Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Global Leader Following Consensus of a Group of Discrete-Time Linear Systems Using Bounded Controls ⋆

Zhiyun Zhao * **Zongli Lin** **,*

* *Department of Automation, Shanghai Jiao Tong University, Shanghai 200240, China (e-mail: zyzhao@sjtu.edu.cn)*
** *Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, P.O. Box 400743, Charlottesville, VA 22904-4743, USA (e-mail: zl5y@virginia.edu)*

Abstract: This paper studies the global leader-following consensus problem for a group of discrete-time linear systems with bounded controls. For each follower agent, we construct a bounded nonlinear feedback control law which uses the information of other agents obtained through multi-hop paths in the communication network. The number of hops each agent uses to obtain its information about other agents is no bigger than the largest algebraic multiplicity of the eigenvalues on the unit circle of the system matrix. We show that these control laws achieve global leader-following consensus when the communication topology is a strongly connected and detailed balanced directed graph and the leader is a neighbor of at least one follower agent.

Keywords: Multi-agent systems, consensus, actuator saturation, bounded controls, discrete-time

1. INTRODUCTION

In recent years, coordinated control of multi-agent systems has drawn substantial attention. As a fundamental approach to achieving group-wide behavior, consensus entails all agents in the system to converge to an agreement state by using only local information. Much effort has been made towards solving consensus problems when the models of agents in the system are in continuous-time (see, *e.g.*, Saber and Murray [2004], Ren, Beard and Atkins [2005], Jin and Murray [2006], Yu et al. [2011], Li et al. [2011]). Fewer results have been obtained on multi-agent systems operating in discrete-time. Examples of these results are Wang and Xiao [2006], Casbeer et al. [2008] and Chen, Lü and Lin [2013]. In particular, Wang and Xiao [2006] presents a so-called "pre-leader-follower" decomposition approach to solving consensus problems for discrete-time multi-agent systems with time delays. Casbeer et al. [2008] considers the consensus problem for agents with discrete-time double-integrator dynamics and shows that consensus can be achieved when the communication topology contains at least one directed spanning tree and average consensus can be achieved when the communication topology is strongly connected and balanced. Chen, Lü and Lin [2013] establishes criteria for consensus of discrete-time multi-agent systems with nonlinear local rules and time-varying delays.

There are also results on consensus of multi-agent systems, both in continuous-time and in discrete-time, that take into consideration input saturation, which is ubiquitous in real world applications. It is clear that global consensus with bounded controls (see Meng et al. [2011] and Yang et al. [2013]), like global stabilization with bounded controls (see, *e.g.*, Teel [1992], Sussmann and Yang [1991], Yang et al. [1997]), is only possible for agents that are not exponentially unstable (that is, all poles of the agent lie on the closed left-half plane in the continuous-time setting, or on or inside the unit circle in the discrete-time setting). In particular, it is established in Meng et al. [2011] that global leader following consensus with bounded controls is possible if the agents are represented by double integrators or by higher order neurally stable linear systems. Yang et al. [2013] deals with the discrete-time counterparts of Meng et al. [2011].

The results of Meng et al. [2011] have been extended to agents that are represented by a chain of integrators of an arbitrary length in Zhao and Lin [2013]. More specifically, a bounded nonlinear feedback control law is constructed for each follower agent in the group, which uses its information about other agents obtained through multi-hop paths in the communication network. The number of hops each agent uses to obtain its information about other agents is no bigger than the number of integrators in the agent. Global leader-following consensus is then established under these feedback control laws when the communication topology among follower agents is a strongly connected and detailed balanced directed graph and the leader is a neighbor of at least one follower agent.

In this paper, we consider the global leader following consensus problem for a multi-agent system where the agents are represented by a general discrete-time linear system with a bounded control. Under the assumption that these linear systems are stabilizable with all their poles

⋆ This work was supported in part by the National Natural Science Foundation of China under grant Nos. 61221003 and 61273105.

lying on or inside the unit circle, we construct, for each follower agent in the system, a bounded nonlinear feedback control law which also uses a multi-hop relay protocol. The number of hops each agent uses to obtain its information about other agents is no bigger than the largest algebraic multiplicity of the eigenvalues of the system matrix on the unit circle. We will show that global leader-following consensus is achieved under these feedback control laws when the communication topology among follower agents forms a strongly connected and detailed balanced directed graph and the leader is a neighbor of at least one follower agent.

The remainder of this paper is organized as follows. In Section 2, we state the problem of global leader-following consensus and recall basic definitions and relevant results in graph theory. In Section 3, we construct a bounded nonlinear feedback control law for each follower agent in the system and prove that these control laws achieve global leader-following consensus. Simulation results are presented in Section 4. Section 5 concludes the paper.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider a group of N follower agents, each described by the dynamics of a discrete-time linear system,

$$x_i(t+1) = Ax_i(t) + bu_i(t), \qquad i = 1, 2, \cdots, N, \qquad (1)$$

where $x_i = [x_{i1}, x_{i2}, \cdots, x_{in}]^{\mathrm{T}} \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$ are respectively the states and control inputs of agent i.

Assumption 1. All eigenvalues of A are inside or on the unit circle and the pair (A, b) is stabilizable.

Let the leader be also described by the dynamics of a discrete-time linear system,

$$x_0(t+1) = Ax_0(t), \qquad (2)$$

where $x_0 = [x_{01}, x_{02}, \cdots, x_{0n}]^{\mathrm{T}} \in \mathbb{R}^n$.

