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# State Estimation Based on Self-Triggered Measurements

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**Abstract:** In this work, the problem of state estimation for nonlinear continuous-time systems from discrete data is tackled in a bounded error context. One assumes that all poorly-known system variables belong to a bounded set with known bounds. Then, a self-triggered algorithm is proposed to improve the performance of the classical set-membership state estimator based on the prediction-correction procedures. In order to cope with pessimism propagation linked to the bounding methods, this algorithm triggers the correction step whenever the size of a part of the estimated state enclosure becomes greater than a time-converging threshold a priori defined by the user. The effectiveness of the proposed self-triggered algorithm is illustrated through numerical simulations.

Keywords: Nonlinear systems; state estimation; bounded-error framework; interval analysis; monotone systems; bounding methods.

## 1. INTRODUCTION

State estimation is an important field of control system theory (Luenberger [1971]),(Isidori [1995]). As a matter of fact, several advanced control and diagnosis approaches are developed under the assumption that the state vector of the continuous-time system is available online. To satisfy this requirement, software sensors called observers are developed to estimate in real-time the state vector (Luenberger [1971]). For example, for linear continuous-time systems one can use the standard Luenberger observer (Luenberger [1971]) or the Kalman filter (Kalman [1960]). Differently, for nonlinear systems, there are different approaches to design nonlinear observers. For instance, one can cite the extended Luenberger observer (Sorenson [1985]), the extended Kalman filter (Misawa and Hedrick [1989]), the high gain observer (Gauthier et al. [1992]) or the sliding mode observer (Drakunov [1983], Slotine et al. [1986]. . . ). All these observer design approaches assume that the model of the real system is perfectly known and the measurements are available in continuous-time. In practice, these assumptions are problematic, especially when dealing with biological or biotechnological systems, because the system parameters are poorly-known and the measurements are generally done in discrete time.

To circumvent this problem, prediction-correction set-membership state estimators are developed during these last years (Jaulin [2002], Raïssi et al. [2004], Goffaux et al. [2009], Meslem et al. [2010a]). This kind of estimators are designed in the unknown but bounded error context (Milanese et al. [1996]). They estimate from discrete data an accurate enclosure of the state flow generated by an uncertain system, where all the uncertain variables are represented by boxes (interval vectors) (Moore [1966], Jaulin et al. [2001]). The main contribution of this work consists in endowing the set-membership state estimator by a self-triggered algorithm in order to apply efficiently the correction procedure. In fact, with this algorithm one can master the propagation of pessimism generated by the bounding methods (Kieffer and Walter [2006], Ramdani et al. [2009], Ramdani et al. [2010]). This algorithm is inspired from the event-triggered control strategy applied to continuous-time systems (Meslem and Prieur [2013]).

Note that, to the best of our knowledge, the convergence issue of the state estimation error in the bounded error context is still not well investigated, in particular when applying the prediction-correction algorithm (Jaulin [2002], Raïssi et al. [2004], Goffaux et al. [2009], Meslem et al. [2010a]). In this work, due to this self-triggered algorithm, the proof of the convergence of the state estimation error is provided under some assumptions. This convergence analysis shows more the importance of our findings.

This paper is organized as follows. In Section 2, basic notions about interval computations are introduced. Then, the core idea of the classical prediction-correction state estimator is recalled in Section 3. The main results of this work are stated and proved in Section 4. An illustrative example is given in Section 5 with several simulation tests. Also, a comparative study with the results obtained by an interval observer is presented and commented in this section. Finally, some concluding remarks and future works are discussed.

## 2. PRELIMINARY NOTIONS ABOUT INTERVAL COMPUTATION

Interval analysis was initially developed to account for the quantification errors introduced by the floating point representation of real numbers with computers and was extended to reliable computations (Moore [1966], Neumaier [1990]). Denote by $[x] = [\underline{x}, \overline{x}]$ a real interval which is a connected and closed subset of $\mathbb{R}$ where the real numbers $\underline{x}$ and $\overline{x}$ are respectively the lower and the upper bound of $[x]$. So, the set of all real intervals of $\mathbb{R}$ is denoted by $\mathbb{IR}$. Over $\mathbb{IR}$ an interval arithmetic was built by an extension of the real arithmetic operations. That means, for each operator $\circ \in \{+, -, \times, \div\}$ and for each couple of intervals $[x]$ and $[y]$ one defines

