ and $s/2$ in $x$ on $[\tau, T] \times [0,1]$. On the other hand, by Lemma 3, $\theta$ is Gevrey of order 1 in $t$ and 1/2 in $x$ on $[\varepsilon, \tau] \times [0,1]$ for all $\varepsilon \in (0, \tau)$. By (32), the two series in (24) and (37) take the same values at $t = \tau$, so that $\theta \in C((0,T], C^k([0,1]))$ for all $k \geq 0$ and $u \in C((0,T])$. Since $s > 1$, to prove that $\theta$ is Gevrey of order $s$ in $t$ and $s/2$ in $x$ on $[\varepsilon, T] \times [0,1]$ for all $\varepsilon \in (0,T)$, it is sufficient to notice that for all $k \geq 0$ and $x \in [0,1]$

$$(-i)^k \partial_t^k \theta(\tau^-, x) = \partial_x^{2k} \theta(\tau^-, x)$$
$$= \partial_x^{2k} \theta(\tau^+, x)$$
$$= (-i)^k \partial_t^k \theta(\tau^+, x).$$

The proof of Theorem 2 is complete. □

## 3. CONTROLLABILITY OF THE BEAM EQUATION

When $\theta$ is decomposed in terms of its real and imaginary parts, $\theta = \alpha + i\beta$, (3) reads

$$\alpha_t + \beta_{xx} = 0$$
$$\beta_t - \alpha_{xx} = 0;$$

differentiating $\alpha$ w.r.t time and eliminating $\beta$ yields

$$\alpha_{tt} + \alpha_{4x} = 0,$$

where $\alpha_{4x} := \partial_x^4 \alpha$. Hence we can adapt the results of the previous sections to derive the exact controllability of the Euler-Bernoulli beam equation

$$\eta_{tt} + \eta_{4x} = 0 \tag{39}$$
$$\big(\eta(t,0), \eta_{xx}(t,0)\big) = (0,0) \tag{40}$$
$$\big(\eta(t,1), \eta_{xx}(t,1)\big) = \big(u_1(t), u_2(t)\big) \tag{41}$$
$$\big(\eta(0,x), \eta_t(0,x)\big) = \big(\eta_0(x), \eta_1(x)\big), \tag{42}$$

where $(t,x) \in (0,T) \times (0,1)$.

When $u_1 = u_2 = 0$, (39)-(42) is a model for a simply supported (or hinged) beam. It is well known, see e.g. (Komornik, 1994, Thm 6.16), that (39)–(42) is null controllable (or, equivalently, exactly controllable) in $H_0^1(0,1) \times H^{-1}(0,1)$ in any time $T > 0$ by using some controls $u_2 \in L^2(0,T)$ and $u_1 \equiv 0$. The aim of this section is to derive in a straightforward way the exact controllability of (39)-(42) with *two* controls $u_1$ and $u_2$ by the flatness approach. There is no loss of generality in assuming that the final state is zero. The following result is a consequence of Theorem 2.

*Theorem 6.* Let $\eta_0 \in H^2(0,1) \cap H_0^1(0,1)$, $\eta_1 \in L^2(0,1)$ and $T > 0$ be given. Pick any $\tau \in (2T/3, T)$ and any $s \in (1,2)$. Then there exist $v_0 \in H^2(\mathbb{R})$ and a function $y : [\tau, T] \to \mathbb{R}$ Gevrey of order $s$ on $[\tau, T]$ such that, setting

$$u(t) = \begin{cases} v(t,1) & \text{if } \ 0 < t \le \tau, \\ \displaystyle\sum_{k \ge 1} \frac{(-i)^k y^{(k)}(t)}{(2k+1)!} & \text{if } \ \tau < t \le T, \end{cases} \tag{43}$$

(where $v$ denotes the solution of (15)-(16)) and $u_1(t) = \text{Re } u(t)$, $u_2(t) = \text{Im } u'(t)$, the solution $\eta$ of (39)-(42) satisfies $\eta(T,.) = \eta_t(T,.) = 0$. Furthermore, $u \in W^{1,4}(0,T)$ and it is Gevrey of order $s$ in $t$ on $[\varepsilon, T]$ for all $\varepsilon \in (0,T)$, $\eta \in C([0,T], L^2(0,1)) \cap C^\infty((0,T] \times [0,1])$, and $\eta$ is Gevrey of order $s$ in $t$ and $s/2$ in $x$ on $[\varepsilon, T] \times [0,1]$ for all $\varepsilon \in (0,T)$.

**Proof.** Let $\varphi = \eta_0$ and let $\psi$ solve $-\psi'' = \eta_1$, $\psi(0) = \psi(1) = 0$. Then $\varphi$ and $\psi$ belong to $H^2(0,1) \cap H_0^1(0,1)$, and the same is true for the (complex-valued) function $\theta_0(x) := \varphi(x) + i\psi(x)$. Pick a function $\zeta \in C^\infty(1,2)$ such that

$$\zeta(x) = \begin{cases} 1 & \text{if } x < 5/4, \\ 0 & \text{if } x > 7/4. \end{cases}$$

Let $v_0 : \mathbb{R} \to \mathbb{C}$ be defined as

$$v_0(x) = \begin{cases} \theta_0(x) & \text{if } 0 < x < 1, \\ -\theta_0(2-x)\zeta(x) & \text{if } 1 < x < 2, \\ 0 & \text{if } x > 2, \\ -v_0(-x) & \text{if } x < 0. \end{cases}$$