The global leader-following consensus problem we are to study is stated as follows. Consider a multi-agent system consisting of the group of follower agents (1) and the leader agent (2) operating on an underlying communication network. For an *a priori* given arbitrarily small scalar $\delta > 0$, construct a bounded state feedback law $u_i = h_i(x_0, x_1, \cdots, x_N)$, $|h_i(x_0, x_1, \cdots, x_N)| \le \delta$ for all $(x_0, x_1, \cdots, x_N) \in \mathbb{R}^{(N+1)n}$, for each follower agent, such that all these feedback laws together achieve global leader-following consensus, that is, for all initial conditions $x_i(0) \in \mathbb{R}^n$, $i = 0, 1, \cdots, N$,

$$\lim_{t\to\infty} (x_i(t) - x_0(t)) = 0, \quad i = 1, 2, \cdots, N.$$

The communication topology among agents is represented by a directed graph $\mathcal{G}_N = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{\nu_1, \nu_2, \cdots, \nu_N\}$ is a finite, nonempty set of nodes (each node denotes a follower agent) and $\mathcal{E} \in \mathcal{V}\times\mathcal{V}$ is a set of edges (each edge denotes an ordered pair of nodes). An edge (ν_j, ν_i) in a directed graph denotes that ν_i has access to the information form ν_j. A directed path in a directed graph is a sequence of edges of the form $(\nu_{i1}, \nu_{i2}), (\nu_{i2}, \nu_{i3}), \cdots$. A directed path $(\nu_i, \nu_{i1}), (\nu_{i1}, \nu_{i2}), \cdots, (\nu_{ik-1}, \nu_j)$ between ν_i and ν_j is called a k-hop path, and ν_i is called a k^{th} neighbor of ν_j.

Let $\mathcal{A}_N = [a_{ij}] \in \mathbb{R}^{N\times N}$ be the adjacency matrix associated with $\mathcal{G}_N$, where $a_{ij} > 0$ if $(\nu_j, \nu_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Here we assume that $a_{ii} = 0$ for all $i = 1, 2, \cdots, N$. Let $\mathcal{L}_N = [l_{ij}] \in \mathbb{R}^{N\times N}$ be the Laplacian matrix associated with $\mathcal{A}_N$, where $l_{ii} = \sum_{i=1}^n a_{ij}$ and $l_{ij} = -a_{ij}$ when $i \ne j$. A directed graph is detailed balanced if there exist some real numbers $v_i > 0, i = 1, 2, \cdots, N$, such that $v_i a_{ij} = v_j a_{ji}$, for all $i, j = 1, 2, \cdots, N$ (Jiang and Wang [2009]). Let $v = [v_1, v_2, \cdots, v_N]^{\mathrm{T}}$ and $\mathrm{diag}\{v\} = \mathrm{diag}\{v_1, v_2, \cdots, v_N\}$.

The leader agent is labeled as ν_0. The communication between a follower agent ν_i and the leader agent ν_0 is denoted as a_{i0}, where $a_{i0} > 0$ if ν_i has access to the information of ν_0 and $a_{i0} = 0$ otherwise. Denote $M = \mathcal{L}_N + \mathrm{diag}\{a_{10}, a_{20}, \cdots, a_{N0}\}$.

Assumption 2. The directed graph $\mathcal{G}_N$ is strongly connected and detailed balanced and $a_{i0} > 0$ for at least one $i, i = 1, 2, \cdots, N$.

Lemma 3. Under Assumption 2, all eigenvalues of M are on the open right-half plane, and the matrix $\mathrm{diag}\{v\}M + M^{\mathrm{T}}\mathrm{diag}\{v\} = 2M^{\mathrm{T}}\mathrm{diag}\{v\}$ is positive definite.

In the above lemma, the fact that all eigenvalues of M are on the open right-half plane is established in Ren and Cao [2011] and the fact that $\mathrm{diag}\{v\}M + M^{\mathrm{T}}\mathrm{diag}\{v\} = 2M^{\mathrm{T}}\mathrm{diag}\{v\}$ is positive definite can be established based on the analysis given in the proof of Lemma 4 in Hu and Hong [2011]. Let $\Gamma = M^{\mathrm{T}}\mathrm{diag}\{v\}$ and γ_{ij} be the $(i, j)^{\text{th}}$ entry of Γ. Let the eigenvalues of Γ be ordered as $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$.

3. MAIN RESULTS

Under an appropriate state transformation, the dynamics of each follower can be rewritten in the following form

$$\Sigma_i : \begin{cases} x_{i\circ}(t+1) = A_\circ x_{i\circ}(t) + b_\circ u(t), \\ x_{i\odot}(t+1) = A_\odot x_{i\odot}(t) + b_\odot u(t), \quad i = 1, 2, \cdots, N, \end{cases}$$

where

(1) $x_{i\circ}(k) \in \mathrm{R}^{n_\circ}$ and $x_{i\odot}(t) \in \mathrm{R}^{n_\odot}$, with $n_\circ + n_\odot = n$,
(2) all the eigenvalues of $A_\circ$ are on the unit circle, and the pair $(A_\circ, b_\circ)$ is controllable, and
(3) all the eigenvalues of $A_\odot$ are strictly inside the unit circle.

Under the same state transformation, the dynamics of the leader agent is written as

$$\Sigma_0 : \begin{cases} x_{0\circ}(t+1) = A_\circ x_{0\circ}(t), \quad x_{0\circ}(t) \in \mathrm{R}^{n_\circ}, \\ x_{0\odot}(t+1) = A_\odot x_{0\odot}(t), \quad x_{0\odot}(t) \in \mathrm{R}^{n_\odot}. \end{cases}$$

Let a bounded state feedback control algorithm $u_i = h_i(x_{0\circ}, x_{1\circ}, \cdots, x_{N\circ})$ cause all states $x_{i\circ}, i = 1, 2, \cdots, N$, to converge to the corresponding state of the leader agent $x_{0\circ}$ asymptotically. Clearly, these same control laws together achieve global leader-following consensus if $u_i(t)$ goes to zero as time goes infinity. Thus, in the remainder of this paper we will assume, without loss of generality, that all the eigenvalues of A are on the unit circle.

Let the complex eigenvalues of A be $\alpha_1 \pm j\beta_1, \alpha_2 \pm j\beta_2, \cdots, \alpha_q \pm j\beta_q$, where α_i's and β_i's are not necessarily distinct. Let there be p eigenvalues on the real axis. That is, $q \ge 0$, $p \ge 0$ and $n = 2q + p$.

We next develop a state feedback control law for each follower agent. We denote the difference between the state of a follower agent and the state of the leader agent as $\bar{x}_i(t) = x_i(t) - x_0(t)$, $i = 1, 2, \cdots, N$. Then,

$$\bar{x}_i(t+1) = A\bar{x}_i(t) + bu_i(t), \quad i = 1, 2, \cdots, N, \qquad (3)$$

where $\bar{x}_i = [\bar{x}_{i1}, \bar{x}_{i2}, \cdots, \bar{x}_{in}]^{\mathrm{T}} \in \mathbb{R}^n$.