$$[x] \circ [y] = \{a \circ b \mid a \in [x],\ b \in [y]\} \tag{1}$$

The width of an interval $[x]$ is defined by $w([x]) = \overline{x} - \underline{x}$. As well, an interval vector or box denoted by $[\mathbf{x}]$ is a subset of $\mathbb{R}^n$ defined as the Cartesian product of $n$ closed intervals. The set of all interval vectors of order $n$ will be denoted by $\mathbb{IR}^n$. The width of an interval vector of dimension $n$ is defined by

$$w([\mathbf{x}]) = \max_{1 \le i \le n} w([x_i])$$

Likewise, we define the vector width of an interval vector by

$$\mathbf{w}_v\big([\mathbf{x}]\big) = \big(w([x_1]), w([x_2]), \ldots, w([x_n])\big)^T$$

That is, the components of the real vector $\mathbf{w}_v$ are the widths of each component of the interval vector $[\mathbf{x}]$.

Now, one can describe uncertain parameters by an upper and lower bound, then rigorous bounds on the range of a real function of these parameters are computed using interval arithmetic. Consider the real function $f : \mathbb{R}^n \to \mathbb{R}$. The range of this function over an interval vector $[\mathbf{x}]$ is given by:

$$f([\mathbf{x}]) = \{f(\mathbf{a}) \mid \mathbf{a} \in [\mathbf{x}]\} \tag{2}$$

Then, one calls an inclusion function denoted by $[f]$ for the real function $f$ an interval application that satisfies the following inclusion

$$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ f([\mathbf{x}]) \subset [f]([\mathbf{x}]) \tag{3}$$

In practice, the simplest manner to obtain an inclusion function $[f]$ for real function $f$ consists in replacing each occurrence of a real variable by the corresponding interval and each standard function by its interval counterpart. The resulting function is called the natural inclusion function and the tightness of the enclosure provided by $[f]$ depends on the formal expression of $f$. In fact, it is well known if the same variable $x_i$ has many occurrences in the mathematical expression of $f$, the *dependence effect* (Moore [1966], Jaulin et al. [2001]) will induce pessimism while computing an enclosure of the range of the real function. Hence, formal pre-processing of the function expression is advisable in order to minimize the number of variable occurrences.

In the sequel, we will show how the joint use of interval computation and the bounding methods for computing rigorous bounds on the reachable set of uncertain nonlinear systems, allows to solve in guaranteed way the state estimation problem for uncertain nonlinear systems from discrete data.

## 3. SET-MEMBERSHIP STATE ESTIMATION

### *3.1 Prediction-Correction state estimator*

In this section, we recall briefly the core idea of the classical prediction-correction state estimator of nonlinear continuous-time systems, which are described by

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{u}) \\ \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{p}, \mathbf{u}) \end{cases} \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector to be estimated from discrete data. The vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^p$ stand for respectively the input and the output of the system. The vector field $\mathbf{f}$ and the output model $\mathbf{g}$ can be linear or nonlinear functions of the state and input with appropriate dimensions. The initial state $\mathbf{x}_0$ and the parameter vector $\mathbf{p} \in \mathbb{R}^{n_p}$ are assumed unknown but bounded with known bounds. That means,

$$\mathbf{x_0} \in [\underline{\mathbf{x}}_0, \overline{\mathbf{x}}_0], \quad \mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}]$$

where $\overline{\mathbf{x}}_0$, $\underline{\mathbf{x}}_0$, $\overline{\mathbf{p}}$ and $\underline{\mathbf{p}}$ are respectively the known upper and lower bound of the initial state and the parameter vector $\mathbf{p}$. Experimental data $\mathbf{y}_j$ are collected at discrete times $t_j,\ j \in \{1, \ldots, N\}$. And the feasible domain for the output values at each time $t_j$ is given by

$$[\mathbf{y}_j] = \mathbf{y}_j + [\mathbf{e}_j] \tag{5}$$

where the box $[\mathbf{e}_j]$ denotes the feasible domain for output error at time $t_j$, which includes both deterministic and random error.

