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New Result on Low-Order Controller Design for First-Order Delay Processes via Eigenvalue Assignment

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Abstract: This article developed a new result on the eigenvalue distribution for a certain class of time delay systems based on the extension of the Hermite-Biehler Theorem. Such result is applied to proportional-integral (PI) controller parameter design for a first-order plant with time delay via eigenvalue assignment. Using the method provided in this paper, one can assign the rightmost eigenvalues of the closed-loop system to desired positions in the complex plane. Further, on the basis of the previous result, this paper also extended the PI control to the proportional-integral-derivative (PID) case.
Keywords: Hermite-Biehler Theorem, eigenvalue assignment, time delay, proportional-integral (PI) control, proportional-integral-derivative (PID) control.

1. INTRODUCTION

Time delay widely exists in lots of process control systems (see Sipahi et al. (2011) and Richard (2003)). It is known that the existence of time delay gives rise to characteristic equations of closed-loop systems with an infinite number of roots (Gu et al. (2003)). Pontryagin studied a class of linear time invariant delay systems several decades ago. Based on Pontryagin's results, a suitable extension of the Hermite-Biehler Theorem can be developed to analyze that if the characteristic equations of these delay systems are Hurwitz, i.e., these equations possess only roots in the open left hand of complex plane (Bellman et al. (1963)). This result has played an important role in stability analysis and controller design of delay systems (Silva et al. (2005)). However, if a given characteristic equation is not Hurwitz, then this result does not provide any information about its root distribution. Though Silva et al. (2005) also revealed the stability-instability boundaries of controller gains based on this result, the pole distribution on the imaginary axis is not exactly depicted. In this paper, on the basis of the extension of the Hermite-Biehler Theorem, we produced a new result applicable to the characteristic equation with only a pair of nonzero imaginary axis roots (IARs) and left hand plane roots (LHPRs). It is worth noting that such result is very significant and can be used to controller design for some plants with time delay. Also, it is one of the staring points of our research.

In industry, proportional-integral-derivative (PID) controllers are most widely used because of simplicity in structure and capability in control (Wang et al. (1997)). An important problem in PID control of time delay systems is to compute the set of all PID parameters which provide internal stability and also some desired performance specifications for the closed-loop system (Bozorg et al. (2011)). In the last years there has been an interest in developing a theoretical analysis in order to determine the set of stabilizing P/PI/PID parameters for a given delay plant using different methods, see. e.g., Silva et al. (2002), Wang (2007), Ou et al. (2009), Oliveira et al. (2009), Hohenbichler (2009), Bozorg et al. (2011), and Padula et al. (2012). Moreover, excepting stabilization of systems, some researchers also consider to guarantee a performance specification, see Bozorg et al. (2011).

As is known that many properties of a closed-loop system depend on the locations of its poles, especially the locations of the rightmost roots in the complex plane. Hence pole placement is one of the mainstream methods in control system design (Åström et al. (2006)). However, for time delay systems, pole placement is not feasible using traditional control methods (Yi et al. (2013)). In this article, we apply the new result produced in this work to PI and PID control for a given first-order delay process. Our objective is to assign the rightmost eigenvalues of the closed-loop system to desired positions in the complex plane. Therefore, this article has the merit of making the rightmost eigenvalues of time delay systems exactly at the positions which are chosen according to the desired performance in industrial engineering.

Some basic properties on the relationship between the poles distribution and the controller gains for delay systems are derived in Michiels et al. (2002) and Michiels et al. (2005). Wang et al. (2009) and Michiels et al. (2010) also

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give some results on quasi-direct pole placement depending on the degrees of freedom in the parameter space.

It has to be stressed that Yi et al. (2013) also study pole placement for time delay systems by using the Lambert W function (for another work with Lambert W function on delay systems, see Yi et al. (2008)). Especially, they have considered PI control for a first-order process with time delay. However, the characteristic equations considered in Yi et al. (2013) are all retarded type quasi-polynomials as requested in the book Yi et al. (2010). When we control a first-order delay plant using a PID controller, the characteristic equation of this closed-loop system is obviously a neutral type quasi-polynomial. This fact means that the method proposed in Yi et al. (2013) can not be used in this case. Thus, another contribution of this work is that one can achieve rightmost poles placement through our approach for a class of time delay systems whenever the characteristic equation is retarded type or neutral type.

2. PRELIMINARIES

Consider a delay system whose characteristic function is

$$\delta(s) = d(s) + n_1(s)e^{-\nu_1 s} + \ldots + n_m(s)e^{-\nu_m s}, \quad (1)$$

where $d(s)$ and $n_j(s)$ for $j = 1, 2, \ldots, m$ are polynomials with real coefficients. For this function, we make the following assumptions.

(A) $0 < \nu_1 < \nu_2 < \ldots < \nu_m = N$;

(B) $\deg[d(s)] = M$ and $\deg[n_j(s)] \leq M$.

Multiplying (1) by $e^{\nu_m s}$, we get the quasi-polynomial as

$$h(s) = d(s)e^{\nu_m s} + n_1(s)e^{(\nu_m-\nu_1)s} + \ldots + n_m(s). \quad (2)$$

Since the term $e^{\nu_m s}$ vanishes nowhere in $\mathbb{C}$, then $\delta(s)$ and $h(s)$ have the same set of roots. Here, we first present a extension of Hermite-Biehler Theorem based on Pontryagin's results. This result can be applied to judge if a given quasi-polynomial as (2) is Hurwitz.