Denote $\tilde{x}_k = [\bar{x}_{1k}, \bar{x}_{2k}, \cdots, \bar{x}_{Nk}]^{\mathrm{T}}$, $k = 1, 2, \cdots, n$, and $\tilde{x} = [\tilde{x}_1^{\mathrm{T}}, \tilde{x}_2^{\mathrm{T}}, \cdots, \tilde{x}_n^{\mathrm{T}}]^{\mathrm{T}}$. Then system (3) can be written as

$$\tilde{x}(t+1) = (A \otimes I_N)\tilde{x}(t) + (b \otimes I_N)u(t),$$

where $u(t) = [u_1(t), u_2(t), \cdots, u_N(t)]^{\mathrm{T}}$ and I_N denotes the N dimensional identity matrix.

According to Yang et al. [1997], there exists a non-singular matrix T such that the dynamics of the transformed state $z = T\tilde{x}$ can be written as

$$z(t+1) = \tilde{A}z(t) + \tilde{B}u(t),$$

where $z = [z_1^{\mathrm{T}}, z_2^{\mathrm{T}}, \cdots, z_n^{\mathrm{T}}]^{\mathrm{T}}$, $z_k = [z_{1k}, z_{2k}, \cdots, z_{Nk}]^{\mathrm{T}}$, $k = 1, 2, \cdots, n$, and the matrices

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

are defined according to the locations of the eigenvalues of A as follows.

Case 1: $p = 0$, or $p = 1$ ($\lambda = 1$), or $p \geq 2$. In this case,

$$A_1 = \begin{bmatrix} I_N & \varepsilon^{2(p+q)-3}\Gamma & \varepsilon^{2(p+q)-5}\Gamma & \cdots & \varepsilon^{2q+1}\Gamma \\ 0 & I_N & \varepsilon^{2(p+q)-5}\Gamma & \cdots & \varepsilon^{2q+1}\Gamma \\ 0 & 0 & I_N & \cdots & \varepsilon^{2q+1}\Gamma \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}_{pN \times pN},$$

$$A_2 = \begin{bmatrix} 0 & \varepsilon^{2q-1}\Gamma & 0 & \varepsilon^{2q-3}\Gamma & \cdots & 0 & \varepsilon\Gamma \\ 0 & \varepsilon^{2q-1}\Gamma & 0 & \varepsilon^{2q-3}\Gamma & \cdots & 0 & \varepsilon\Gamma \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \varepsilon^{2q-1}\Gamma & 0 & \varepsilon^{2q-3}\Gamma & \cdots & 0 & \varepsilon\Gamma \end{bmatrix}_{pN \times 2qN},$$

$$A_3 = \begin{bmatrix} \alpha_1 I_N & -\beta_1 I_N & 0 & -\beta_1\varepsilon^{2q-3}\Gamma & \cdots & 0 & -\beta_1\varepsilon\Gamma \\ \beta_1 I_N & \alpha_1 I_N & 0 & \alpha_1\varepsilon^{2q-3}\Gamma & \cdots & 0 & \alpha_1\varepsilon\Gamma \\ 0 & 0 & \alpha_2 I_N & -\beta_2 I_N & \cdots & 0 & -\beta_2\varepsilon\Gamma \\ 0 & 0 & \beta_2 I_N & \alpha_2 I_N & \cdots & 0 & \alpha_2\varepsilon\Gamma \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_q I_N & -\beta_q I_N \\ 0 & 0 & 0 & 0 & \cdots & \beta_q I_N & \alpha_q I_N \end{bmatrix},$$

$$B_1 = [\, I_N \;\; I_N \;\; \cdots \;\; I_N \,]^{\mathrm{T}}_{pN \times N},$$

$$B_2 = [\, -\beta_1 I_N \;\; \alpha_1 I_N \;\; -\beta_2 I_N \;\; \alpha_2 I_N \;\; \cdots \;\; -\beta_q I_N \;\; \alpha_q I_N \,]^{\mathrm{T}}_{2qN \times N},$$

where $\varepsilon > 0$ is a design parameter whose value is to be determined later.

Case 2: $p = 1$ ($\lambda = -1$). In this case,

$$A_1 = -I_N, \; A_2 = \begin{bmatrix} 0 & \varepsilon^{2q-1}\Gamma & 0 & \varepsilon^{2q-3}\Gamma & \cdots & 0 & \varepsilon\Gamma \end{bmatrix}^{\mathrm{T}}_{N \times 2qN},$$
$$B_1 = -I_N,$$

and A_3 and B_2 are as defined in Case 1.

Such a transformation T is explicitly constructed in Yang et al. [1997] as $T = R_2 R_1^{-1}$, where $R_1 = [b \otimes I_N,\ (Ab) \otimes I_N,\ \cdots,\ (A^{n-1}b) \otimes I_N]$ and $R_2 = [\tilde{B},\ \tilde{A}\tilde{B},\ \cdots,\ \tilde{A}^{n-1}\tilde{B}]$.

We use an example to help illustrate the state transformation. Let the dynamics of the agent be represented by

$$\begin{cases} x_{i1}(t+1) = x_{i1}(t) + x_{i2}(t) + x_{i3}(t), \\ x_{i2}(t+1) = x_{i2}(t) + x_{i3}(t), \\ x_{i3}(t+1) = x_{i3}(t) + u_i(t). \end{cases}$$

In this case, the linear state transformation matrix T is constructed as

$$T = \begin{bmatrix} \varepsilon^4\Gamma^2 & (\varepsilon+\varepsilon^3)\Gamma - \varepsilon^4\Gamma^2 & I \\ 0 & \varepsilon\Gamma & I \\ 0 & 0 & I \end{bmatrix}.$$

We note here that such a state transformation is constructed from matrix Γ. From the expression of T, we can further see that the state vector z_i is a linear combination of the states of agents that are within k hops away from agent i, where k is less than the largest algebraic multiplicity of the eigenvalues of the system matrix on the unit circle.

