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Delay-Dependent Regional Stabilization of Nonlinear Quadratic Time-Delay Systems ⋆

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Abstract: This paper addresses the synthesis of delay-dependent local stabilizing controllers for, possibly open-loop unstable, nonlinear quadratic systems with a varying time-delay in the state. We develop methods for designing static nonlinear quadratic state feedback controllers that guarantee the local asymptotic stability of the closed-loop system zero equilibrium point in some polytopic region of the state-space while ensuring a region of stability inside this polytope. Control designs based on either the Razumikhin or the Lyapunov-Krasovskii approaches are considered. The proposed designs are delay-dependent and are formulated in terms of linear matrix inequalities. A numerical example is presented to illustrate the application of the stabilization methods.

Keywords: Time-delay systems, quadratic systems, local stabilization, stability region, delay-dependent stability, Razumikhin theorem, Lyapunov-Krasovskii functional.

1. INTRODUCTION

Stability analysis and stabilization of nonlinear time-delay systems are problems of considerable theoretical and practical relevance which are receiving increasing attention of control system researchers. These problems are known to be hard, in particular when one is interested in computing estimates of the system domain of attraction (DOA) and/or designing locally stabilizing controllers with a guaranteed region of stability.

Over the past decade, several approaches of global stabilization have been proposed in the literature, such as, state feedback linearization (Zhang and Cheng [2005]), sum of squares (Papachristodoulou [2005]), backstepping (Mazenc and Bliman [2006]) and forwarding (Jankovic [2009]). In addition, input-to-state stability and stabilization have been studied in, e.g. Pepe and Jiang [2006], Fridman *et al.* [2008] and Pepe [2009]. In the context of DOA estimation, Melchor-Aguilar and Niculescu [2007] have presented techniques to compute DOA estimates for linear time-delay systems with a Lipschitz-type nonlinearity, for both constant and time-varying delays. On the other hand, Coutinho and de Souza [2008] have developed methods for DOA estimation and $\mathscr{L}_2$ analysis for a class of nonlinear systems subject to a constant time-delay and polytopic-type parameter uncertainty, which includes systems with rational functions of the state and uncertain parameters as well as some trigonometric nonlinearities. As for the issue of designing local stabilizing controllers for open-loop unstable nonlinear time-delay systems while providing a guaranteed region of stability, linear matrix inequality (LMI) based methods of *delay-independent* state feedback stabilization have been recently proposed in de Souza and Coutinho [2012] for the class of nonlinear quadratic time-delay systems, namely state-delayed systems with quadratic nonlinearities in the state variables and bilinear terms in the state and control signal.

Inspired by de Souza and Coutinho [2012], in this paper we address the problem of *delay-dependent* local stabilization of, possibly open-loop unstable, nonlinear quadratic systems with a time-varying delay in the state. The motivation for considering delayed-state systems with quadratic nonlinearities is that they can represent a large number of processes, and include the so-called bilinear time-delay systems as a special case (see, for instance, Amato *et al.* [2007], Coutinho and de Souza [2012], and the references therein). We develop LMI based synthesis methods of static nonlinear quadratic state feedback controllers to ensure the local asymptotic stability of the origin with a guaranteed region of stability inside some polytopic region of the state-space. Two control design approaches are proposed. The first one, referred to as the *Razumikhin approach*, is based on the Razumikhin stability theorem, whereas the second one, referred to as the *Lyapunov-Krasovskii approach*, builds on a Lyapunov-Krasovskii functional.

Notation. $\mathbb{R}^+ = [0,\infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real matrices, I_n is the $n\times n$ identity matrix, 0_n and $0_{m\times n}$ are the $n\times n$ and $m\times n$ matrices of zeros, respectively, and $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix. For a real matrix S, S' denotes its transpose, $\mathrm{He}(S)$ stands for $S+S'$, and $S>0$ means that S is symmetric and positive-definite. For a symmetric block matrix, the symbol $\star$ stands for the transpose of the blocks outside the main diagonal block. The Banach space of continuous functions $\phi : [-d,0] \to \mathbb{R}^n$ with finite norm $\|\phi\|_d =: \sup_{-d\le t\le 0} \|\phi(t)\|$ is denoted by $\mathscr{C}_d^n$, where $\|\cdot\|$ is the Euclidean vector norm, and $x_t \in \mathscr{C}_d^n$ is a segment of the function $x(\cdot)$ given by $x_t(s) = x(t+s)$, $\forall\, s \in [-d,0]$.

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2. PROBLEM FORMULATION

Consider the following class of nonlinear quadratic systems with a time-delayed state:

$$\begin{cases} \dot{x}(t) = A(x(t))x(t) + A_d(x(t))\tilde{x}(t) + B(x(t))u(t)\,, \\ x(t) = \phi(t),\ \forall\, t \in [-d,0], \quad \tilde{x}(t) := x(t-\tau(t)) \end{cases} \tag{1}$$

where $x(t) \in \mathscr{X} \subset \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $\mathscr{X}$ is some polytopic region of the state-space containing the system origin (to be specified later), $\phi(\cdot) \in \mathscr{C}_d^n$ is the system initial function, and $\tau(t)$ is a varying time-delay satisfying

$$0 < \tau(t) \le d, \quad \dot{\tau}(t) \le h < \infty, \ \forall\, t \ge 0\,, \tag{2}$$

with d and h being given positive scalars. The matrices $A(\cdot)$, $A_d(\cdot)$ and $B(\cdot)$ are real affine functions, that is:

$$\begin{cases} A(x) = A_0 + \sum_{i=1}^{n} x_i A_i, \quad A_d(x) = A_{d_0} + \sum_{i=1}^{n} x_i A_{d_i}, \\ B(x) = B_0 + \sum_{i=1}^{n} x_i B_i, \end{cases} \tag{3}$$

where x_i denotes the i-th component of x, and A_i, A_{d_i} and B_i are given constant matrices.