Let $h(j\omega) = h_r(\omega) + jh_i(\omega)$ by substituting $s = j\omega$, $\omega \in \mathbb{R}$, into $h(s)$ in (2). $h_r'(\omega)$ and $h_i'(\omega)$ denote the first derivatives with respect to ω of $h_r(\omega)$ and $h_i(\omega)$, respectively. The following theorem presents the necessary and sufficient conditions for $h(s)$ in (2) being Hurwitz.

Theorem 1: (Bellman et al. (1963), Silva et al. (2005)) The quasi-polynomial $h(s)$ in (2) is Hurwitz if and only if

(i) $h_r(\omega)$ and $h_i(\omega)$ have only real roots and these interlace;

(ii) $h_i'(\omega_o)h_r(\omega_o) - h_i(\omega_o)h_r'(\omega_o) > 0$, for some $\omega_o \in \mathbb{R}$.

This result has played an important role in studying the stabilization problem of a given plant with time delay. From this theorem, it is clear that to ensure that $h_r(\omega)$ and $h_i(\omega)$ possess only real zeros is a key step. This property can be ensured by another result of Pontryagin as follows.

Theorem 2: (Bellman et al. (1963)) Under Assumption (A) and (B) for $h(s)$ in (2), let η be a constant such that the coefficients of terms of highest degree in $h_r(\omega)$ and $h_i(\omega)$ do not vanish at $\omega = \eta$. Then, the necessary and sufficient condition under which $h_r(\omega)$ or $h_i(\omega)$ has only real roots is that, in the interval $-2l\pi + \eta \leq \omega \leq 2l\pi + \eta$, $h_r(\omega)$ or $h_i(\omega)$ has exactly $4lN + M$ real roots for a sufficiently large integer l.

3. A NEW RESULT ON THE LOCATION OF ZEROS

In this section, we present a result for the quasi-polynomial (2) possessing only LHPRs and a pair of nonzero IARs based on the extension of Hermite-Biehler Theorem. It is worth mentioned that this result plays a fundamental role in solving the problem of ascertaining the parameter set of some low-order controllers via eigenvalue assignment for a system with time delay.

Theorem 3: For a quasi-polynomial $h(s)$ in (2), all but a pair of roots $s = \pm j\omega^*$, $\omega^* \in \mathbb{R}^+$, are in the open left hand plane if and only if

(i) $h_i(\omega^*) = h_r(\omega^*) = 0$;

(ii) $h_i(\omega)$ and $h_r(\omega)$ has only real roots denoted as ω_i and ω_r respectively and, excepting the root $\omega_i = \omega^*$ and the root $\omega_r = \omega^*$, all the other nonnegative roots interlace;

(iii) $h_i'(\omega_o)h_r(\omega_o) - h_i(\omega_o)h_r'(\omega_o) > 0$ for some $\omega_o \in \mathbb{R}$.

Proof: Sufficiency. From Condition (i), it is easily seen that $s = j\omega^*$ is a root of $h(s)$ in (2). By the property of complex conjugates, $s = -j\omega^*$ is also a root of this quasi-polynomial. Since $h(s)$ is analytic in the entire complex plane, then it can be expanded as a Taylor series about the point $s = j\omega^*$ as the form

$$h(s) = (s - j\omega^*)\bar{h}_1(s) \quad (3)$$

where

$$\bar{h}_1(s) = h'(j\omega^*) + h''(j\omega^*)\frac{(s - j\omega^*)}{2!} + \ldots$$

From (3) we can ascertain that $s = -j\omega^*$ must be a root of $\bar{h}_1(s)$ as it is a root of $h(s)$. Moreover, it is found that $\bar{h}_1(s)$ is also analytic in the complex plane. Then we have

$$h(s) = (s - j\omega^*)(s + j\omega^*)\bar{h}(s) \quad (4)$$

where

$$\bar{h}(s) = \bar{h}_1'(-j\omega^*) + \bar{h}_1''(-j\omega^*)\frac{(s + j\omega^*)}{2!} + \ldots \quad (5)$$

Thus, excepting $s = \pm j\omega^*$, the other roots of $h(s)$ can be determined by $\bar{h}(s) = 0$. By substituting $s = j\omega$ and denoting $\bar{h}_r(\omega)$ and $\bar{h}_i(\omega)$ as the real and imaginary part of $\bar{h}(j\omega)$, we get that

$$h_r(\omega) = (-\omega^2 + \omega^{*2})\bar{h}_r(\omega), \quad (6)$$

$$h_i(\omega) = (-\omega^2 + \omega^{*2})\bar{h}_i(\omega). \quad (7)$$

Excepting $\omega = \pm\omega^*$, the other zeros of $h_i(\omega)$ and $h_r(\omega)$ can be ascertained by the equations $\bar{h}_i(\omega) = 0$ and $\bar{h}_r(\omega) = 0$ respectively. It is noted that $h_i(\omega)$ is an odd function and $h_r(\omega)$ is an even function. Therefore, $\bar{h}_i(\omega)$ is also an odd function and $\bar{h}_r(\omega)$ is also an even function. All the real zeros of $h_i(\omega)$, $h_r(\omega)$, $\bar{h}_i(\omega)$ and $\bar{h}_r(\omega)$ are symmetrical about the origin, respectively. Then, from Condition (ii), we can judge that $\bar{h}_i(\omega)$ and $\bar{h}_r(\omega)$ have only real zeros with $\omega \in \mathbb{R}$ and these zeros interlace. By means of Condition (iii), for some $\omega_o \in \mathbb{R}$, we have