Based on the transformed states of the equation (3), we are now ready to construct the following bounded consensus control algorithm for each follower agent

$$u_i = -\sum_{l=1}^{q} \varepsilon^{l-1}\sigma\left(\varepsilon^l \sum_{j=1}^{N} \gamma_{ij} z_{n-2l+2,i}\right) - \sum_{l=q+1}^{q+p} \varepsilon^{l-1}\sigma\left(\varepsilon^l \sum_{j=1}^{N} \gamma_{ij} z_{n-l-q+1,i}\right), \; i = 1, 2, \cdots, N, \qquad (4)$$

where $\sigma : R \to R$ is a saturation function defined as $\sigma(s) = \mathrm{sign}(s)\min\left\{|s|, \frac{\delta}{2}\right\}$, and $0 < \varepsilon \leq \min\{\frac{1}{2}, \frac{1}{2N\lambda_N + 2N - 2}\}$. It is then easy to verify that $|u_i| \leq \frac{1-\varepsilon^{p+q}}{1-\varepsilon}\frac{\delta}{2} < \delta$.

Theorem 4. Let Assumptions 1 and 2 hold. Then, under the bounded control laws (4), the group of follower agents (1) and the leader agent (2) achieve global leader-following consensus. Moreover, each control input converges to zero as time goes to infinity.

Proof: We start from the states z_{n-1} and z_n and prove that, in a finite time, z_n will enter and remain in a bounded set, where the first term of the control law remains in the bounded set $[-\frac{\delta}{2}, \frac{\delta}{2}]$. Following similar analysis, we can prove that the states $z_k, k = 1, 2, \cdots, p, p+2, p+4, \cdots, n$, will all enter one by one and remain in a bounded set, such that each term in the control law remains in the bounded set $[-\frac{\delta}{2}, \frac{\delta}{2}]$. Then the closed-loop system become a linear system, for which global leader-following consensus can be reality established.

Notice that the control laws (4) can be rewritten as

$$\begin{aligned} u &= [u_1, u_2, \cdots, u_N]^{\mathrm{T}} \\ &= -\sum_{l=1}^{q} \varepsilon^{l-1}\sigma(\varepsilon^l\Gamma z_{n-2l+2}) - \sum_{l=q+1}^{q+p} \varepsilon^{l-1}\sigma(\varepsilon^l\Gamma z_{n-q-l+1}) \\ &= -\sum_{l=1}^{q} \varepsilon^{l-1}\sigma(\theta_{n-2l+2}) - \sum_{l=q+1}^{q+p} \varepsilon^{l-1}\sigma(\theta_{n-q-l+1}), \end{aligned}$$

where $\theta_{k,i}$ is the i^{th} element of

$$\theta_k = \begin{cases} \varepsilon^{n-q-l+1}\Gamma z_k, & k = 1, 2, \cdots, p, \\ \varepsilon^{\frac{1}{2}(n-l+2)}\Gamma z_k, & k = p+2, p+4, \cdots, n. \end{cases}$$

Here we have slightly abused the notation by using σ to denote both a scalar valued and a vector valued saturation function, that is, for $s = [s_1, s_2, \cdots, s_n]^{\mathrm{T}}$, $\sigma(s) = [\sigma(s_1), \sigma(s_2), \cdots, \sigma(s_n)]^{\mathrm{T}}$.

We first consider the evolutions of z_{n-1} and z_n, which are governed by

$$\begin{aligned} z_{n-1}(t+1) &= \alpha_q z_{n-1}(t) - \beta_q(z_n(t) + u), \\ z_n(t+1) &= \beta_q z_{n-1}(t) + \alpha_q(z_n(t) + u). \end{aligned}$$

Construct a Lyapunov function $V_1 = \frac{1}{2}z_{n-1}^{\mathrm{T}}\Gamma z_{n-1} + \frac{1}{2}z_n^{\mathrm{T}}\Gamma z_n$, which is positive definite. Then,

$$\begin{aligned}\Delta V_1 =& \frac{1}{2}(\alpha_q z_{n-1} - \beta_q(z_n+u))^{\mathrm{T}}\Gamma(\alpha_q z_{n-1} - \beta_q(z_n+u)) \\ &+\frac{1}{2}(\beta_q z_{n-1} + \alpha_q(z_n+u))^{\mathrm{T}}\Gamma(\beta_q z_{n-1} + \alpha_q(z_n+u)) \\ &-\frac{1}{2}z_{n-1}^{\mathrm{T}}\Gamma z_{n-1} - \frac{1}{2}z_n^{\mathrm{T}}\Gamma z_n \\ =& z_n^{\mathrm{T}}\Gamma u + \frac{1}{2}u^{\mathrm{T}}\Gamma u \\ =& -\sum_{i=1}^{N}\varepsilon^{-1}\theta_{n,i}\left(\sigma(\theta_{n,i}) + \sum_{l=2}^{q}\varepsilon^{l-1}\sigma(\theta_{n-2l+2,i})\right. \\ &\left.+\sum_{l=q+1}^{q+p}\varepsilon^{l-1}\sigma(\theta_{n-q-l+1,i})\right) + \frac{1}{2}u^{\mathrm{T}}\Gamma u.\end{aligned}$$

Here, and hereafter in a similar situation, we have suppressed the dependence on t of the state variables. If $|\theta_{n,i}| \geq \frac{\Delta}{2}$ for at least one $i, i = 1, 2, \cdots, N$, then

$$\begin{aligned}\Delta V_1 \leq& -\sum_{|\theta_{n,i}|\geq\frac{\delta}{2}}\varepsilon^{-1}\theta_{n,i}\left(\sigma(\theta_{n,i}) + \sum_{l=2}^{q}\varepsilon^{l-1}\sigma(\theta_{n-2l+2,i})\right. \\ &\left.+\sum_{l=q+1}^{q+p}\varepsilon^{l-1}\sigma(\theta_{n-q-l+1,i})\right) - \sum_{|\theta_{ni}|<\frac{\delta}{2}}\varepsilon^{-1}\theta_{n,i}\left(\sigma(\theta_{n,i}) + \right. \\ &\left.\sum_{l=2}^{q}\varepsilon^{l-1}\sigma(\theta_{n-2l+2,i}) + \sum_{l=q+1}^{q+p}\varepsilon^{l-1}\sigma(\theta_{n-q-l+1,i})\right) + \frac{u^{\mathrm{T}}\Gamma u}{2} \\ <& -\sum_{|\theta_{n,i}|\geq\frac{\delta}{2}}\left(\varepsilon^{-1} - \frac{1-\varepsilon^{p+q-1}}{1-\varepsilon}\right)\frac{\delta^2}{4} \\ &+\sum_{|\theta_{n,i}|<\frac{\delta}{2}}\frac{(1-\varepsilon^{p+q-1})\delta^2}{4(1-\varepsilon)} + \frac{N\lambda_N\delta^2}{2} \\ <& -\frac{\delta^2}{4}\left(\varepsilon^{-1} - \frac{(N-1)(1-\varepsilon^{p+q-1})}{1-\varepsilon} - 2N\lambda_N\right) \\ <& \, 0,\end{aligned}$$