*The prediction stage (**Pred**):* In this context the prediction procedure has to compute an outer enclosure of all possible state trajectories generated by the uncertain system (4) between two measurement time instants $t_j$ and $t_{j+1}$. To accomplish this task, one can use either the validated methods for initial value problems for ordinary differential equations (Rihm [1994], Nedialkov et al. [2001]) based on interval analysis or the hybrid bounding methods (Ramdani et al. [2009], Ramdani et al. [2010]) based on the comparison theorems of differential inequalities (Müller [1926], Marcelli and Rubbioni [1997], Smith [1995]). In this work, the bounding methods are used to carry out the prediction stage. So, an outer enclosure of the state trajectories generated by (4) over a time interval $[t_j, t_{j+1}]$ is obtained by integrating the following bounding system

$$\begin{cases} \dot{\overline{\mathbf{x}}} = \overline{\mathbf{f}}(\overline{\mathbf{x}}, \underline{\mathbf{x}}, \overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{u}),\ \overline{\mathbf{x}}(t_j) = \overline{\mathbf{x}}_j, \\ \dot{\underline{\mathbf{x}}} = \underline{\mathbf{f}}(\overline{\mathbf{x}}, \underline{\mathbf{x}}, \overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{u}),\ \underline{\mathbf{x}}(t_j) = \underline{\mathbf{x}}_j, \end{cases} \tag{6}$$

where the vector functions $\overline{\mathbf{f}}$, $\underline{\mathbf{f}}$ are built in order to frame the field vector $\mathbf{f}$ for all $\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and for all $\mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}]$. To get more explanations about the construction of the bounding system (6) the reader can refer to Ramdani et al. [2009], Ramdani et al. [2010] and references therein. Thus, one can claim that all possible state trajectories generated by the uncertain system (4) are framed by the solution of the deterministic system (6). That means,

$$\begin{aligned} &\mathbf{x}(t_j) \in [\underline{\mathbf{x}}(t_j), \overline{\mathbf{x}}(t_j)], \mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}] \\ &\qquad \Rightarrow \forall t \in [t_j, t_{j+1}], \mathbf{x}(t) \in [\underline{\mathbf{x}}(t), \overline{\mathbf{x}}(t)] \end{aligned} \tag{7}$$

To sum up, the prediction stage computes an outer enclosure, here denoted by $[\mathbf{x}(t)]^p$, of the all state trajectories generated by the system (4) on the period $[t_j, t_{j+1}]$.

*The correction stage (**Corr**):* At each measurement time instant $t_j$, an other outer enclosure of the state vector denoted by $[\mathbf{x}(t_j)]^{inv}$ is computed now by solving the following set inversion problem

$$[\mathbf{x}(t_j)]^{inv} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{g}(\mathbf{x}, \mathbf{p}, \mathbf{u}) \in [\mathbf{y}(t_j)]\} \tag{8}$$

Then, the inconsistent state vectors belonging to the two outer enclosures are discarded as follows

$$[\mathbf{x}(t_j)]^c = [\mathbf{x}(t_j)]^{inv} \cap [\mathbf{x}(t_j)]^p \tag{9}$$

Now, in order to improve the accuracy of the state enclosure, the next step of the prediction-correction procedure is initialized with the corrected state enclosure $[\mathbf{x}(t_j)]^c$. It is worth pointing out that to solve the set inversion problem (8), one can either use an advanced version of the **SIVIA** algorithm based on interval analysis or an adequate interval **Contractor**. For more details about these consistency techniques and their implementation the reader can refer to Jaulin et al. [2001] and references therein.

The below algorithm synthesizes the operating principle of this kind of the Set-Membership State Estimators (**SM-SE**).

**Algorithm**: **SM-SE**$([\mathbf{x}_0], [\mathbf{p}], \mathbf{f}, \mathbf{g}, t_0, \ldots, t_N,)$

- **For** $j = 0$ to $j = N - 1$
  - $[\mathbf{x}(t)]^p := \mathbf{Pred}([\mathbf{x}_0], [\mathbf{p}], \mathbf{f}, [t_j, t_{j+1}])$
  - $[\mathbf{y}_{j+1}] := \mathbf{y}_{j+1} + [\mathbf{e}_{j+1}]$
  - $[\mathbf{x}(t_{j+1})]^c := \mathbf{Corr}([\mathbf{x}(t_{j+1})]^p, [\mathbf{y}_{j+1}], [\mathbf{p}], \mathbf{g})$
  - $[\mathbf{x}_0] := [\mathbf{x}(t_{j+1})]^c$
- **End**

However, to the best of our knowledge, no convergence analysis of the estimation error of the type of estimators is done up to now, see for example Jaulin [2002], Raïssi et al. [2004], Goffaux et al. [2009], Meslem et al. [2010a]. Thus, the main contribution of this paper concerns this matter. In fact, we propose to endow the correction stage by a self-triggered algorithm in order to control the pessimism propagation and so to improve the accuracy of the estimate state enclosure. Moreover, under some assumptions, a convergence analysis of the estimation error is presented in the next section.