$$h_i'(\omega_o)h_r(\omega_o) - h_i(\omega_o)h_r'(\omega_o)$$
$$=(-\omega_o^2 + \omega^{*2})^2[\bar{h}_i'(\omega_o)\bar{h}_r(\omega_o) - \bar{h}_i(\omega_o)\bar{h}_r'(\omega_o)] > 0 \quad (8)$$

which means that $\bar{h}_i'(\omega_o)\bar{h}_r(\omega_o) - \bar{h}_i(\omega_o)\bar{h}_r'(\omega_o) > 0$. According to Theorem 1, we can conclude that all the zeros of $\bar{h}(s)$ in (5) are in the open left hand complex plane.

Necessity. When $h(s)$ in (2) has a pair roots $s = \pm j\omega^*$, we have the expressions (4), (6) and (7). Then the conditions (i)-(iii) can be easily verified. ■

4. PI CONTROL OF FIRST-ORDER DELAY SYSTEMS

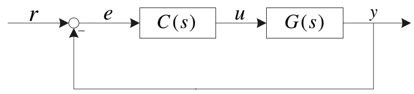


Fig. 1. Unit feedback control system

Consider a unit feedback control system shown in Fig. 1, where $G(s)$ is a first-order delay plant described as

$$G(s) = \frac{K}{Ts+1}e^{-\theta s} \tag{9}$$

and $C(s)$ is a PI controller whose transfer function is

$$C(s) = k_p + \frac{k_i}{s}. \tag{10}$$

In $G(s)$, $K \in \mathbb{R}^+$ is the steady-state gain, $\theta \in \mathbb{R}^+$ is the time delay and $T \in \mathbb{R}$ is the time constant.

In this section, by using Theorem 2 and Theorem 3, we present a method to design PI controllers for a first-order delay system so that the rightmost poles of the closed-loop system can be located at the desired positions $s = -\sigma^* \pm j\omega^*$, where $\sigma^*, \omega^* \in \mathbb{R}^+$.

The closed-loop characteristic equation of the system is

$$\delta(s) := s(Ts+1) + K(k_p s + k_i)e^{-\theta s} = 0. \tag{11}$$

Taking $s = \lambda - \sigma^*$ into (11), we get that

$$\bar{\delta}(\lambda) := (\bar{T}_1\lambda + 1)(\bar{T}_2\lambda + 1) - \bar{K}(\bar{k}_p\lambda + \bar{k}_i)e^{-\theta\lambda} = 0. \tag{12}$$

where $\bar{T}_1 = -1/\sigma^*$, $\bar{k}_p = k_p$, and $\bar{k}_i = k_i - \sigma^* k_p$. Moreover, $\bar{T}_2$ and $\bar{K}$ are given by

$$\begin{cases} \bar{T}_2 = \dfrac{T}{1-\sigma^* T},\ \bar{K} = \dfrac{Ke^{\theta\sigma^*}}{\sigma^*(1-\sigma^* T)}, & \text{for } \sigma^* T \neq 1; \\ \bar{T}_2\lambda + 1 = T\lambda,\ \bar{K} = \dfrac{Ke^{\theta\sigma^*}}{\sigma^*}, & \text{for } \sigma^* T = 1. \end{cases} \tag{13}$$

Multiplying $\bar{\delta}(\lambda)$ by $e^{\theta\lambda}$, we have

$$\bar{H}(\lambda) := (\bar{T}_1\lambda + 1)(\bar{T}_2\lambda + 1)e^{\theta\lambda} - \bar{K}(\bar{k}_p\lambda + \bar{k}_i) = 0. \tag{14}$$

Then substituting $\lambda = jz/\theta$ into (14), it yields

$$\bar{H}(j\frac{z}{\theta}) = \bar{H}_r(z) + j\bar{H}_i(z). \tag{15}$$

where

$$\bar{H}_r(z) = \xi(z)\cos[z + \varphi(z)] - \bar{K}\bar{k}_i, \tag{16}$$

$$\bar{H}_i(z) = \xi(z)\sin[z + \varphi(z)] - \frac{z}{\theta}\bar{K}\bar{k}_p. \tag{17}$$

Here

$$\begin{cases} \xi(z) = \sqrt{(\dfrac{\bar{T}_1^2 z^2}{\theta^2} + 1)(\dfrac{\bar{T}_2^2 z^2}{\theta^2} + 1)}, & \text{for } \sigma^* T \neq 1; \\ \xi(z) = \dfrac{Tz}{\theta}\sqrt{(\dfrac{\bar{T}_1^2 z^2}{\theta^2} + 1)}, & \text{for } \sigma^* T = 1, \end{cases} \tag{18}$$

$$\begin{cases} \varphi(z) = \arctan(\dfrac{\bar{T}_1 z}{\theta}) + \arctan(\dfrac{\bar{T}_2 z}{\theta}), & \text{for } \sigma^* T \neq 1; \\ \varphi(z) = \arctan(\dfrac{\bar{T}_1 z}{\theta}) + \dfrac{\pi}{2}, & \text{for } \sigma^* T = 1. \end{cases} \tag{19}$$

Definition 1: Consider the equation $[\theta\xi(z)/z]' = 0$. The value of its positive real root is denoted by ρ. If it has no positive real roots, let $\rho = 0$. Then define l^*, Z^*, and Q as follows.