where we have used the facts that $\varepsilon \leq \frac{1}{2N\lambda_N + 2N - 2}$ and $|u_i| < \delta$. The above derivation implies that z_n will keep decreasing and $\theta_{n,i}$, $i = 1, 2, \cdots, N$, will enter and remain inside the interval $(-\frac{\delta}{2}, \frac{\delta}{2})$ in a finite time, after which, the evolutions of z_{n-3} and z_{n-2} and the control laws u respectively become

$$\begin{aligned}z_{n-3}(t+1) =& \alpha_{q-1}z_{n-3}(t) - \beta_{q-1}\left(z_{n-2}(t) - \sum_{l=2}^{q}\varepsilon^{l-1}\right. \\ &\left.\times\sigma(\varepsilon^l\Gamma z_{n-2l+2}(t)) - \sum_{l=q+1}^{q+p}\varepsilon^{l-1}\sigma(\varepsilon^l\Gamma z_{n-q-l+1}(t))\right), \\ z_{n-2}(t+1) =& \beta_{q-1}z_{n-3}(t) + \alpha_{q-1}\left(z_{n-2}(t) - \sum_{l=2}^{q}\varepsilon^{l-1}\right. \\ &\left.\times\sigma(\varepsilon^l\Gamma z_{n-2l+2}(t)) - \sum_{l=q+1}^{q+p}\varepsilon^{l-1}\sigma(\varepsilon^l\Gamma z_{n-q-l+1}(t))\right),\end{aligned}$$

and

$$u = -\varepsilon\Gamma z_n - \sum_{l=2}^{q}\varepsilon^{l-1}\sigma(\varepsilon^l\Gamma z_{n-2l+2}) - \sum_{l=q+1}^{q+p}\varepsilon^{l-1}\sigma(\varepsilon^l\Gamma z_{n-q-l+1}).$$

Following a similar analysis as in the analysis of the evolutions of z_{n-1} and z_n, we can show that all $\theta_{k,i}$, $k = p+2, p+4, \cdots, n, i = 1, 2, \cdots, N$, will enter and remain inside the interval $(-\frac{\delta}{2}, \frac{\delta}{2})$ in a finite time.

We next consider the evolution of z_p. We first consider Case 1, $p = 0$, or $p = 1, (\lambda = 1)$ or $p \geq 2$. The evolution of z_p is governed by

$$z_p(t+1) = z_p(t) - \sum_{l=q+1}^{p+q}\varepsilon^{l-1}\sigma\left(\varepsilon^l\Gamma z_{n-q-l+1}(t)\right) = z_p(t) + \bar{u}.$$

where

$$\bar{u} = -\sum_{l=q+1}^{p+q}\varepsilon^{l-1}\sigma\left(\varepsilon^l\Gamma z_{n-q-l+1}\right) = -\sum_{l=q+1}^{p+q}\varepsilon^{l-1}\sigma(\theta_{n-q-l+1}).$$

Construct a Lyapunov function $V_2 = \frac{1}{2}z_p^{\mathrm{T}}\Gamma z_p$, which is positive definite. Then,

$$\begin{aligned}\Delta V_2 =& \frac{1}{2}(z_p+\bar{u})^{\mathrm{T}}\Gamma(z_p+\bar{u}) - \frac{1}{2}z_p^{\mathrm{T}}\Gamma z_p \\ =& z_p^{\mathrm{T}}\Gamma\bar{u} + \frac{1}{2}\bar{u}^{\mathrm{T}}\Gamma\bar{u} \\ =& -\sum_{i=1}^{N}\varepsilon^{-1-q}\theta_{p,i}\left(\varepsilon^q\sigma(\theta_{p,i}) + \sum_{l=q+2}^{q+p}\varepsilon^{l-1}\sigma(\theta_{n-q-l+1,i})\right) + \frac{\bar{u}^{\mathrm{T}}\Gamma\bar{u}}{2}.\end{aligned}$$

If $|\theta_{p,i}| \geq \frac{\delta}{2}$ for at least one $i, i = 1, 2, \cdots, N$, then

$$\begin{aligned}\Delta_{V_2} \leq& -\sum_{|\theta_{p,i}|\geq\frac{\delta}{2}}\varepsilon^{-1-q}\theta_{p,i}\left(\varepsilon^q\sigma(\theta_{p,i}) + \sum_{l=q+2}^{q+p}\varepsilon^{l-1}\sigma(\theta_{n-q-l+1,i})\right) \\ &-\sum_{|\theta_{n,i}|<\frac{\delta}{2}}\varepsilon^{-1-q}\theta_{p,i}\left(\varepsilon^q\sigma(\theta_{p,i}) + \sum_{l=q+2}^{q+p}\varepsilon^{l-1}\sigma(\theta_{n-q-l+1,i})\right) \\ &+\frac{1}{2}\bar{u}^{\mathrm{T}}\Gamma\bar{u} \\ <& -\sum_{|\theta_{n,i}|\geq\frac{\delta}{2}}\left(\varepsilon^{-1} - \frac{1-\varepsilon^{p-1}}{1-\varepsilon}\right)\frac{\delta^2}{4} + \sum_{|\theta_{n,i}|<\frac{\delta}{2}}\frac{(1-\varepsilon^{p-1})\delta^2}{4(1-\varepsilon)} \\ &+\frac{1}{2}N\lambda_N\varepsilon^{2q}\delta^2 \\ <& -\left(\varepsilon^{-1} - \frac{(N-1)(1-\varepsilon^{q-1})}{1-\varepsilon} - 2N\lambda_N\varepsilon^{2q}\right)\frac{\delta^2}{4} \\ \leq& \, 0,\end{aligned}$$