## 4. SELF-TRIGGERED ALGORITHM TO SCHEDULE THE CORRECTION STAGE

Let us start by introducing the assumptions needed to prove the convergence of the state estimation error, then we state the main contributions of this work (new **SM-SE** algorithm, convergence analysis).

*Hypothesis 1. The state equation of* (4) *can be divided as follows*

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}, \mathbf{u}) \quad (10)$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, \mathbf{x}_1, \mathbf{p}, \mathbf{u}) \quad (11)$$

*where* $\mathbf{x}_1 \in \mathbb{R}^{n_1}, \mathbf{x}_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$ *and* $(\mathbf{f}_1, \mathbf{f}_2)^T = \mathbf{f}$. *Then, we assume*

- *the subsystem* (11) *is monotone stable over a given domain* $\mathbb{D} \subset \mathbb{R}^{n_1+n_2+n_p+m}$. *For more details about monotone systems, the reader can refer to Smith [1995], Angeli and Sontag [2003].*
- *all state variables of the vector* $\mathbf{x}_1$ *and all parameters of the vector* $\mathbf{p}$ *act positively or negatively on all the state variables of the vector* $\mathbf{x}_2$.

*Hypothesis 2. There exists a positive time-decreasing threshold* $\xi$ *such that: at each measurement time instant* $t_j > 0$ *the following inequality is satisfied*

$$w([\mathbf{x}_1(t_j)]^{inv}) < \xi(t_j) \quad (12)$$

*That is, the width of the first part of the enclosure of the state vector* $[\mathbf{x}_1(t_j)]^{inv}$*, obtained via a set inversion of the feasible domain of measurement* (5) *according to the output model* $\mathbf{g}$*, must be lower than an instrumental threshold a priori defined by the user.*

These hypotheses are satisfied by a large class of biological systems (Sontag [2005], Angeli and Sontag [2008]), thermal systems (Meslem et al. [2010b]), biotechnological systems (Hadj-Sadok and Gouzé [2001], Moisan et al. [2009]) and all monotone systems closed by a negative feedback (Angeli and Sontag [2003]).

So, under Hypothesis 1, applying the bounding methods (Ramdani et al. [2009], Ramdani et al. [2010]) to ((10), (11)), we obtain the following bracketing systems to carry out the prediction stage.

$$\begin{cases} \dot{\overline{\mathbf{x}}}_1 = \overline{\mathbf{f}}_1(\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2, \underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{u}) \\ \dot{\underline{\mathbf{x}}}_1 = \underline{\mathbf{f}}_1(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2, \overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{u}) \end{cases} \quad (13)$$

$$\begin{cases} \dot{\overline{\mathbf{x}}}_2 = \overline{\mathbf{f}}_2(\overline{\mathbf{x}}_2, \mathbf{x}_1^+, \mathbf{p}^+, \mathbf{u}) \\ \dot{\underline{\mathbf{x}}}_2 = \underline{\mathbf{f}}_2(\underline{\mathbf{x}}_2, \mathbf{x}_1^-, \mathbf{p}^-, \mathbf{u}) \end{cases} \quad (14)$$

where the vector $\mathbf{x}_1^+$ is either equal to the upper bound $\overline{\mathbf{x}}_1$ or to the lower bound $\underline{\mathbf{x}}_1$ (idem for $\mathbf{x}_1^-$). Also the vector $\mathbf{p}^+$ is either equal to the upper bound $\overline{\mathbf{p}}$ or to the lower one $\underline{\mathbf{p}}$ (idem for $\mathbf{p}^-$).

We can now introduce the first contribution of this paper, namely the new Set-Membership State Estimation algorithm where the correction stage is managed by a Self-Triggered procedure (**ST-SM-SE**).