(i) $l^* \in \mathbb{Z}^+$ is a number which satisfies that

$$\rho + \varphi(\rho) \leq 2(l^* - 1)\pi + \pi/2. \tag{20}$$

(ii) for a fixed l^* satisfying (20), we define $Z^* \in \mathbb{R}^+$ ascertained by the following equation:

$$Z^* + \varphi(Z^*) = 2l^*\pi + \pi/2. \tag{21}$$

(iii) $0 = z_0 < z_1 \leq \ldots \leq z_{Q-1}$ are the real roots of $\bar{H}_i(z)$ in the interval $[0, Z^*)$.

Lemma 1: The quasi-polynomial (14) possesses only real roots if and only if the $\bar{k}_p$ value ensures

$$Q = \begin{cases} 2l^* + 2 & \text{for } 0 < \sigma^* T \leq 1 \\ 2l^* + 3 & \text{for } \sigma^* T > 1 \text{ or } T < 0, \end{cases} \tag{22}$$

where Q and l^* are given by Definition 1.

Proof: By Theorem 2, $\bar{H}_i(z)$ has only real roots if there are $4l+2$ real roots in $[2l\pi - \pi/2, 2l\pi + \pi/2]$ for a sufficiently large integer l. Since $\bar{H}_i(z)$ is an odd function, i.e., all its real roots are symmetrical about the origin, then the condition which $\bar{H}_i(z)$ has only real roots can be equivalent to that $\bar{H}_i(z)$ has only $2l+2$ real roots in $[0, 2l\pi + \pi/2)$.

According to (17), the roots of $\bar{H}_i(z)$ can be given by

$$\frac{z\bar{K}\bar{k}_p}{\theta\xi(z)} = \sin[z + \varphi(z)]. \tag{23}$$

It is clear that $z_0 = 0$ is one root of (17). From the expression (19), one can find that for $z \to +\infty$,

$$\begin{cases} \varphi(z) = 0, & \text{if } 0 < \sigma^* T \leq 1; \\ \varphi(z) = -\pi, & \text{if } \sigma^* T > 1 \text{ or } T < 0. \end{cases} \tag{24}$$

By Definition 1, we can ascertain that $|z\bar{K}\bar{k}_p|/|\theta\xi(z)|$ is a strictly decreasing function with $z \geq Z^*$. Therefore, when $|Z^*\bar{K}\bar{k}_p|/|\theta\xi(Z^*)| < 1$, the intersection points of the plots $z\bar{K}\bar{k}_p/[\theta\xi(z)]$ and $\sin[z + \varphi(z)]$ will be $2(l - l^*)$ for $0 < \sigma^* T \leq 1$ and $2(l - l^*) - 1$ for $\sigma^* T > 1$ or $\sigma^* T < 0$ in the interval $[Z^*, 2l\pi + \pi/2)$, respectively.

From the analysis above, it is obtained that $\bar{H}_i(z)$ has $2l+2$ real roots in the interval $[0, 2l\pi + \pi/2)$ if (22) holds. ■

Lemma 2: For a fixed value of $\bar{k}_p$ which can ensure that the expression (22) holds, in order that the quasi-polynomial (14) possesses only LHPRs and a pair of nonzero IARs, the range of $\bar{k}_i$ is given by

(i) $\bar{k}_i \neq \frac{\sigma^*(1-\sigma^* T)}{Ke^{\theta\sigma^*}}$;

(ii) if $\max\limits_{\mathcal{S}(t)=0,-1}[p(z_t)] \leq \min\limits_{\mathcal{S}(t)=0,1}[p(z_t)]$ then

$$\bar{K}\bar{k}_i = \left\{\max_{\mathcal{S}(t)=0,-1}[p(z_t)],\ \min_{\mathcal{S}(t)=0,1}[p(z_t)]\right\} \tag{25}$$

where

$$p(z) = \xi(z)\{\cos[\varphi(z)]\cos(z) - \sin[\varphi(z)]\sin(z)\}, \tag{26}$$

$\mathcal{S}(t) = \mathrm{sgn}[\bar{H}_i'(z_t)]$, and $t = 0, 1, \ldots, Q-1$;

(iii) if $\mathcal{S}(t) = 0$, then it requires $\bar{H}_i''(z_t)\bar{H}_r'(z_t) > 0$.

Proof: To meet the requirement of this lemma, by Theorem 3 we have to satisfy the following conditions:

$$\bar{H}_r(z_{k^*}) = 0, \quad \text{for a } z_{k^*} \in \{z_k\},\ z_{k^*} \neq 0 \tag{27}$$

$$\bar{H}_i'(z_k)\bar{H}_r(z_k) > 0, \quad \text{for } z_k \neq z_{k^*}, \tag{28}$$

$$\bar{H}_i''(z_{k^*})\bar{H}_r'(z_{k^*}) > 0, \text{ if } \bar{H}_r'(z_{k^*}) = 0. \tag{29}$$

where $z = z_k$, $k \in \mathbb{N}$, are the nonnegative real roots of $\bar{H}_i(z)$ arranged in ascending order. Note that the condition

(28) and (29) can guarantee that all the roots of $\bar{H}_r(z)$ but $z = z_{k^*}$ in $[0, +\infty)$ are real and they interlace with those of $\bar{H}_i(z)$ excepting $z = z_{k^*}$. Also, the condition (28) ensures Condition (iii) of Theorem 3 as $\bar{H}_i(z_k) = 0$.