The equilibrium solution $x \equiv 0$ of system (1) with $u \equiv 0$ is allowed to be unstable and it is assumed that the following linear system:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_{d_0}\tilde{x}(t) + B_0 u(t)\,, \\ x(t) = \phi(t),\ \forall\, t \in [-d,0], \end{cases}$$

is delay-dependent stabilizable via a linear state feedback $u(t) = Kx(t)$. For notation simplicity, in the sequel the argument t of $x(t)$, $\tilde{x}(t)$, $u(t)$ and $\tau(t)$ will be often omitted.

Given upper bounds d and h on the delay and its time-derivative, respectively, this paper focuses on designing a nonlinear quadratic static state feedback control law as follows:

$$u = K(x)x, \quad K(x) = K_0 + \sum_{i=1}^{n} x_i K_i, \tag{4}$$

that ensures the local asymptotic stability of the equilibrium solution $x \equiv 0$ of the closed-loop system. In addition, we also aim at deriving a *stability region* for the controlled system, namely a set of initial functions ϕ for the closed-loop system of (1) and (4) such that $x(t) \to 0$ as $t \to \infty$.

The problem of local stabilization while providing a stability region will be referred to as *regional stabilization*, and it will be addressed in this paper via two approaches. The first one builds on the Razumikhin stability theorem while the second approach is based on Lyapunov-Krasovskii functionals, with both being of delay-dependent type.

3. PRELIMINARY RESULTS

The polytopic state-space domain $\mathscr{X}$ will play an important role on deriving numerically tractable solutions to the regional stabilization problem for both the Razumikhin and Lyapunov-Krasovskii approaches. In this paper, it is assumed that $\mathscr{X}$ is a given symmetric polytope (with respect to the origin). Depending on the convenience, the polytope $\mathscr{X}$ will be represented either in terms of the convex hull of its κ vertices

$$\mathscr{X} = \mathrm{Co}\{v_1, v_2, \ldots, v_\kappa\}, \tag{5}$$

or alternatively in terms of its faces

$$\mathscr{X} = \left\{x \in \mathbb{R}^n : |c_i' x| \le 1, \ i = 1, \ldots, n_f\right\}, \tag{6}$$

with $c_i \in \mathbb{R}^n$, $i = 1, \ldots, n_f$ defining the faces of $\mathscr{X}$.

In order to obtain an LMI formulation for the regional stabilization methods to be developed in the next two sections, the closed-loop system of (1) with the control law in (4) is written in the following form:

$$\dot{x} = \bar{A}(x)x + A_d(x)\tilde{x}, \tag{7}$$

where

$$\bar{A}(x) = A(x) + (B_0 + \Pi(x)' B_x)K(x), \tag{8}$$

$$B_x = \left[\, B_1' \ \cdots \ B_n' \,\right]', \quad \Pi(x) = \left[\, x_1 I_n \ \cdots \ x_n I_n \,\right]'. \tag{9}$$

We end this section by recalling four results that are instrumental to derive LMI based conditions for regional stabilization of the nonlinear quadratic state-delayed system in (1)-(3). The first result is a version of Finsler's lemma to handle constrained inequalities; see, for instance, de Oliveira and Skelton [2001]. The second one is a well-known inequality for completing the squares; see, for instance, Li and de Souza [1997]. The other two results are respectively the Razumikhin and Lyapunov-Krasovskii stability theorems for retarded functional differential equations; see, for instance, Hale and Lunel [1993].

Lemma 1. Given matrix functions $N(v) \in \mathbb{R}^{s\times m}$, $S(v) = S(v)' \in \mathbb{R}^{m\times m}$ and $\eta(v) \in \mathbb{R}^m$, with $v \in \mathbb{V} \subseteq \mathbb{R}^v$, then

$$\eta(v)' S(v) \eta(v) < 0, \ \forall\, v \in \mathbb{V} : N(v)\eta(v) = 0, \ \eta(v) \ne 0$$

if there exists a matrix $L \in \mathbb{R}^{m\times s}$ such that

$$S(v) + \mathrm{He}\big(LN(v)\big) < 0, \ \forall\, v \in \mathbb{V}. \qquad \square$$

Lemma 2. For any vectors w, $y \in \mathbb{R}^n$ and any symmetric positive definite matrix $T \in \mathbb{R}^{n\times n}$, the following holds:

$$-2w'y \le w'Tw + y'T^{-1}y\,. \qquad \square$$

For the next two results, consider the following functional differential equation of retarded type:

$$\begin{cases} \dot{x} = f(x_t), \ \ x \in \mathscr{X} \subset \mathbb{R}^n, \ \ x_t \in \mathscr{C}_d^n, \\ x(s) = \phi(s), \ \forall\, s \in [-d,0], \ \ \phi \in \mathscr{C}_d^n, \end{cases} \tag{10}$$

where $f : \mathscr{C}_d^n \to \mathbb{R}^n$ is continuous and $f(0) = 0$, and it is assumed that for any $\phi \in \mathscr{C}_d^n$, (10) possesses a unique solution.