**Algorithm**: **ST-SM-SE**$([\mathbf{x}_0], [\mathbf{p}], \mathbf{f}, \mathbf{g}, h_0, \ldots, h_k, \xi)$

- **For** $j = 0$ to $j = k - 1$
  - $t_{j+1} := t_j + h_j$
  - $[\mathbf{x}(t)]^p := \mathbf{Pred}([\mathbf{x}_0], [\mathbf{p}], \mathbf{f}, [t_j, t_{j+1}])$
  - $[\mathbf{x}_0] := [\mathbf{x}(t_{j+1})]^p$
  - **If** $w([\mathbf{x}_1(t_{j+1})]^p) \geq \xi(t_{j+1})$
    - $[\mathbf{y}_{j+1}] := \mathbf{y}_{j+1} + [\mathbf{e}_{j+1}]$
    - $[\mathbf{x}(t_{j+1})]^c := \mathbf{Corr}([\mathbf{x}(t_{j+1})]^p, [\mathbf{y}_{j+1}], [\mathbf{p}], \mathbf{g})$
    - $[\mathbf{x}_0] := [\mathbf{x}(t_{j+1})]^c$
  - **End**
- **End**

where $h_j (j = 0, \ldots, k)$ stand for the integration steps, which are not necessarily equal. As aforementioned, this algorithm is designed in order to guarantee the convergence of the state estimation error with a small number of measurements. The next section is devoted to prove this convergence.

### 4.1 Convergence analysis

Consider the estimated enclosure of the state trajectories

$$[\mathbf{x}(t)] = \big([\mathbf{x}_1(t)], [\mathbf{x}_2(t)]\big)^T$$

computed by the **ST-SM-SE** algorithm. Denote by $\mathcal{E}(t)$ the state estimation error defined as follows

$$\mathcal{E}(t) = \max\left\{e_1(t) = w([\mathbf{x}_1(t)]^p), e_2(t) = w([\mathbf{x}_2(t)]^p)\right\}$$

Hereafter, we present the main result of this work.

*Theorem 3. If Hypotheses 1 and 2 hold, then the state estimation error* $\mathcal{E}(t)$ *generated by the self-triggered algorithm* ***ST-SM-SE*** *converges towards a ball of diameter*

$$\mathcal{R} \leq \max\left\{\xi(\infty), f\big(\xi(\infty)\big)\right\}$$

*where* $f$ *is a positive bounded function.*

**Proof.** It is clear that Hypothesis 2 coupled with **ST-SM-SE** algorithm ensures that the size of the first part of the predicted state enclosure $[\mathbf{x}_1(t)]^p$ stays always lower than $\xi(t)$ after the first integration step:

$$\forall t \geq h_0, \quad e_1(t) = w([\mathbf{x}_1(t)]^p) < \xi(t)$$

That means, thanks to the measurement one can directly reduce the width of $[\mathbf{x}_1(t)]^p$. In fact, when the width of $[\mathbf{x}_1(t)]^p$ exceeds $\xi(t)$ at $t_j$, the **ST-SM-SE** algorithm activates the correction stage; and so under Hypothesis 2 we obtain

$$\begin{aligned} [\mathbf{x}_1(t_j)]^c &= [\mathbf{x}_1(t_j)]^p \cap [\mathbf{x}_1(t_j)]^{inv} \\ w([\mathbf{x}_1(t_j)]^c) &\leq w([\mathbf{x}_1(t_j)]^{inv}) < \xi(t_j) \end{aligned}$$

Thereafter, for the next integration step $h_{j+1}$, the **ST-SM-SE** algorithm reinitializes the prediction stage by

$$[\mathbf{x}_1(t_j)]^p = [\mathbf{x}_1(t_j)]^c$$

which implies that $w([\mathbf{x}_1(t_j)]^p) < \xi(t_j)$. So, we can claim that in the steady state the width of the box $[\mathbf{x}_1(t)]^p$ is upper bounded by $\xi(\infty)$.

Now, consider the enclosure of the second part of the state vector $[\mathbf{x}_2(t)]^p$ engendered by the system (14), which is not assumed directly corrected by the measurement. According to Hypothesis 1, we can frame the box $[\mathbf{x}_2(t)]^p$ by the state trajectories generated by the following stable system

$$\begin{cases} \dot{\overline{\mathbf{z}}}_2 = \overline{\mathbf{f}}_2(\overline{\mathbf{z}}_2, \mathbf{z}_1^+(\xi), \mathbf{p}^+, \mathbf{u}) \\ \dot{\underline{\mathbf{z}}}_2 = \underline{\mathbf{f}}_2(\underline{\mathbf{z}}_2, \mathbf{z}_1^-(\xi), \mathbf{p}^-, \mathbf{u}) \end{cases} \tag{15}$$