Substituting $z = z_k$ into the equation $\bar{H}_i(z) = 0$ and the function $p(z)$ in (26), then we can get that

$$p(z_k) = \frac{z_k \bar{K} \bar{k}_p \sin[\varphi(z_k)]}{\theta \cos[\varphi(z_k)]} + \frac{\xi(z_k)\cos(z_k)}{\cos[\varphi(z_k)]}. \tag{30}$$

According to (23), it is found that

$$\cos[z_k + \varphi(z_k)] = \pm \frac{\sqrt{\xi^2(z_k) - \bar{K}^2 \bar{k}_p^2 \frac{z_k^2}{\theta^2}}}{\xi(z_k)}. \tag{31}$$

From the property of trigonometric function, we have

$$\cos(z_k) = \cos[z_k + \varphi(z_k) - \varphi(z_k)]$$
$$= \cos[z_k + \varphi(z_k)]\cos[\varphi(z_k)] + \sin[z_k + \varphi(z_k)]\sin[\varphi(z_k)].$$

Substituting (23) and (31) into this function, and then taking it into (30), we can obtain that

$$p(z_k) = \pm\sqrt{\xi^2(z_k) - \bar{K}^2 \bar{k}_p^2 \frac{z_k^2}{\theta^2}} \tag{32}$$

It is mentioned from (31) that whether the value of $p(z_k)$ is positive or negative is decided by the value of $z_k + \varphi(z_k)$. It can be judged that for a value of $\bar{k}_p$ which can ensure that (22) holds, we have the following properties: 1) when $z_k \geq z_{Q-2}$, $\text{sgn}[\bar{H}_i'(z_{k+1})] = -\text{sgn}[\bar{H}_i'(z_k)]$ and $|p(z_k)|$ is a strictly increasing function with z_k; 2) when $z_k \geq z_{Q-1}$, $\text{sgn}[p(z_k)] = \text{sgn}[\bar{H}_i'(z_k)]$; 3) $z_{Q-2}, z_{Q-1} \in (\rho, Z^*)$. It is sufficient that $\bar{H}_i'(z_k)\bar{H}_r(z_k) > 0$ for all $z_k \in [Z^*, +\infty)$ if $\bar{H}_i'(z_k)\bar{H}_r(z_k) \geq 0$ for z_{Q-2} and z_{Q-1}. Then, it is obtained that the conditions (27)-(29) will be satisfied if and only if the conditions (i)-(iii) of this lemma hold. ■

Lemma 3: The quasi-polynomial (14) has a pair of roots $\lambda = \pm j\omega^*$, $\omega^* \in \mathbb{Z}^+$, if and only if $\bar{k}_p$ and $\bar{k}_i$ satisfy

(i) $\cos(\omega^*\theta) = \text{Re}\left[-\frac{(j\omega^* - \sigma^*)(jT\omega^* + 1 - T\sigma^*)}{Ke^{\theta\sigma^*}(j\omega^* \bar{k}_p + \bar{k}_i)}\right]$;

(ii) $\sin(\omega^*\theta) = \text{Im}\left[\frac{(j\omega^* - \sigma^*)(jT\omega^* + 1 - T\sigma^*)}{Ke^{\theta\sigma^*}(j\omega^* \bar{k}_p + \bar{k}_i)}\right]$.

Proof: As is known from the property of complex conjugates that if $\lambda = j\omega^*$ is a root of the quasi-polynomial (14), say, so is $\lambda = -j\omega^*$. Then substituting $\lambda = j\omega^*$ into $\bar{H}(\lambda) = 0$, we can get the conditions of this lemma. ■

Now we will present the main result of this section.

Theorem 4: For a PI control of a given plant with transfer function $G(s)$ as in (9), in order to assign the rightmost eigenvalues of the closed-loop systems to the desired positions $s = -\sigma^* \pm j\omega^*$, where $\sigma^*, \omega^* \in \mathbb{R}^+$, in the complex plane, the values of k_p and k_i are given by

$$k_p = \bar{k}_p', \ k_i = \bar{k}_i' + \sigma^* \bar{k}_p', \tag{33}$$

where

$$(\bar{k}_p', \bar{k}_i') \subset \{\Theta_1 \cap \Theta_2\} \tag{34}$$

Here Θ_1 is the set of $(\bar{k}_p, \bar{k}_i)$ obtained by Lemma 1 and Lemma 2; Θ_2 is the set of $(\bar{k}_p, \bar{k}_i)$ gotten by Lemma 3.

Proof: According to (11) and (12), it is found that for a given set of (k_p, k_i) values, in the complex plane the positions of the roots of (11) in s are equivalent to the positions that all the roots of (12) with respect to λ renovate a left shift σ^* in the horizontal direction. It is known that (12) and (14) have the same roots. Then by Lemma 1 and Lemma 2, the set of $(\bar{k}_p, \bar{k}_i)$ values in Θ_1 can ensure that a pair nonzero imaginary axis roots of (12) in λ are at the rightmost. By Lemma 3, the set of $(\bar{k}_p, \bar{k}_i)$ values in Θ_2 can guarantee that (12) has a pair roots $\lambda = \pm j\omega^*$. Therefore the rightmost eigenvalues of (12) will be $\lambda = \pm j\omega^*$ if the set of $(\bar{k}_p, \bar{k}_i)$ values is in $\Theta_1 \cap \Theta_2$. Moreover, it is clear that $\bar{k}_p = k_p$ and $\bar{k}_i = k_i - \sigma^* k_p$. From the discussion above, we get the result of this theorem. ■

Example 1: Consider an open-loop stable plant provided by Yi et al. (2013):

$$G(s) = \frac{e^{-0.2s}}{0.5s + 1} \tag{35}$$

Choose a PI controller to let the rightmost eigenvalues of the closed-loop system be at $s = -1.25 \pm 2.1651i$.