Lemma 3. Let u, $v : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous, positive definite functions, u non-decreasing, and v strictly increasing. Suppose $p : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function satisfying $p(s) > s$ for $s > 0$. Suppose there exists a continuously differentiable function $V_R : \mathscr{X} \to \mathbb{R}$ such that

(a) $u(\|x\|) \le V_R(x) \le v(\|x\|), \ \forall\, x \in \mathscr{X}$;

(b) the time-derivative $\dot{V}_R(x(t))$ of $V_R(x(t))$ along the solution of (10) satisfies

$$\dot{V}_R(x(t)) < 0, \ \text{ if } \ V_R(x(t+s)) \le p\big(V_R(x(t))\big),$$
$$\forall\, s \in [-d,0], \ t \ge 0, \ \forall\, x(t) \in \mathscr{X}, \ x(t) \ne 0.$$

Let $\mathscr{R}_R(c) := \{\phi \in \mathscr{C}_d^n : \phi(s) \in \mathscr{B}_R(c), \ \forall\, s \in [-d,0]\}$ and $\mathscr{B}_R(c) := \{x \in \mathbb{R}^n : V_R(x) \le c\}$, where $c > 0$ is such that $\mathscr{B}_R(c) \subset \mathscr{X}$. Then, the solution $x \equiv 0$ of (10) is locally asymptotically stable in $\mathscr{X}$. Moreover, $x_t \in \mathscr{R}_R(c)$, $\forall t \ge 0$ and $\lim_{t\to\infty} x(t) = 0$ for any $\phi \in \mathscr{R}_R(c)$. $\square$

Lemma 4. Let u, v, $w : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous, positive definite functions, u, w non-decreasing, and v strictly increasing. Suppose there exists a continuously differentiable functional

$V_K : \mathscr{C}_d^n \to \mathbb{R}$ such that

(a) $u(\|x(t)\|) \leq V_K(x_t) \leq v(\|x_t\|_d), \ \forall x(t) \in \mathscr{X}$;

(b) the time-derivative $\dot{V}_K(x_t)$ of $V_K(x_t)$ along the solution of (10) satisfies

$$\dot{V}_K(x_t) < -w(\|x(t)\|), \ \forall \, x(t) \in \mathscr{X}.$$

Let $\mathscr{R}_K(\gamma) := \{\, \phi \in \mathscr{C}_d^n : V_K(\phi) \leq \gamma\}$ with $\gamma > 0$ such that $\mathbb{B}_K(\gamma) := \{\, \phi(s) \in \mathbb{R}^n, \ s \in [-d, 0] : \phi \in \mathscr{R}_K(\gamma)\} \subset \mathscr{X}$. Then, the solution $x \equiv 0$ of (10) is locally asymptotically stable in $\mathscr{X}$. Moreover, $x_t \in \mathscr{R}_K(\gamma), \forall t \geq 0$, and $\lim_{t\to\infty} x(t) = 0$ for any initial function $\phi \in \mathscr{R}_K(\gamma)$.

4. RAZUMIKHIN APPROACH

Letting $x(t)$ be the trajectory of the closed-loop system in (7) with an initial function $\phi(\cdot) \in \mathscr{C}_d^n$, it holds for $t \geq d$ that:

$$x(t-\tau) = x(t) - \int_{-\tau}^{0} \dot{x}(t+\alpha) d\alpha .$$

Then, from (7) we get

$$\dot{x}(t) = (\bar{A}(x) + A_d(x))x - A_d(x) \int_{-\tau}^{0} \big[\bar{A}\big(x(t+\alpha)\big)x(t+\alpha) - A_d\big(x(t+\alpha)\big)x(t+\alpha-\tau)\big] \, d\alpha. \tag{11}$$

Note that (11) with the following initial function:

$$x(t) = \psi(t), \ \forall \, t \in [-2d, 0] \tag{12}$$

is a nonlinear quadratic system with distributed time-delay which has the property that the trajectories of (7) are also trajectories of (11)-(12); see, e.g. Hale and Lunel [1993]. Hence, the local asymptotic stability of (11)-(12) will ensure the local asymptotic stability of (7). In this section, we will derive conditions for local asymptotic stability of the system (11)-(12) via the Razumikhin stability theorem in order to solve the problem of regional stabilization of the system (1). To this end, consider the following quadratic Lyapunov-Razumikhin function candidate:

$$V_R(x) = x'Px, \quad P > 0 \tag{13}$$

with $P \in \mathbb{R}^{n \times n}$ to be determined. In addition, the stability region we will consider is defined in terms of the following normalized level set of $V_R(x)$:

$$\begin{cases} \mathscr{R}_R = \{\phi \in \mathscr{C}_d^n : \phi(s) \in \mathscr{B}_R, \, \forall \, s \in [-d, 0]\}, \\ \mathscr{B}_R = \{x \in \mathbb{R}^n : x'Px \leq 1\}. \end{cases} \tag{14}$$

Before presenting the regional stabilization result we introduce the following notation, where $m = (n-1)n$:

$$\Psi_1(x) = \begin{bmatrix} 0_{n^2 \times n} & \Pi(x) & -I_{n^2} \\ 0_{m \times n} & 0_{m \times n} & \mathscr{N}(x) \end{bmatrix}, \tag{15}$$

$$\Psi_2(x) = \begin{bmatrix} \Pi(x) & -I_{n^2} \\ 0_{m \times n} & \mathscr{N}(x) \end{bmatrix}, \tag{16}$$

$$\mathscr{N}(x) = \begin{bmatrix} x_2 I_n & -x_1 I_n & 0_n & \cdots & 0_n \\ 0_n & x_3 I_n & -x_2 I_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & x_n I_n & -x_{(n-1)} I_n \end{bmatrix}. \tag{17}$$