where we have used the facts that $\varepsilon \leq \frac{1}{2N\lambda_N + 2N - 2}$ and $|\bar{u}| \leq \varepsilon^q\delta$. Therefore, the state z_p will keep decreasing and $\theta_{p,i}$ will enter and remain inside the interval $(-\frac{\delta}{2}, \frac{\delta}{2})$ in a finite time, after which the evolutions of z_{p-1} and the control laws u respectively become,

$$z_{p-1}(t+1) = z_{p-1}(t) - \sum_{l=q+2}^{q+p}\varepsilon^{l-1}\sigma(\varepsilon^l\Gamma z_{n-q-l+1}),$$

and

$$u = -\sum_{l=1}^{q}\varepsilon^{2l-1}\Gamma z_{n-2l+2} - \varepsilon^{2q+1}\Gamma z_p - \sum_{l=q+2}^{q+p}\varepsilon^{l-1}\sigma\left(\varepsilon^l\Gamma z_{n-q-l+1}\right).$$

Following a similar analysis as in the analysis of the evolution of z_p, we can show that all $\theta_{k,i}$, $k = 1, 2, \cdots, p, i = 1, 2, \cdots, N$, will enter and remain inside the interval $(-\frac{\delta}{2}, \frac{\delta}{2})$ in a finite time, after which the closed-loop system (3) becomes

$$z_1(t+1) = \left(I - \varepsilon^{2(p+q)-1}\Gamma\right)z_1(t),$$

$$
\begin{aligned}
z_2(t+1) &= \left(I - \varepsilon^{2(p+q)-3}\Gamma\right) z_2(t) - \varepsilon^{2(p+q)-1}\Gamma z_1(t),\\
&\vdots\\
z_p(t+1) &= \left(I - \varepsilon^{2q+1}\Gamma\right) z_p(t) - \sum_{l=q+1}^{q+p} \varepsilon^{2l-1}\Gamma z_{n-q-l+1},\\
z_{p+1}(t+1) &= \alpha_1 z_{p+1}(t) - \beta_1\left(I - \varepsilon^{2q-1}\Gamma\right) z_{p+2}(t)\\
&\quad +\beta_1 \sum_{l=q+1}^{q+p} \varepsilon^{2l-1}\Gamma z_{n-q-l+1},\\
z_{p+2}(t+1) &= \beta_1 z_{p+1}(t) + \alpha_1(I - \varepsilon^{2q-1}\Gamma) z_{p+2}(t)\\
&\quad -\alpha_1 \sum_{l=q+1}^{q+p} \varepsilon^{2l-1}\Gamma z_{n-q-l+1},\\
z_{p+3}(t+1) &= \alpha_2 z_{p+3}(t) - \beta_2(I - \varepsilon^{2q-3}\Gamma) z_{p+4}\\
&\quad +\beta_2\left(\varepsilon^{2q-1}\Gamma z_{p+2}(t) + \sum_{l=q+1}^{q+p}\varepsilon^{2l-1}\Gamma z_{n-q-l+1}\right),\\
z_{p+4}(t+1) &= \beta_2 z_{p+3}(t) + \alpha_2\left(I - \varepsilon^{2q-3}\Gamma\right) z_{p+4}\\
&\quad -\alpha_2\left(\varepsilon^{2q-1}\Gamma z_{p+2}(t) + \sum_{l=q+1}^{q+p}\varepsilon^{2l-1}\Gamma z_{n-q-l+1}\right),\\
&\vdots\\
z_{n-1}(t+1) &= \alpha_q z_{n-1}(t) - \beta_q(I - \varepsilon\Gamma) z_n(t) +\\
&\quad \beta_q\left(\sum_{l=2}^{q}\varepsilon^{2l-1}\Gamma z_{n-2l+2} + \sum_{l=q+1}^{q+p}\varepsilon^{2l-1}\Gamma z_{n-q-l+1}\right),\\
z_n(t+1) &= \beta_q z_{n-1}(t) + \alpha_q(I - \varepsilon\Gamma) z_n(t) -\\
&\quad \alpha_q\left(\sum_{l=2}^{q}\varepsilon^{2l-1}\Gamma z_{n-2l+2} + \sum_{l=q+1}^{q+p}\varepsilon^{2l-1}\Gamma z_{n-q-l+1}\right).
\end{aligned}
$$

It is trivial to show that $\lim_{t\to\infty} z_k(k) = 0, k = 1, 2, \cdots, p$, since Γ is positive definite and $\varepsilon \leq \frac{1}{2N\lambda_N+2N-2}$.

We then consider the evolutions of z_{p+1} and z_{p+2},

$$
\begin{aligned}
z_{p+1}(t+1) &= \alpha_1 z_{p+1}(t) - \beta_1\big(z_{p+2}(t) - \varepsilon^{2q-1}\Gamma z_{p+2}(t)\big)\\
&\quad +\beta_1 \sum_{l=q+1}^{q+p} \varepsilon^{2l-1}\Gamma z_{n-q-l+1}, \qquad (6)\\
z_{p+2}(t+1) &= \beta_1 z_{p+1}(t) + \alpha_1\big(z_{p+2}(t) - \varepsilon^{2q-1}\Gamma z_{p+2}(t)\big)\\
&\quad +\alpha_1 \sum_{l=q+1}^{q+p} \varepsilon^{2l-1}\Gamma z_{n-q-l+1}. \qquad (7)
\end{aligned}
$$