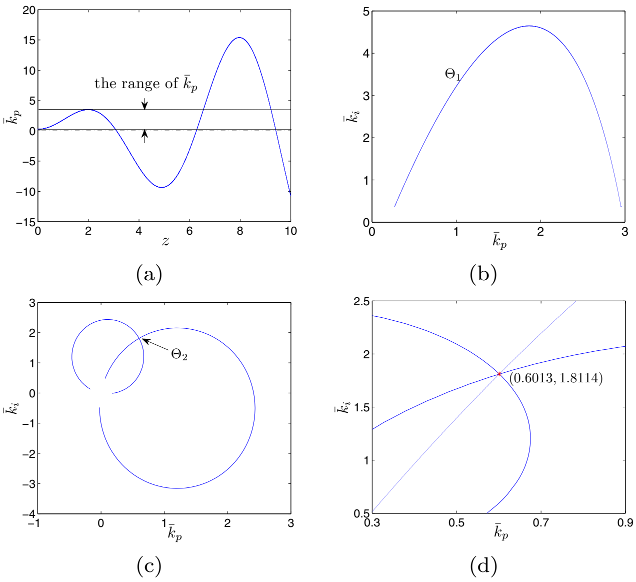


Fig. 2. The parameters $\bar{k}_p$ and $\bar{k}_i$ for Example 1.

Substituting $s = \lambda - 1.25$ into the characteristic equation of the closed-loop system, we have

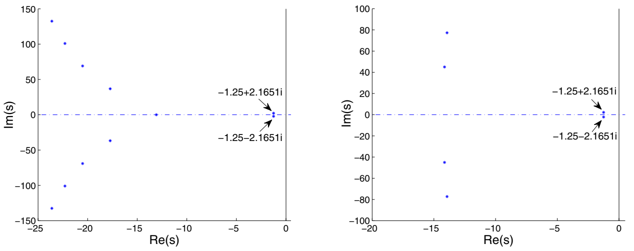
$$\bar{\delta}(\lambda) = (-0.8\lambda + 1)(1.3333\lambda + 1) - 2.7393(\bar{k}_p\lambda + \bar{k}_i)e^{-0.2\lambda} = 0. \tag{36}$$

By the form of the expression (12), it is seen that $\bar{K} = 2.7393$, $\bar{T}_1 = -0.8$ *sec*, $\bar{T}_2 = 1.3333$ *sec*, and $\theta = 0.2$ *sec*. From (17), we have the expression $\bar{H}_i(z)$ in which

$$\xi(z) = \sqrt{(16z^2 + 1)(44.4444z^2 + 1)}, \tag{37}$$

$$\varphi(z) = -\arctan(4z) + \arctan(6.6667z). \tag{38}$$

According to Definition 1, we can calculate that $l^* = 1$ and $Z^* = 7.8412$. By Lemma 1, the necessary condition for values of $\bar{k}_p$ is that they can ensure that the function $\bar{H}_i(z)$ has 4 real roots in the interval $[0, 7.8412)$. Seeing the intersection points of the plot $\bar{k}_p$ and the plot $\xi(z)\sin[z + \varphi(z)]/(\bar{K}z)$ in Fig. 2(a), one can calculate that such a condition holds within the interval $0.2677 \leq \bar{k}_p \leq 3.4933$. By Lemma 1 and Lemma 2, we can get the set of $(\bar{k}_p, \bar{k}_i)$ values denoted as Θ_1 in Fig. 2(b), to ensure that the function $\bar{\delta}(\lambda)$ in (36) has only LHPRs and a pair of nonzero IARs. Moreover, by Lemma 3 we can obtain the set of $(\bar{k}_p, \bar{k}_i)$ values shown in Fig. 2(c) and denoted as Θ_2 such that the function $\bar{\delta}(\lambda)$ has a pair of IARs at $\lambda = \pm 2.1651i$. It is sufficient that the set of $(\bar{k}_p, \bar{k}_i)$ values in $\Theta_1 \cap \Theta_2$ is $(0.6013, 1.8114)$, see Fig. 2(d). This set of $(\bar{k}_p, \bar{k}_i)$ can make the equation (36) only possessing LHPRs and a pair of IARs $\lambda = \pm 2.1651i$. By Theorem 4 one can get that $k_p = 0.6013$ and $k_i = 2.5630$ for the PI controller which can assign the rightmost roots of the closed-loop system at $s = -1.25 \pm 2.1651i$, see Fig. 3(a). The root distribution is obtained by using the bifurcation analysis package DDE-BIFTOOL in Matlab (Engelborghs et al. (2002)).



(a) (b)

Fig. 3. The characteristic root distribution of the closed-loop system: (a) for Example 1; (b) for Example 2.