Theorem 1. Consider the system defined in (1)-(3) and let $\mathscr{X}$ be a given polytopic region defined by either (5) or (6). Suppose there exist real matrices $F_0, F_1, \ldots, F_n$, L_1, L_2, P_1, P_2, and Q with appropriate dimensions and satisfying the following LMIs:

$$\Phi_1(v_i) + \mathrm{He}(L_1 \Psi_1(v_i)) > 0, \ i = 1, \ldots, \kappa, \tag{18}$$

$$\Phi_2(v_i) > 0, \ i = 1, \ldots, \kappa, \tag{19}$$

$$\begin{bmatrix} \mathscr{U}(Q, F(v_i), d) + \mathrm{He}\big(L_2 \Psi_2(v_i)\big) & d \, \mathscr{A}_d \mathscr{P} \\ d \, \mathscr{P} \mathscr{A}_d' & -d \, \mathscr{P} \end{bmatrix} < 0, \quad i = 1, \ldots, \kappa, \tag{20}$$

$$1 - c_i' Q c_i \geq 0, \quad i = 1, \ldots, n_f, \tag{21}$$

where

$$\Phi_1(x) = \begin{bmatrix} Q & \star & \star \\ A(x)Q + B_0 F(x) & P_1 & \star \\ B_x F(x) & 0 & 0 \end{bmatrix}, \tag{22}$$

$$\Phi_2(x) = \begin{bmatrix} Q & \star \\ A_d(x) Q & P_2 \end{bmatrix}, \quad F(x) = F_0 + \sum_{i=1}^{n} x_i F_i, \tag{23}$$

$$\mathscr{U}(Q, F(x), \beta) = \begin{bmatrix} \mathrm{He}\big(A_1(x) + B_0 F(x) + \beta Q\big) & \star \\ B_x F(x) & 0_{n^2} \end{bmatrix}, \tag{24}$$

$$A_1(x) = A(x) + A_d(x), \tag{25}$$

$$\mathscr{A}_d = \begin{bmatrix} A_d & A_d \end{bmatrix}, \quad \mathscr{P} = \mathrm{diag}\{P_1, P_2\}. \tag{26}$$

Then, the control law $u = F(x)Q^{-1}x$ ensures that the equilibrium solution $x \equiv 0$ of the controlled system is locally asymptotically stable in $\mathscr{X}$. In addition, $x_t \in \mathscr{R}_R$, $\forall \, t \geq 0$, and $\lim_{t\to\infty} x(t) = 0$, for any initial function $\phi \in \mathscr{R}_R$, where $\mathscr{R}_R$ is as in (14) with $P = Q^{-1}$. □

A proof of Theorem 1 is presented in Appendix A.

It is desirable to obtain a stability region as large as possible, i.e., the largest possible size $\mathscr{B}_R$ inside $\mathscr{X}$. Note that, since the volume of $\mathscr{B}_R$ is proportional to $\sqrt{\det(Q)}$, the maximization of the volume of $\mathscr{B}_R$ is a nonconvex problem. However, monotonic transformations such as $\log(\det(Q))$ can render this problem convex (see, e.g., Boyd *et al.* [1994]). Hence, the following LMI optimization problem is proposed to maximize the size of the stability region $\mathscr{R}_R$ for a given polytopic state-space domain $\mathscr{X}$ and an upper bound d on $\tau(t)$:

$$\min_{L_1, L_2, P_1, P_2, Q, F_0, \ldots, F_n} -\log\det\big(Q\big) \ \text{ subject to (18)-(21)} . \tag{27}$$

5. LYAPUNOV-KRASOVSKII APPROACH

In the sequel an LMI method of regional stabilization is developed based on the Lyapunov-Krasovskii stability theorem. For the sake of easier readability, the local stabilization and stability region results are separately presented.

5.1 Local Stabilization

Consider the following Lyapunov-Krasovskii functional (LKF) candidate for the closed-loop system in (7):

$$V_K(x_t) = V_1(x) + V_2(x_t) + V_3(x_t), \tag{28}$$

where

$$V_1(x) = x'P_1 x, \quad V_2(x_t) = \int_{t-\tau(t)}^{t} x(\alpha)' P_2 x(\alpha) d\alpha,$$

$$V_3(x_t) = \int_{-d}^{0} \int_{t+\beta}^{t} \dot{x}(\alpha)' P_3 \dot{x}(\alpha) d\alpha d\beta \tag{29}$$

with P_1, P_2 and P_3 being symmetric positive-definite matrices.

The time-derivative of $V_K(x_t)$ along the trajectory of (7) is as follows:

$$\dot{V}_K(x_t) = 2x'P_1\big(\bar{A}(x)x + A_d(x)\tilde{x}\big) + x'P_2x - (1-\dot{\tau})\tilde{x}'P_2\tilde{x} + d\big(\bar{A}(x)x + A_d(x)\tilde{x}\big)'P_3\big(\bar{A}(x)x + A_d(x)\tilde{x}\big) - \int_{t-d}^{t} \dot{x}(\alpha)'P_3\dot{x}(\alpha)d\alpha. \quad (30)$$

Introducing the following variable transformations:

$$\begin{cases} \xi_1(t) = P_1x(t), & \xi_2(t) = P_1\tilde{x}(t), \quad \xi_3(\alpha) = \tau P_1\dot{x}(\alpha), \\ Q_1 = P_1^{-1}, & Q_2 = Q_1P_2Q_1, \end{cases} \quad (31)$$

it follows that $\dot{V}_K(x_t)$ in (30) can be written as

$$\dot{V}_K(x_t) = \frac{1}{d}\int_{t-d}^{t} \hat{\xi}(\alpha)'\big[\Xi(x,\tau,\dot{\tau}) + d\mathscr{A}(x)'P_3\mathscr{A}(x)\big]\hat{\xi}(\alpha)d\alpha \quad (32)$$

where

$$\begin{cases} \hat{\xi}(\alpha) = \big[\ \xi_1(t)'\ \ \xi_2(t)'\ \ \xi_3(\alpha)'\ \big]', \\ \mathscr{A}(x) = \big[\bar{A}(x)Q_1\ \ A_d(x)Q_1\ \ 0_n\big], \end{cases} \quad (33)$$

and $\Xi(x,\tau,\dot{\tau}) = [\Xi_{i,j}]_{i,j=1,2,3}$ is a symmetric block matrix function whose nonzero blocks are given by:

$$\begin{cases} \Xi_{11} = \mathrm{He}\big(\bar{A}(x)Q_1\big) + Q_2, & \Xi_{21} = Q_1A_d(x)', \\ \Xi_{22} = -(1-\dot{\tau})Q_2, & \Xi_{33} = -(d/\tau^2)Q_1P_3Q_1. \end{cases} \quad (34)$$