We first consider the linear system

$$
\begin{aligned}
z_{p+1}(t+1) &= \alpha_1 z_{p+1}(t) - \beta_1\big(z_{p+2}(t) - \varepsilon^{2q-1}\Gamma z_{p+2}(t)\big), \quad (8)\\
z_{p+2}(t+1) &= \beta_1 z_{p+1}(t) + \alpha_1\big(z_{p+2}(t) - \varepsilon^{2q-1}\Gamma z_{p+2}(t)\big), \quad (9)
\end{aligned}
$$

for which we construct a Lyapunov function $V_3 = \frac{1}{2}z_{p+1}^{\mathrm{T}}\Gamma z_{p+1} + \frac{1}{2}z_{p+2}^{\mathrm{T}}\Gamma z_{p+2}$, which is positive definite. Then,

$$
\begin{aligned}
\Delta V_3 &= -\varepsilon^{2q-1}(z_{p+2}\Gamma)^{\mathrm{T}}\Gamma z_{p+2} + \frac{1}{2}\varepsilon^{4q-2}(z_{p+2}\Gamma)^{\mathrm{T}}\Gamma(\Gamma z_{p+2})\\
&= -\varepsilon^{-1}\theta_{p+2}^{\mathrm{T}}\theta_{p+2} + \frac{1}{2}\varepsilon^{2q-2}\theta_{p+2}^{\mathrm{T}}\Gamma\theta_{p+2}\\
&\leq -\left(\varepsilon^{-1} - \frac{1}{2}\varepsilon^{2q-2}\lambda_N\right)\theta_{p+2}^{\mathrm{T}}\theta_{p+2}\\
&\leq 0,
\end{aligned}
$$

where we have used the facts that $\varepsilon \leq \frac{1}{2N\lambda_N+2N-2}$ and $q \geq 1$ (the states z_{p+1} and z_{p+2} do not exist when $q = 0$). Therefore, we have $\Delta V_3 \leq 0$ and $\Delta V_3 \equiv 0$ only when $z_{p+1}(t) = z_{p+2}(t) \equiv 0$. That is, the linear discrete-time system (8)-(9) is asymptotically stable, which in turn implies that the states $z_{p+1}(t)$ and $z_{p+2}(t)$ of system (6)-(7) approach zero as time goes to infinity. Following a similar analysis of the evolutions of z_{p+1} and z_{p+2}, we can show that $\lim_{t\to\infty} z_k(t) = 0$, $k = p+3, p+4, \cdots, n$, which means $\lim_{t\to\infty}(x_{ik}(t) - x_{0k}(t)) = 0$, $k = 1, 2, \cdots, n, i = 1, 2, \cdots, N$, and $\lim_{t\to\infty} u_i(t) = 0, i = 1, 2, \cdots, N$.

We next consider Case 2, $p = 1$ and $\lambda_1 = -1$, and

$$z_1(t+1) = -z_1(t) + \varepsilon^q\sigma(\varepsilon^{q+1}\Gamma z_1(t)).$$

Construct a Lyapunov function $V_4 = \frac{1}{2}z_1^{\mathrm{T}}\Gamma z_1$. Then,

$$
\begin{aligned}
\Delta V_4 &= -z_1^{\mathrm{T}}\Gamma\varepsilon^q\sigma\left(\varepsilon^{q+1}\Gamma z_1\right) + \frac{1}{2}\varepsilon^{2q}\sigma(\varepsilon^{q+1}\Gamma z_1)^{\mathrm{T}}\Gamma\sigma\left(\varepsilon^{q+1}\Gamma z_1\right)\\
&= -\varepsilon^{-1}\theta_1\sigma(\theta_1) + \frac{1}{2}\varepsilon^{2q}\sigma(\theta_1)^{\mathrm{T}}\Gamma\sigma(\theta_1)\\
&\leq -\left(\varepsilon^{-1} - \frac{1}{2}\varepsilon^{2q}\lambda_N\right)\sigma(\theta_1)^{\mathrm{T}}\sigma(\theta_1)\\
&< 0,
\end{aligned}
$$

where we have used the fact that $\varepsilon \leq \frac{1}{2N\lambda_N+2N-1}$. Thus, we have $\lim_{t\to\infty} x_{i1}(t) = 0$.

As we have proven in Case 1, $\lim_{t\to\infty} z_k(t) = 0$, $k = 2, 3, \cdots, n$, which in turn means that $\lim_{t\to\infty}(x_{ik}(t) - x_{0k}(t)) = 0$, $k = 1, 2, \cdots, n, i = 1, 2, \cdots, N$, and $\lim_{t\to\infty} u_i(t) = 0, i = 1, 2, \cdots, N$.

4. SIMULATION RESULTS

Consider a group of 3 follower agents, each described by

$$
\begin{cases}
x_{i1}(t+1) = x_{i1}(t) + x_{i2}(t) + x_{i3}(t),\\
x_{i2}(t+1) = x_{i2}(t) + x_{i3}(t),\\
x_{i3}(t+1) = x_{i3}(t) + u_i(t),\ i = 1, 2, 3.
\end{cases}
$$

The dynamics of the leader agent is described as

$$
\begin{cases}
x_{01}(t+1) = x_{01}(t) + x_{02}(t) + x_{03}(t),\\
x_{02}(t+1) = x_{02}(t) + x_{03}(t),\\
x_{03}(t+1) = x_{03}(t).
\end{cases}
$$

The communication topology among the followers is represented by a directed graph which satisfies Assumption 2. The associated adjacency matrix $\mathcal{A}_{\mathcal{N}}$ is given by

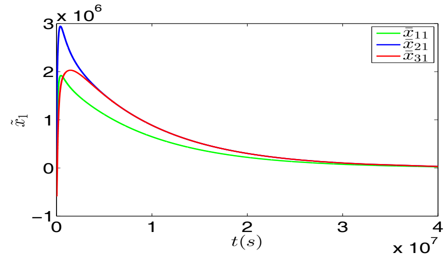
$$
\mathcal{A}_{\mathcal{N}} = \begin{bmatrix} 0 & -1 & -1\\ -2 & 0 & -2\\ -1 & -1 & 0 \end{bmatrix},
$$

and $a_{01} = 1, a_{02} = a_{03} = 0$. Choose $v_1 = v_3 = 0.2, v_2 = 0.1$, then $\lambda_N = 0.7464$. Choose $\varepsilon = 0.1$ to satisfy $\varepsilon < \frac{1}{2N\lambda_N+2N-2}$.