5. EXTENSION TO PID CONTROL

Along the same lines of the previous section, the method for PI control can be extended to PID control. In PID control, the characteristic equation is given by

$$\gamma(s) = s(Ts+1) + K(k_p s + k_i + k_d s^2)e^{-\theta s}. \tag{39}$$

Taking $s = \lambda - \sigma^*$ into (39), we get that

$$\begin{aligned}\tilde{\gamma}(\lambda) :=& (\bar{T}_1\lambda + 1)(\bar{T}_2\lambda + 1) \\ &- \bar{K}(\tilde{k}_p\lambda + \tilde{k}_i + \tilde{k}_d\lambda^2)e^{-\theta\lambda} = 0.\end{aligned} \tag{40}$$

where $\tilde{k}_p = k_p - 2\sigma^* k_d$, $\tilde{k}_i = k_i - \sigma^* k_p + \sigma^{*2} k_d$, $\tilde{k}_d = k_d$, and $\bar{K}$, $\bar{T}_1$, $\bar{T}_2$ are the same as those in (12). Multiplying $\tilde{\gamma}(\lambda)$ by $e^{\theta\lambda}$, we have

$$\begin{aligned}\tilde{H}(\lambda) :=& (\bar{T}_1\lambda + 1)(\bar{T}_2\lambda + 1)e^{\theta\lambda} \\ &- \bar{K}(\tilde{k}_p\lambda + \tilde{k}_i + \tilde{k}_d\lambda^2) = 0.\end{aligned} \tag{41}$$

Then substituting $\lambda = jz/\theta$ into (41), it yields

$$\tilde{H}(j\frac{z}{\theta}) = \tilde{H}_r(z) + j\tilde{H}_i(z). \tag{42}$$

where

$$\tilde{H}_r(z) = \xi(z)\cos[z + \varphi(z)] - \bar{K}(\tilde{k}_i - \tilde{k}_d\frac{z^2}{\theta}), \tag{43}$$

$$\tilde{H}_i(z) = \xi(z)\sin[z + \varphi(z)] - \frac{z}{\theta}\bar{K}\tilde{k}_p. \tag{44}$$

Here $\xi(z)$ and $\varphi(z)$ are given by (18) and (19).

Remark 1: It is easily found that the form of $\tilde{H}_i(z)$ in (44) is the same as that of $\bar{H}_i(z)$ in (17). Thus, a necessary condition for the admissible range of $\tilde{k}_p$ can be ascertained by (22). The only difference is that here Q is the number of the real roots of $\tilde{H}_i(z)$ in the interval $[0, Z^*)$.

Lemma 4: The quasi-polynomial (41) has a pair of roots $\lambda = \pm j\omega^*$, $\omega^* \in \mathbb{Z}^+$, if and only if $\tilde{k}_p = \alpha$ and $\tilde{k}_i - \omega^{*2}\tilde{k}_d = \beta$, where α and β satisfy

(i) $\cos(\omega^*\theta) = \mathrm{Re}\left[-\frac{(j\omega^* - \sigma^*)(jT\omega^* + 1 - T\sigma^*)}{Ke^{\theta\sigma^*}(j\omega^*\alpha + \beta)}\right]$;

(ii) $\sin(\omega^*\theta) = \mathrm{Im}\left[\frac{(j\omega^* - \sigma^*)(jT\omega^* + 1 - T\sigma^*)}{Ke^{\theta\sigma^*}(j\omega^*\alpha + \beta)}\right]$.

Proof: This proof is similar to Lemma 3. ■

Lemma 5: For $\tilde{k}_p = \alpha$, when the value of α is in the admissible range of $\tilde{k}_p$ given by Remark 1, the quasi-polynomial (14) only has LHPRs and a pair of nonzero IARs $\lambda = \pm j\omega^*$ if and only if

(i) for $\forall t = 0, 1, \ldots, Q' - 1$,

$$\begin{cases} |\tilde{k}_d| \le |\dfrac{T}{Ke^{\theta\sigma^*}}|, \\ \tilde{k}_i - \omega^{*2}\tilde{k}_d = \beta, \\ \bar{K}[\tilde{k}_i - A(z_t)\tilde{k}_d + B(z_t)] \cdot \mathcal{S}(t) < 0, \ z_t \neq \theta\omega^*; \end{cases} \tag{45}$$

(ii) if $\mathcal{S}(t) = 0$, then it requires $\tilde{H}_i''(z_t)\tilde{H}_r'(z_t) > 0$.

Here, $A(z_t) = z_t^2/\theta^2$, $B(z_t) = -\xi(z_t)\cos[z_t + \varphi(z_t)]/\bar{K}$, $\mathcal{S}(t) = \mathrm{sgn}[\tilde{H}_i'(z_t)]$. $0 = z_0 < z_1 \le \ldots \le z_{Q'-1}$ are the real roots of $\tilde{H}_i(z)$. The value of Q' satisfies

$$z_{Q'-1} = \max\{z_{Q-1}, z_{Q_1-1}\}, \tag{46}$$

where z_{Q-1} is the largest root of $\tilde{H}_i(z)$ in $[0, Z^*)$ and z_{Q_1-1} is its second smallest root in the interval $(\theta\omega^*, +\infty)$.