Note that the vector $\hat{\xi}(\alpha)$ satisfies the equality constraint

$$\int_{t-\tau}^{t} Q_1\mathscr{I}\hat{\xi}(\alpha)d\alpha = 0, \quad \mathscr{I} = [\,I_n\ -I_n\ -I_n\,]. \quad (35)$$

As $1/(d\tau) \le 1/\tau^2$, $Q_1 > 0$ and $P_3 > 0$, it follows that

$$\frac{1}{d}\int_{t-d}^{t}\xi_3(\alpha)'\Xi_{33}\xi_3(\alpha)d\alpha \le -\frac{1}{\tau}\int_{t-\tau}^{t}\xi_3(\alpha)'\frac{Q_1P_3Q_1}{d}\xi_3(\alpha)d\alpha.$$

Now, considering that Ξ_{11}, Ξ_{21}, Ξ_{22} and $\mathscr{A}(x)$ are independent of α, $\dot{\tau} \le h$ and (32), it can be easily verified that the latter integral inequality implies the following:

$$\dot{V}_K(x_t) \le \frac{1}{\tau}\int_{t-\tau}^{t}\hat{\xi}(\alpha)'\big(\hat{\Xi}(x) + d\mathscr{A}'P_3\mathscr{A}\big)\hat{\xi}(\alpha)d\alpha, \quad (36)$$

where $\hat{\Xi}(x) := \Xi(x,d,h)$ and $\mathscr{A} := \mathscr{A}(x)$.

Next, in the light of (35) and (36), by Lemma 1 it follows that $\dot{V}_K(x_t)$ is negative definite if the next inequality holds:

$$\frac{1}{\tau}\int_{t-\tau}^{t}\hat{\xi}(\alpha)'\big[\hat{\Xi}(x) + d\mathscr{A}'P_3\mathscr{A} + \mathrm{He}\big(R(x)\mathscr{I}\big)\big]\hat{\xi}(\alpha)d\alpha < 0, \quad \forall\, x_t \not\equiv 0, \quad (37)$$

for a free multiplier $R(x) = \big[\,R_1(x)'\ \ R_2(x)'\ \ R_3(x)'\,\big]'$, where

$$R_j(x) = R_{j0} + \sum_{i=1}^{n} x_iR_{ji}, \quad R_{ji} \in \mathbb{R}^{n\times n}, \quad j = 1,2,3. \quad (38)$$

In view of (37), in order to present the stabilization result of this section we introduce the following block matrix functions:

$$\Lambda(x) = [\Lambda_{ij}]_{i,j=1,\ldots,6}, \quad \Omega(x) = [\Omega_{ij}]_{i=1,\ldots,4,\,j=1,\ldots,6}$$

with $\Lambda(x)$ being symmetric and where the nonzero blocks of $\Lambda(x)$ and $\Omega(x)$ are as follows:

$$\Lambda_{11} = \mathrm{He}\big(A(x)Q_1 + B_0Y(x) + R_1(x)\big) + Q_2,$$
$$\Lambda_{21} = \Lambda_{61} = B_xY(x),\ \ \Lambda_{31} = Q_1A_d(x)' + R_2(x) - R_1(x)',$$
$$\Lambda_{33} = (h-1)Q_2 - \mathrm{He}\big(R_2(x)\big),\ \ \Lambda_{41} = R_3(x) - R_1(x)',$$
$$\Lambda_{43} = -R_3(x) - R_2(x)',\ \ \Lambda_{44} = -(\sigma d)^{-1}Q_1 - \mathrm{He}\big(R_3(x)\big),$$
$$\Lambda_{51} = A(x)Q_1 + B_0Y(x),\ \ \Lambda_{53} = A_d(x)Q_1,\ \ \Lambda_{55} = -(\sigma/d)Q_1,$$
$$\Omega_{11} = \Omega_{35} = \Pi(x),\ \ \Omega_{12} = \Omega_{36} = -I_{n^2},$$
$$\Omega_{22} = \Omega_{46} = \mathscr{N}(x),$$

where σ is a given positive scalar, $\mathscr{N}(x)$ is as in (17) and

$$Y(x) = Y_0 + \sum_{i=1}^{n} x_iY_i, \quad Y_i \in \mathbb{R}^{n_u\times n}. \quad (39)$$

Theorem 2. Consider the system in (1)-(3) and let $\mathscr{X}$ be a given polytopic region defined by either (5) or (6). Suppose that for a given scalar $\sigma > 0$ there exist real matrices $Q_1 > 0$, $Q_2 > 0$, Y_j, R_{1j}, R_{2j} and R_{3j} for $j = 0,1,\ldots,n$, and M, satisfying the following LMIs:

$$\Lambda(v_i) + \mathrm{He}\big(M\Omega(v_i)\big) < 0, \quad i = 1,\ldots,\kappa. \quad (40)$$

Then, the control law $u = Y(x)Q_1^{-1}x$, with $Y(x)$ as defined in (39), ensures that the equilibrium solution $x \equiv 0$ of the controlled system is locally asymptotically stable in $\mathscr{X}$.