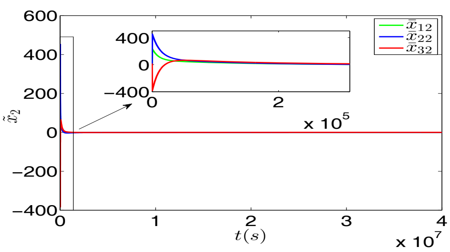
Also, let $\delta = 2$. For the simulation purpose, we pick initial states of the follower agents and the leader agent as

$$
x_0(0) = \begin{bmatrix} 0.1\\ 1\\ 6 \end{bmatrix}, x_1(0) = \begin{bmatrix} -6\\ 16\\ 26 \end{bmatrix}, x_2(0) = \begin{bmatrix} 20\\ -10\\ 36 \end{bmatrix}, x_3(0) = \begin{bmatrix} 20\\ 18\\ 12 \end{bmatrix}.
$$

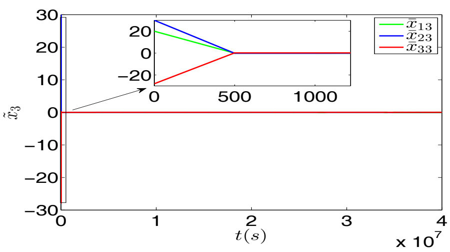
Under the feedback control laws (4), the evolutions of the differences between the states of the follower agents and the corresponding state of the leader agent are shown in Fig. 1(a), 1(b) and 1(c), respectively. Shown in Fig. 1(d) are the inputs the follower agents. We can see that the leader-following consensus is achieved.



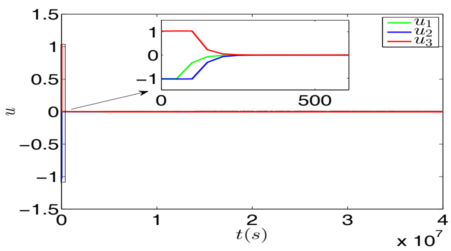
(a) The difference between $\bar{x}_{i1}$ and $\bar{x}_{01}$, $i = 1, 2, 3$.



(b) The difference between $\bar{x}_{i2}$ and $\bar{x}_{02}$, $i = 1, 2, 3$.



(c) The difference between $\bar{x}_{i3}$ and $\bar{x}_{03}$, $i = 1, 2, 3$.



(d) The evolution of u_i, $i = 1, 2, 3$.

Fig. 1. The evolutions of the agents.

5. CONCLUSIONS

In this paper we have studied the global leader-following consensus problem for a group of discrete-time linear systems with bounded control. We constructed, for each follower agent, a bounded nonlinear feedback control, which uses the information of other agents obtained through multi-hop paths in the communication network. We established that global leader-following consensus is achieved under the feedback control laws we have constructed when the communication topology among follower agents forms a strongly connected and detailed balanced directed graph and the leader is a neighbor of at least one follower agent.

REFERENCES

R.O. Saber and R.M. Murray. Consensus problems in networks of agents with swiching topology and time-delays. *IEEE Trans. on Auto. Contr.*, 49(9): 1520–1533, 2004.

W. Ren, R.W. Beard and E.M. Atkins. A survey of consensus pronlems in multi-agent coordination. *Proc. of American Control Conference*, 1859–1864, Portland, OR, USA, 2005.

Z. Jin and R.M. Murray. Multi-hop relay protocols for fast consensus seeking. *Proc. of the 45th IEEE Conference on Decision and Control*, 1001–1006, 2006, San Diego, CA, USA, 2006.

W. Yu, G. Chen, W. Ren, J. Kurths and W. Zheng. Distributed higher order consensus protocols in multiagent dynamical systems. *IEEE Trans. on Circuits and Systems*, 58(8): 1924–1932, 2011.

Y. Li, J. Xiang and W. Wei. Consensus problems for linear time-invariant multi-agent systems with saturation constraints. *IET Control Theory and Applications*, 5(6): 823–829, 2011.

L. Wang and F. Xiao. A new approach to consensus problems for discrete-time multiagent systems with time-delays. *Proc. of American Control Conference*, 2118–2123, Minneapolis, Minnesota, USA, 2006.

D.W. Casbeer, R. Beard and A.L. Swindlehurst. Discrete double integrator consensus. *Proc. of 47th IEEE Conference on Decision and Control*, 2264–2269, Cancun, Mexico, 2008.

Y. Chen, J. Lü and Z. Lin. Consensus of discrete-time multi-agent systems with nonlinear transmission nonlinearities. *Automatica*, 49(6): 1768–1775, 2013.

Z. Meng, Z. Zhao and Z. Lin. On global leader-following consensus of identical linear dynamic systems subject to actuator saturation. *Sys. & Contr. Letters*, 62(2): 132–142, 2013.

T. Yang, Z. Meng, D.V. Dimarogonas and K.H. Johansson. Global consensus for discrete-time multi-agent systems with input saturation constraints. *Automatica*, 50: 499–506, 2014.

H. Sussmann and Y. Yang. On the stabilizability of multiple integrators by means of bounded feedback controls. *Proc. of 30th IEEE Conference on Decision and Control*, 70–72, 1991.

H.J. Sussmann, E.D. Sontag and Y. Yang. A general result on stabilization of linear systems using bounded controls. *IEEE Trans. on Auto. Contr.*, 39(12): 2411–2425, 1994.

A.R. Teel. Global stabilization and restricted tracking for multiple integrators with bounded controls. *Sys. & Contr. Letters*, 18(3): 165–171, 1992.

Y. Yang, E.D. Sontag and H. Sussmann. Global stabilization of linear discrete-time systems with bounded feedback. *Sys. & Contr. Letters*, 30(5): 273–281, 1997.

Z. Zhao and Z. Lin. Global leader-following consensus of a group of multiple integrator agents using bounded controls. *Proc. of 32nd Chinese Control Conference*, 7172–7178, Xian, China, 2013.

F. Jiang and L. Wang. Finite-time information consensus for multi-agent systems with fixed and switching topologies. Physica D, 238(16): 1550–1560, 2009.

W. Ren and Y. Cao. Distributed coordination of multi-agent networks: emergent problems, models, and issues. Springer, 2011.

J. Hu and Y. Hong. Leader-following coordination of multi-agent systems with coupling time delays. Physica A: Statistical Mechanics and its Applications, 374(2): 853–863, 2007.