where $(\mathbf{z}_1^+, \mathbf{z}_1^-) \in \mathbb{R}^{2n}$ and either $\mathbf{z}_1^+(\xi) = \mathbf{x}_1 + \xi$ or $\mathbf{z}_1^+(\xi) = \mathbf{x}_1 - \xi$ (same for $\mathbf{z}_1^-$). The vector $\mathbf{x}_1$ contains the first part of the state variables of the nominal system (4). Then, at the equilibrium, we obtain

$$[\mathbf{x}_2(t_e)]^p \subset [\overline{\mathbf{z}}_2(t_e), \underline{\mathbf{z}}_2(t_e)] = [\mathbf{z}_2(t_e)]$$

which implies that for all $t \geq t_e$, $e_2(t)$ is upper bounded as follows

$$\forall t \geq t_e, \; e_2(t) \leq w([\mathbf{z}_2(t_e)]) = f\big(\xi(\infty)\big)$$

where $f(.)$ is a positive bounded function in $\xi$ because subsystem (15) is stable on the domain $\mathbb{D}^2$ as claimed by Hypothesis 1. Finally, we can say, under Hypotheses 1 and 2 the state estimation error generated by the self-triggered algorithm **ST-SM-SE** converges towards a ball with diameter $\mathcal{R} = \max\{\xi(\infty), f\big(\xi(\infty)\big)\}$. This completes the proof. □

## 5. ILLUSTRATIVE EXAMPLE

Usually in biotechnology field, the models of bio-process are poorly-known and the measurements of their outputs are available in discrete time. So, in this context, the proposed algorithm **ST-SM-SE** is suitable to resolve the state estimation problem. Consider the case of a bioreactor with two state variables. A bioreactor is a reactor wherein microorganism grows by consuming a substrate. As a rule, the concentrations of microorganism and substrate in the bioreactor are assumed to be weak. This allows us to use the assumption that the dynamics of the microorganism and that of substrate are observed with constant volume.

Hereafter, denote by $x_1$ and $x_2$ the concentration of the microorganism and the substrate, respectively. Moreover, we consider that the growth rate of the microorganism is described by the Contois law (Contois [1959]) and the dynamical model of the bioreactor is presented by (Gauthier et al. [1992])

$$\begin{cases} \dot{x}_1 = \dfrac{a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_1 \\ \dot{x}_2 = -\dfrac{a_3 a_1 x_1 x_2}{a_2 x_1 + x_2} - u x_2 + u a_4 \end{cases} \tag{16}$$

where the initial state of (16) is uncertain but belonging to the box $(x_1(0), x_2(0))^T \in [0.001, 0.1] \times [0.001, 0.1]$. The parameters $a_2$ and $a_3$ are considered perfectly known $a_2 = a_3 = 1$ and the parameters $a_1$, $a_4$ are unknown but bounded with known bounds. We have $\mathbf{p} = (a_1, a_4)^T \in [0.9, 1.1] \times [0.09, 0.11]$. The system input is defined as follows $u(t) = 0.08$ for $t \leq 10$, $u(t) = 0.02$ for $10 \leq t \leq 20$ and again $u(t) = 0.08$ for $t \geq 20$. Finally, the system output is given by

$$y(t) = x_1(t) \tag{17}$$

and the feasible domain of the output error is determined as follows

$$[e(t)] = [-5\% y_m(t), +5\% y_m(t)]$$

where $y_m$ stands for the real measurement.

For this example, Hypothesis 1 is satisfied. It is straightforward to show that the subsystem $\dot{x}_2$ is monotone and stable for all $(x_1, x_2) \in \mathbb{R}_+^2$, for all $(a_1, a_4)^T \in [\mathbf{p}]$ and for the given input $u$. First, this subsystem is monotone because it is mono-variable. In fact, it is well known that all first order system is monotone. Second, as the partial derivative $\frac{\partial \dot{x}_2}{\partial x_2}$ is strictly negative for all $(x_1, x_2) \in \mathbb{R}_+^2$, for all $(a_1, a_4)^T \in [\mathbf{p}]$ and for the given input $u$, then we can claim that the positive function $V(\Delta x) = \Delta x^2$ is a Lyapunov function of this subsystem where $\Delta x = x_2 - x_2^e$ and $x_2^e$ is its equilibrium point.