Proof. Firstly, it follows that $V_K(x_t)$ in (28) satisfies

$$\varepsilon_1\|x(t)\|^2 \le V_R(x_t) \le \varepsilon_2\|x_t\|_d^2$$

for some positive scalars ε_1 and ε_2. Secondly, by convexity, (40) holds for all $x \in \mathscr{X}$, and thus consider (40) with $v_i = x$, $\forall\, x \in \mathscr{X}$. Post- and pre-multiplying this inequality by respectively $\hat{\Upsilon}(x) = \mathrm{diag}\{\,\Upsilon(x), I_n, I_n, \Upsilon(x)\,\}$ and $\hat{\Upsilon}(x)'$, where $\Upsilon(x) = \big[\,I_n\ \ \Pi(x)'\,\big]'$, and considering that $\Omega(x)\hat{\Upsilon}(x) = 0$ and $Y(x) = K(x)Q_1$, leads to

$$\begin{bmatrix} \mathscr{Z}(x) + \mathrm{He}\big(R(x)\mathscr{I}\big) & \mathscr{A}(x)' \\ \mathscr{A}(x) & -(\sigma/d)Q_1 \end{bmatrix} < 0, \ \ \forall\, x \in \mathscr{X}, \quad (41)$$

where $\mathscr{Z}(x)$ is equal to $\hat{\Xi}(x)$ with $P_3 = P_1/\sigma$.

Applying Schur's complement, (41) is equivalent to

$$\mathscr{Z}(x) + d\mathscr{A}(x)'(P_1/\sigma)\mathscr{A}(x) + \mathrm{He}\big(R(x)\mathscr{I}\big) < 0, \ \ \forall\, x \in \mathscr{X},$$

which in light of (37) implies that $\dot{V}_k(x_t)$ is negative definite. Thus, by Lemma 4 the controlled system is locally asymptotically stable in $\mathscr{X}$. $\nabla\nabla\nabla$

Observe that in Theorem 2 the matrix P_3 in (29) is constrained to be equal to P_1/σ in order to obtain a set of state-dependent LMIs in (40), where σ is a positive scalar to be chosen. This relaxation may be conservative for stability analysis, but the extra degree of freedom introduced by the control design may overcome this problem. In addition, for reasons to be clarified in the next subsection, the parametrization $P_3 = P_1/\sigma$ shows to be advantageous for maximizing the stability region.

5.2 *Stability Region*

Assuming that the conditions in Theorem 2 hold and considering Lemma 4, then the set $\mathscr{R}_K(\gamma) = \{\,\phi \in \mathscr{C}_d^n : V_K(\phi) \le \gamma\}$ with $\gamma > 0$ such that $\mathbb{B}_K(\gamma) = \{\,\phi(s) \in \mathbb{R}^n,\ s \in [-d,0] : \phi \in \mathscr{R}_K(\gamma)\,\} \subset \mathscr{X}$ and $V_K(x)$ as in (28) is a contractive positively invariant set. However, verifying the condition $\mathbb{B}_K(\gamma) \subset \mathscr{X}$ is numerically hard. To overcome this difficulty, a bounding set $\mathscr{B}_1(\gamma)$ of $\mathbb{B}_K(\gamma)$ is introduced such that the condition $\mathscr{B}_1(\gamma) \subset \mathscr{X}$ can be verified in terms of LMIs, and for that we consider

$$\mathscr{B}_1(\gamma) := \{x \in \mathbb{R}^n : V_1(x) \le \gamma\} \quad (42)$$

with $V_1(x)$ as defined in (29). Note that $V_1(x(t)) \le V_K(x_t)$ for all $x_t \in \mathscr{C}_d^n$ and thus $\mathbb{B}_K(\gamma) \subseteq \mathscr{B}_1(\gamma)$ holds if $V_K(x_t) \le \gamma$. To this end, we introduce a matrix W to be determined and the constraints as follow:

$$x'P_2x \le x'Wx \le 1, \;\; \dot{x}(\alpha)'P_3\dot{x}(\alpha) = \dot{x}(\alpha)'(\sigma Q_1)^{-1}\dot{x}(\alpha) \le 1 .$$

Hence, taking into account (28) and (29), it follows that

$$V_K(x_t) \le x(t)'Q_1^{-1}x(t) + d + 0.5d^2. \tag{43}$$

Thus, the condition $\mathbb{B}_K(\gamma) \subseteq \mathscr{B}_1(\gamma)$ is guaranteed if

$$x'Q_1^{-1}x + d + 0.5d^2 \le \gamma, \;\forall\, x \in \mathbb{R}^n : x'Wx \le 1 . \tag{44}$$

By the $\mathscr{S}$-procedure, (44) holds if

$$\gamma - d - 0.5d^2 - 1 \ge 0, \quad W - Q_1^{-1} \ge 0 . \tag{45}$$

On the other hand, as $V_1(x) = x'P_1x$, the inclusion $\mathscr{B}_1(\gamma) \subset \mathscr{X}$ is equivalent to (see, e.g., Boyd *et al.* [1994]):

$$1 - c_i'(\gamma Q_1)c_i \ge 0, \;\; i = 1, \ldots, n_f. \tag{46}$$

In light of the previous arguments, we define the stability region for the Lyapunov-Krasovskii approach as follows:

$$\begin{cases} \mathscr{R}_K = \{\, \phi \in \mathscr{C}_d^n : \phi(s) \in \mathscr{B}_K, \forall s \in [-\tau, 0],\ \dot{\phi} \in \mathscr{D}_K \,\}, \\ \mathscr{B}_K = \{\, x \in \mathbb{R}^n : x'Wx \le 1 \,\}, \\ \mathscr{D}_K = \{\, \varphi \in \mathscr{C}_d^n : \|\varphi\|_d^2 \le \lambda \,\}, \;\; \lambda = \sigma\lambda_1, \end{cases} \tag{47}$$

with λ_1 being the smallest eigenvalue of Q_1, and subject to (45), (46) and $W - P_2 \ge 0$. Indeed, it is easy to verify that $x_t \in \mathscr{R}_K$, $\forall t \ge 0$, and $\lim_{t\to\infty} x(t) = 0$ for any initial function $\phi \in \mathscr{R}_K$.