Then $v_0 \in H^2(\mathbb{R})$, supp $v_0 \subset [-7/4, 7/4]$, and $v_0(-x) = -v_0(x)$ for all $x \in \mathbb{R}$. Clearly, $w_0 = (v_0)_{xx} \in L^2(\mathbb{R})$ satisfies also supp $w_0 \subset [-7/4, 7/4]$, and $w_0(-x) = -w_0(x)$ for a.e. $x \in \mathbb{R}$. Let $v$ (resp. $w$) denote the solution of (15)-(16) (resp. of (15)-(16) with $w_0$ substituted to $v_0$). We know from (18) that $v, w \in C^\infty(\mathbb{R}_t \setminus \{0\} \times \mathbb{R}_x)$, with $w(t,x) = v_{xx}(t,x)$, and that

$$\left( \int_{-\infty}^{\infty} ||v||^4_{L^\infty(\mathbb{R})} + ||w||^4_{L^\infty(\mathbb{R})} dt \right)^{\frac{1}{4}} \le c||v_0||_{H^2(\mathbb{R})}.$$

Since $v_t = iv_{xx} = w$, we infer that

$$v(.,1) \in W^{1,4}(0,\tau) \subset C([0,\tau]).$$

Let $\eta(t,x) = \text{Re } \theta(t,x)$, where $\theta_0$ is as above and $u$ is as in (43). Then the conclusion of Theorem 2 (with the new function $v$) is still valid, so that the regularity properties of $u$ and $\eta$ are established. Next,

$$(\partial_t^2 + \partial_x^4)\theta = (-i\partial_t + \partial_x^2)(i\partial_t + \partial_x^2)\theta = 0$$

so that (39) holds. (40) (resp. (41)) follows from (3) and (4) (resp. from (3) and (5)). (42) is clear from the construction of $\theta_0$, and we infer from Theorem 2 and (3) that $\eta(T,.) = \eta_t(T,.) = 0$. □

## 4. NUMERICAL EXPERIMENTS

We illustrate the approach on a numerical example. The parameters are $\tau = 0.35$, $T = 0.5$ and $s = 1.9$; the (discontinuous) initial condition is

$$\text{Re } \theta_0(x) := \begin{cases} 0 & \text{if } x \in (0, 0.5), \\ 1 & \text{if } x \in (0.5, 1) \end{cases}$$

$$\text{Im } \theta_0(x) := \begin{cases} 0 & \text{if } x \in (0, 0.2) \cup (0.7, 1), \\ 1 & \text{if } x \in (0.2, 0.7). \end{cases}$$

To compute the control $u(t)$ on $[0,\tau]$ and the coefficients $y_k$ we make use of the convolution formula (23)

$$v(t,x) = \int_{-1}^{1} E(t, x-y) v_0(y) dy$$
$$= \int_0^1 \underbrace{\big(E(t,x-y) - E(t,x+y)\big)}_{=: F(tx,y)} \theta_0(y) dy,$$

where we have used (14). Then by (22)

$$u(t,x) = v(t,1) = \int_0^1 F(t,1,y)\theta_0(y) dy,$$

and by (24)

$$y_k = \frac{\partial_x^{2k+1} v(\tau, 0)}{(-i)^k} = \frac{1}{(-i)^k} \int_0^1 \partial_x^{2k+1} F(\tau, 0, y) \theta_0(y) dy.$$

All the integrals where numerically computed with the Matlab `quadgk` function. The series (35) for $\bar{y}(t)$ was truncated at $k = 15$, and so was the series (38) for $u(t)$ on $(\tau, T]$.

Fig. 1 displays the evolution of $\theta$ on $[0,T]$. The regularizing effect of the control on $(0,\tau)$ is clearly visible.

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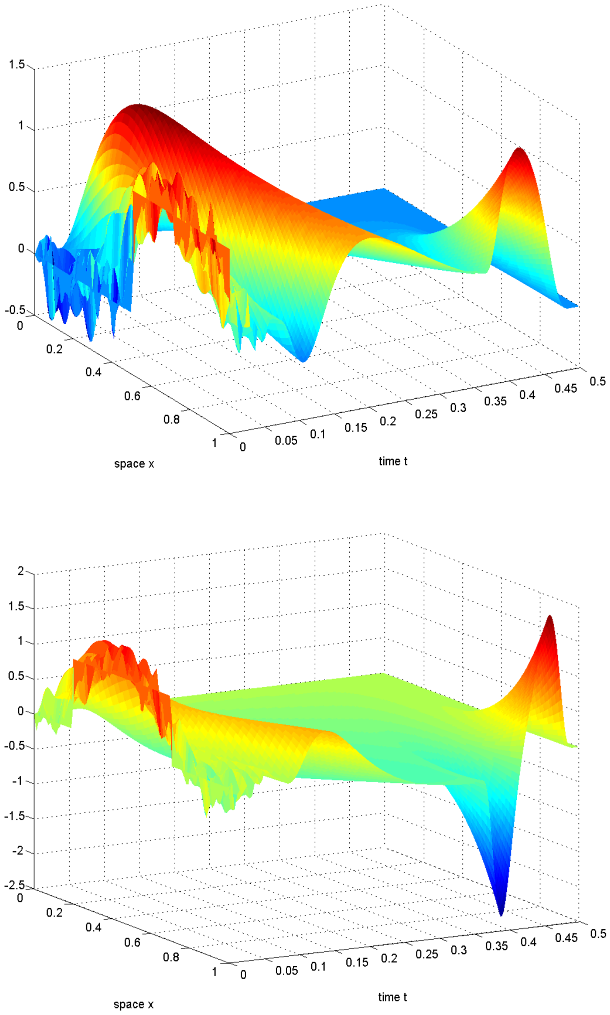


Fig. 1. Re $\theta$ (top) and Im $\theta$ (bottom).

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