Now, applying the bounding method to the uncertain system (16) we obtain the following bracketing systems to use in the prediction stage

$$\begin{cases} \dot{\overline{x}}_1 = -u\overline{x}_1 + \dfrac{\overline{a}_1 \overline{x}_1 \overline{x}_2}{a_2 \overline{x}_1 + \overline{x}_2} \\ \dot{\underline{x}}_1 = -u\underline{x}_1 + \dfrac{\underline{a}_1 \underline{x}_1 \underline{x}_2}{a_2 \underline{x}_1 + \underline{x}_2} \end{cases} \tag{18}$$

$$\begin{cases} \dot{\overline{x}}_2 = -u\overline{x}_2 - \dfrac{a_3 \underline{a}_1 \underline{x}_1 \overline{x}_2}{a_2 \underline{x}_1 + \overline{x}_2} + u\overline{a}_4 \\ \dot{\underline{x}}_2 = -u\underline{x}_2 - \dfrac{a_3 \overline{a}_1 \overline{x}_1 \underline{x}_2}{a_2 \overline{x}_1 + \underline{x}_2} + u\underline{a}_4 \end{cases} \tag{19}$$

Moreover, from the output equation (17), it is clear that the size of $[x_1(t_j)]^{inv}$ is directly controlled by the feasible domain of measurement. In fact, at each time measurement $t_j$, we have $[x_1(t_j)]^{inv} = [y(t_j)]$. Then, for this example, in order to satisfy Hypothesis 2, it is enough to choose the following threshold

$$\xi(t) = \epsilon_1 \exp(-\epsilon_2 t) + \epsilon_3 \overline{x}_1(t)$$

where $\epsilon_1$, $\epsilon_2$ are positive constants and $\epsilon_3$ must be upper than 0.1, which represents the maximal percentage of the output error. So, we have

$$w([x_1(t_j)]^{inv}) = w([y(t_j)]) = 0.1 y_m(t_j) < 0.1\overline{x}_1(t_j) < \xi(t_j)$$

Furthermore, the convergence of $\xi(t)$ is guaranteed indirectly by the convergence of $\overline{x}_2(t)$. In fact, when the trajectory $\overline{x}_2(t)$ reaches its equilibrium, denoted here by $\overline{x}_2^e$, one can show by simple computation that

$$\overline{x}_1 \leq (\frac{\overline{a}_1}{u} - 1)\overline{x}_2^e$$

In order to show the efficiency of the **ST-SM-SE** algorithm, we test it in different simulation conditions; and then we compare its performance with those of interval observer presented in Meslem et al. [2008]. Note that, here, the measurements are obtained by simulating (16) with initial conditions $x_1 = x_2 = 0.05$ and the parameters $a_1, a_4$ are considered equal to $a_1 = 1, a_4 = 0.1$. The time decreasing threshold is chosen as follows

$$\xi(t) = 0.1\exp(-0.2t) + 0.2\overline{x}_1$$

- Test 1: without uncertainties on the parameters

For this first test, only initial state is considered uncertain. As shown in Fig. 1, the **ST-SM-SE** algorithm uses only

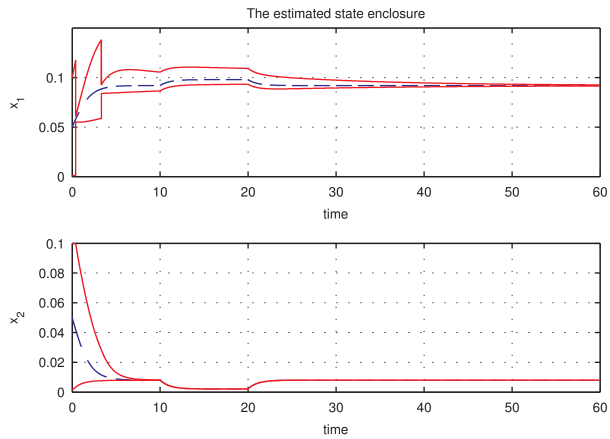


Fig. 1. The lower and upper estimated bounds of the state variables. Only initial state is considered uncertain.

two measurements (at $t = 0.4h$ and $t = 3.32h$) to compute a tight enclosure of all possible state trajectories generated by (16). Moreover, in this case, the state estimation error converges towards zero, see Fig. 1. This result is expected because there are no uncertainties on the parameters, which implies that the bounding systems (18), (19) have the same equilibrium point with the nominal system (16).