Defining $G = Q_1WQ_1$ and $\rho = \gamma^{-1}$, and considering that $Q_2 = Q_1P_2Q_1$, the following optimization problem is proposed to approximately jointly maximize the sets $\mathscr{B}_K$ and $\mathscr{D}_K$:

$$\begin{aligned} &\min_{\substack{\rho, G, Q_1, Q_2, M, Y_0, \ldots, Y_n \\ R_{1i}, R_{2i}, R_{3i}, i = 1, \ldots, n}} \operatorname{trace}(G) - \log\det(Q_1) : \;\; \text{s.t. } (40), \\ &Q_i \ge 0, \; i = 1,2, \;\; \rho - c_i'Q_1c_i \ge 0, \;\; i = 1, \ldots, n_f, \\ &1 - \rho(d + 0.5d^2) - \rho \ge 0, \;\; G - Q_i \ge 0, \;\; i = 1,2 . \end{aligned} \tag{48}$$

The achieved stability region is as in (47) with $W = Q_1^{-1}GQ_1^{-1}$.

Remark 1. We have considered the LKF in (28) in order to achieve a numerically tractable local stabilization method with a guaranteed stability region. It is not clear if the use of a more complex LKF will allow for deriving a less conservative numerically tractable stabilization method.

Remark 2. Notice that the optimization problem (48) is not jointly convex in σ and Q_1. However, for a given σ the matrix inequalities in (40) become LMIs. Thus, a direct LMI-based approach to solve (48) is to perform a line search on $\sigma > 0$.

Remark 3. It turns out that the stability region in (47) depends not only on the delay upper bound d, but also on $\|\dot{\phi}\|_d^2$. Notice from the optimization problem in (48) that we approximately maximize the smallest eigenvalue of Q_1 and minimize $\operatorname{trace}(G)$. Since $W = Q_1^{-1}GQ_1^{-1}$, we approximately maximize the size of $\mathscr{B}_K$. In addition, $\mathscr{D}_K$ is also being maximized, which is a desirable feature.

6. AN EXAMPLE

Consider an open-loop unstable quadratic time-delay system as in (1) with the following matrices:

$$\begin{aligned} A(x) &= \begin{bmatrix} 0.2x_1 & 0 \\ 0 & 1 + 0.2x_2 \end{bmatrix}, \;\; B(x) = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ A_d(x) &= \begin{bmatrix} -1 + 0.2x_1 & -1 \\ 0 & -0.9 + 0.2x_2 \end{bmatrix}, \end{aligned} \tag{49}$$

and $\tau(t)$ being a time-varying delay satisfying $0 < \tau(t) \le d$ and $\dot{\tau}(t) \le h$, $\forall\, t \ge 0$.

In this example, the objective is to design a quadratic control law as in (4) which guarantees the local asymptotic stability of the closed-loop system while maximizing the stability region of the equilibrium point $x \equiv 0$. It should be emphasized that the pair $\big(A(0), B\big)$ is not stabilizable and thus the results proposed in de Souza and Coutinho [2012] cannot be applied.

In the following, we apply the Razumikhin and the Lyapunov-Krasovskii approaches of Theorems 1 and 2, respectively, to derive a stabilizing quadratic state feedback controller for the system (1) with (49) and

$$d = 0.3, \;\; h = 1, \;\; \mathscr{X} = \{ x \in \mathbb{R}^2 : |x_i| \le 1, \; i = 1,2 \}.$$

Fig. 1 shows the sets $\mathscr{B}_R$ and $\mathscr{B}_K$ of the stability regions $\mathscr{R}_R$ and $\mathscr{R}_K$, respectively, given in (14) and (47), obtained by Theorems 1 and 2 for the above setup, where σ in Theorem 2 has been chosen to enlarge the size of the set $\mathscr{D}_K$ as defined in (47). In particular, we have obtained $\mathscr{D}_K = \{\, \dot{\phi} \in \mathscr{C}_{0.3}^2 : \|\dot{\phi}(t)\|_{0.3}^2 \le 1.5 \,\}$.

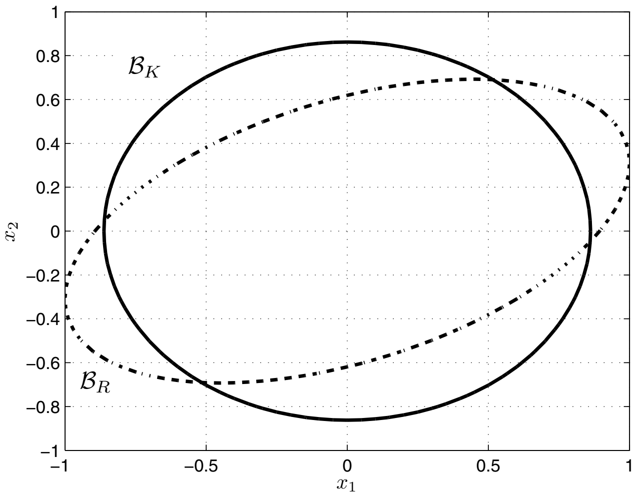

Fig. 1. Sets $\mathscr{B}_R$ and $\mathscr{B}_K$ of stability regions $\mathscr{R}_R$ and $\mathscr{R}_K$.

7. CONCLUSION

This paper has dealt with the problem of state feedback local stabilization of, possibly open-loop unstable, nonlinear quadratic state-delayed systems with a time-varying delay. In particular, we have proposed LMI methods of delay-dependent local stabilization via static nonlinear quadratic state-feedback based on either the Razumikhin or the Lyapunov-Krasovskii stability theorems that provide a maximized stability region for the closed-loop system. It turns out that the stability region obtained via the Lyapunov-Krasovskii approach is dependent on the maximal magnitude of the initial function time-derivative, which may be restrictive in some practical applications.