Proof: This lemma is the extension of Lemma 2. One can obtain it alone the same lines as the proof of Lemma 2. ■

Theorem 5: For a PID control of a given plant (9), in order to assign the rightmost eigenvalues of the closed-loop system to the desired positions $s = -\sigma^* \pm j\omega^*$, $\sigma^*, \omega^* \in \mathbb{R}^+$, the values of k_p, k_i, and k_d are given by

$$k_p = \tilde{k}_p' + 2\sigma^*\tilde{k}_d', \ k_i = \tilde{k}_i' + \sigma^*\tilde{k}_p' + \sigma^{*2}\tilde{k}_d', \ k_d = \tilde{k}_d' \tag{47}$$

where

$$(\tilde{k}_p', \tilde{k}_i', \tilde{k}_d') \subset \Omega \tag{48}$$

Here Ω denotes the set of $(\tilde{k}_p, \tilde{k}_i, \tilde{k}_d)$ given by Lemma 5.

Proof: The proof is similar to that of Theorem 4. Due to space limitations , we omit the details. ■

Example 2: In this example, we also consider the delay plant (35). Here we adopt PID control so that the rightmost eigenvalues of the closed-loop system are at $s = -\sigma^* \pm \omega^* i$, where $\sigma^* = 1.25$ and $\omega^* = 2.1651$.

Substituting $s = \lambda - 1.25$ into the characteristic equation of the closed-loop system, we have

$$\begin{aligned}\tilde{\gamma}(\lambda) =& (-0.8\lambda + 1)(1.3333\lambda + 1) \\ &- 2.7393(\tilde{k}_p\lambda + \tilde{k}_i + \tilde{k}_d\lambda^2)e^{-0.2\lambda}.\end{aligned} \tag{49}$$

The parameters $\bar{T}_1$, $\bar{T}_2$, $\bar{K}$ and θ are the same as those in Example 1. Meanwhile, the expressions $\xi(z)$ and $\varphi(z)$ in $\tilde{H}_i(z)$ in (44) are also given by (37) and (38), which brings the same values of l^* and Z^* as those in Example 1. By Remark 1 and Example 1, the admissible range of $\tilde{k}_p$ is $[0.2677, 3.4933]$. According to Lemma 4, it is clear that the function $\tilde{\gamma}(\lambda)$ has a pair of IARs at $\lambda = \pm 2.1651i$ if

$$\tilde{k}_p = 0.6013, \ \tilde{k}_i - 2.1651^2\tilde{k}_d = 1.8114 \tag{50}$$

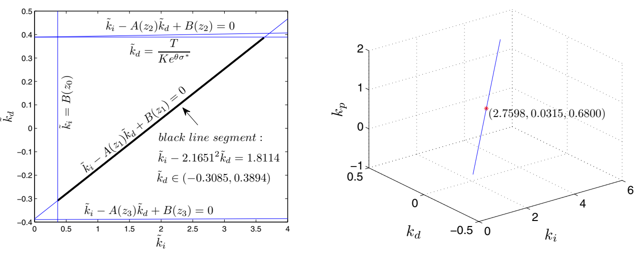
When $\tilde{k}_p = 0.6013$, we have

$$\tilde{H}_i(z) = -8.2356z + \sqrt{(16z^2+1)(44.4444z^2+1)} \cdot \sin[z - \arctan(4z) + \arctan(6.6667z)].$$

In the interval $[0, 7.8412)$, $\tilde{H}_i(z)$ has 4 real and distinct roots, i.e., $z_0 = 0$, $z_1 = 0.4330$, $z_2 = 3.0060$ and $z_3 = 6.3162$. Hence, the value of Q in Lemma 5 is $Q = 4$. Additionally, it is found that $\omega^* = z_1/\theta$, so the value of Q_1 in Lemma 5 is $Q_1 = 3$. Then from (46), we have $Q' = 4$. By Lemma 5, for $\tilde{k}_p = 0.6013$, one can obtain the region of $(\tilde{k}_i, \tilde{k}_d)$ which satisfies the line $\tilde{k}_i - 2.1651^2\tilde{k}_d = 1.8114$ for $\tilde{k}_d \in (-0.3085, 0.3894)$ shown in Fig. 4(a) (the thick black line segment). Finally, when $\tilde{k}_d$ is in $(-0.3085, 0.3894)$, on the basis of Theorem 5 and the expression (50), we have

$$\begin{cases} -0.1701 < k_p < 1.5748 \\ k_i = 2.5001k_p + 1.0597 \\ k_d = 0.4k_p - 0.2405 \end{cases} \tag{51}$$

Such values of (k_p, k_i, k_d) can ensure that the rightmost eigenvalues of the closed-loop system with plant (49) are at $s = -1.25 \pm 2.1651i$. It is seen that the range of (k_p, k_i, k_d) is depicted as a line segment in three dimensional space in Fig. 4(b). Now, by choosing a set of (k_p, k_i, k_d) values as $(0.6800, 2.7598, 0.0315)$ which is also the point on the line shown in Fig. 4(b), the characteristic root distribution of the closed-loop system is shown in Fig. 3(b).



(a) (b)

Fig. 4. The parameter range for Example 2: (a) the region of $(\tilde{k}_i, \tilde{k}_d)$ for $\tilde{k}_p = 0.6013$; (b) the region of (k_p, k_i, k_d).

6. CONCLUSIONS

This paper produced a new result on the root distribution of a class of quasi-polynomial based on the extension of the Hermite-Biehler Theorem. Such result is then used to PI/PID controller design for a first-order plant with time delay via pole placement. Numerical examples are also provided to illustrate the effectiveness of the presented conclusions. Our next-step work is to explore the possibility of extending the proposed method for high-order delay plants. It is our belief that the results of this paper will form a new tool for PID controller design and analysis.

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