- Test 2: $(a_1, a_4) \in [0.95, 1.05] \times [0.095, 0.105]$

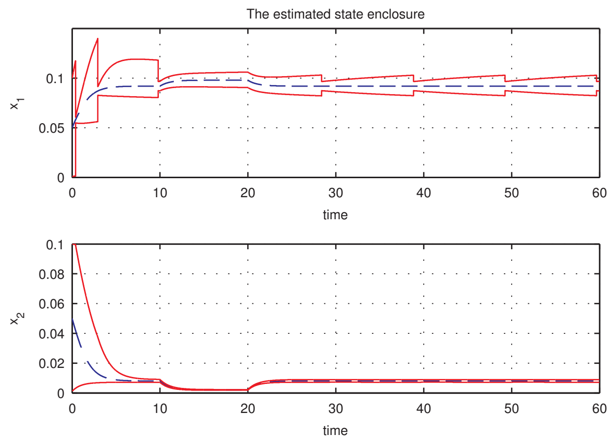


Fig. 2. The lower and upper estimated bounds of the state variables. The uncertain parameter box is considered equal to $[a_1] \times [a_4] = [0.95, 1.05] \times [0.095, 0.105]$.

Despite the presence of uncertainties, the **ST-SM-SE** algorithm computes an accurate enclosure of all the state trajectories of (16) by using only seven measurements, see Fig. 2. This figure shows also, how the self-triggered procedure imposes the convergence of the state estimation error. For this case the state estimation error never exceeds the tuning threshold $\xi(\infty)$.

It is worth noting that, for the first test, if one applies the classical **SM-SE** algorithm one will use regularly measurements even if it is not necessary as shown in Fig. 1. In fact, by construction, the **SM-SE** algorithm still continues to carry out the correction stage without improving the accuracy of the estimated state enclosure. Moreover, with this algorithm one can neither claim that the size of $[x_1(t)]$ do not exceed the a priori defined threshold $\xi(t)$, nor to give an upper estimation of the width of the estimated state enclosure.

Now, we devote the next two tests to compare the performance of the **ST-SM-SE** algorithm with those of interval observer (Meslem et al. [2008]). To do that, we use a smaller time-converging threshold defined by

$$\xi(t) = 0.05\exp(-0.5t) + 0.11\overline{x}_1$$

in order to better reduce the pessimism and so the comparison with interval observer makes sense.

- Test 3: comparison with interval observer (without uncertainties on the parameters)

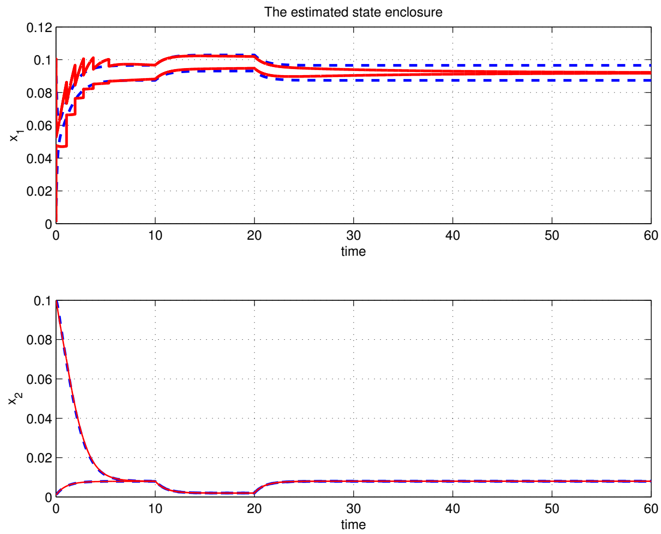


Fig. 3. The lower and upper estimated bounds of the state variables. Only initial state is considered uncertain.

In this case, the **ST-SM-SE** algorithm uses six measurements to get a tight estimated enclosure of the state trajectories with an estimation error decreasing towards zero as shown by the continuous curves in Fig. 3. On the other side, in the same conditions, the interval observer cannot compute an enclosure of the state trajectories with a converging size towards zeros as illustrated by the dashed curves in Fig. 3. This result is expected. In fact, by construction, we ask to the interval observer to generate a state enclosure such that the estimated enclosure of the output converges to the feasible domain of measurements.

## 6. CONCLUSION

In this paper, a new set-membership state estimation algorithm was presented. The main advantage of this algorithm is its ability to cope with pessimism propagation. Indeed, when one uses set-membership approaches the last phenomenon is usually source of an important conservatism. Moreover, with this algorithm, one uses measurements only when it is necessary; and in this event based framework, a convergence analysis of the state estimation error was presented. For future work, we will attempt to generalize the use of this estimation algorithm for a large class of nonlinear systems. To reach this objective, we must find a relaxed version of Hypotheses 1 and 2.

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