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Appendix A. PROOF OF THEOREM 1

Firstly, note that the time-derivative of $V_R(x)$ in (13) along the solution of (11) is given by

$$\dot V_R(x) = 2x'P\big(\bar A(x)+A_d(x)\big)x - 2x'PA_d(x)\cdot \int_{-\tau}^{0}\big[\bar A(x(t+\alpha))x(t+\alpha)A_d(x(t+\alpha))x(t+\alpha-\tau)\big]d\alpha. \tag{A.1}$$

Applying Lemma 2 to the two terms of the above integral, it follows that for any $n\times n$ matrices $P_1>0$ and $P_2>0$, we get:

$$\begin{aligned}\dot V_R(x) \le{}& 2x'P\big(\bar A(x)+A_d(x)\big)x + \int_{-\tau}^{0} x'PA_d(x)(P_1+P_2)A_d(x)'Px\,d\alpha\\ &+\int_{-\tau}^{0}\big[x(t+\alpha)'\bar A(x(t+\alpha))'P_1^{-1}\bar A(x(t+\alpha))x(t+\alpha)\\ &+x(t+\alpha-\tau)'A_d(x(t+\alpha))'P_2^{-1}\cdot A_d(x(t+\alpha))x(t+\alpha-\tau)\big]d\alpha\\ ={}& 2x'P\big[\bar A(x)+A_d(x)+0.5\tau A_d(x)(P_1+P_2)A_d(x)'P\big]x\\ &+\int_{-\tau}^{0}\big[x(t+\alpha)'\bar A(x(t+\alpha))'P_1^{-1}\bar A(x(t+\alpha))x(t+\alpha)\\ &+x(t+\alpha-\tau)'A_d(x(t+\alpha))'P_2^{-1}\cdot A_d(x(t+\alpha))x(t+\alpha-\tau)\big]d\alpha \end{aligned}\tag{A.2}$$

We will now show that the conditions in Theorem 1 imply that

$$\bar A(x)'P_1^{-1}\bar A(x) < P,\ \ \forall\, x\in\mathscr{X}, \tag{A.3}$$

$$A_d(x)'P_2^{-1}A_d(x) < P,\ \ \forall\, x\in\mathscr{X}. \tag{A.4}$$

To this end, define

$$\begin{cases} F(x) := F_0 + x_1F_1 + \ldots + x_nF_n = K(x)Q,\\ Q = P^{-1},\end{cases} \tag{A.5}$$

where $F_0, F_1, \ldots, F_n$ are $n_u\times n$ real matrices to be found. Note that this leads to the controller gain parameterization $K(x) = F(x)Q^{-1}$. Then, it can be verified that (A.3) is equivalent to

$$\begin{cases} \zeta_1'\Phi_1(x)\zeta_1 > 0,\ \forall\ \zeta_1 = \big[\ \zeta_{11}'\ \ \zeta_{12}'\ \ (\Pi(x)\zeta_{12})\big]' \neq 0,\\ \qquad\qquad \big[\ \zeta_{11}'\ \ \zeta_{12}'\big]' \in \mathbb{R}^{2n},\ \ \forall\, x\in\mathscr{X}\end{cases} \tag{A.6}$$

where $\Pi(x)$ and $\Phi_1(x)$ are as defined in (9) and (22), respectively. Since the vector ζ_1 is such that $\Psi_1(x)\zeta_1=0$, where $\Psi_1(x)$ is given in (15), by Lemma 1 it follows that (18) guarantees that (A.6), or equivalently (A.3), is satisfied.

Next, by Schur's complements, (A.4) is equivalent to

$$\Phi_2(x) > 0,\ \ \forall\, x\in\mathscr{X}$$

with $\Phi_2(x)$ as in (23), which is ensured to hold by (19).

On the other hand, in light of Lemma 3 assume that for some real number $\delta > 1$,

$$V_R(x(s)) < \delta V_R(x(t)),\ \forall\ s\in[t-2d,t]. \tag{A.7}$$

Considering (A.3), (A.4) and (A.7), the following can be obtained from (A.2):

$$\begin{aligned}\dot V_R(x) < x'\big[&2P\big(\bar A(x)+A_d(x)\big) + 2\delta\,\tau P\\ &+\tau PA_d(x)(P_1+P_2)A_d(x)'P\big]x.\end{aligned}$$

Introducing the variable transformation $\xi = Px$ and considering (A.5), the latter inequality can be written as:

$$\dot V_R(x) < \zeta_2'\big[\mathscr{U}(Q,F(x),\delta\tau) + \tau\mathscr{A}_d(x)\mathscr{P}\mathscr{A}_d(x)'\big]\zeta_2, \tag{A.8}$$

where $\zeta_2 = \big[\ I_n\ \ \Pi(x)'\ \big]'\xi$, $\mathscr{U}(\cdot)$ is given in (24), and $\mathscr{A}_d(x)$ and $\mathscr{P}$ are given in (26).

Since the vector ζ_2 is such that $\Psi_2(x)\zeta_2=0$, with the matrix $\Psi_2(x)$ as defined in (16), by Lemma 1 it follows from (A.8) that $\dot V_R(x) < 0$ for all nonzero $x\in\mathscr{X}$ if

$$\mathscr{U}(Q,F(x),\delta\tau) + \tau\mathscr{A}_d(x)\mathscr{P}\mathscr{A}_d(x)' + \mathrm{He}\big(L_2\Psi_2(x)\big) < 0 \tag{A.9}$$

over $\mathscr{X}$ for some matrix L_2 of appropriate dimensions.

Considering that $\tau(t)\le d$ and since (20) is strict, (20) implies that there exists a small scalar $\delta > 1$ such that (A.9) is satisfied.

Finally, (21) is equivalent to $\mathscr{B}_R \subset \mathscr{X}$ (see, e.g. Boyd *et al.* [1994]). Hence, by Lemma 3 it follows that Theorem 1 holds.