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Rejection of Heave-Induced Pressure Fluctuations at the Casing Shoe in Managed Pressure Drilling

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Abstract—**In this paper, we apply recent results on disturbance rejection in the interior domain of a class of 2×2 linear hyperbolic systems of partial differential equations to the problem of rejecting heave-induced pressure fluctuations at the casing shoe in managed pressure drilling. We show that a PDE model of the drilling dynamics belongs to this class of systems, and derive both state feedback and output feedback control laws for the drilling system. The performance of the control laws are demonstrated in simulations that show efficient rejection of pressure oscillations at a desired location in the well.**

I. Introduction

THE undesired pressure fluctuations of concern in this paper, emerges when drilling offshore from a rig floating at the sea. During drilling operations, a drilling fluid called mud is pumped down through the drill string, through the drill bit at the bottom of the well, and up the annulus around the drill string. The mud serves several functions, like cooling down the drill bit and carrying cuttings out of the system. The mud also works to keep the pressure in the annulus at a desired level. This latter purpose is a crucial part of drilling, as the pressure needs to be kept within certain bounds to avoid fracturing of the formation or collapse of the well. Technologies developed with the aim of improving the pressure control throughout the well are often referred to as Managed Pressure Drilling (MPD).

When drilling offshore, however, the floating rig naturally moves up and down with the waves. During drilling, an active mechanism is used to keep the string from moving with the rig. However, every $27 - 29$ metres, it is necessary to stop the drilling to extend the drill string. During this procedure, the heave compensation mechanism is deactivated and the string is rigidly attached to the rig. The drill string then moves with the rig and acts as a piston on the mud in the well. Left uncompensated, this piston effect results in severe pressure fluctuations throughout the well, often exceeding the standard limits for pressure regulation accuracy in MPD, which are about ± 2.5 bar.

The heave problem has previously been addressed in [1] using a lumped model and simplifying assumptions with regards to available measurements, in [2] where a linearization technique was used that neglected the friction terms, making the system decoupling trivial, and in [3] using a simplified friction model and a model reduction scheme based on Laguerre polynomials. In [4], theory derived for linear 2×2 hyperbolic PDEs was used to design a control law for rejection of heave-induced pressure fluctuations. In all these works, the location of rejection was limited to the bottom of the well. There may be situations, however, where pressure regulation at other points in the well is preferable, e.g. at the bottom of a casing string (also known as a *casing-shoe*), and this is the main motivation for this paper.

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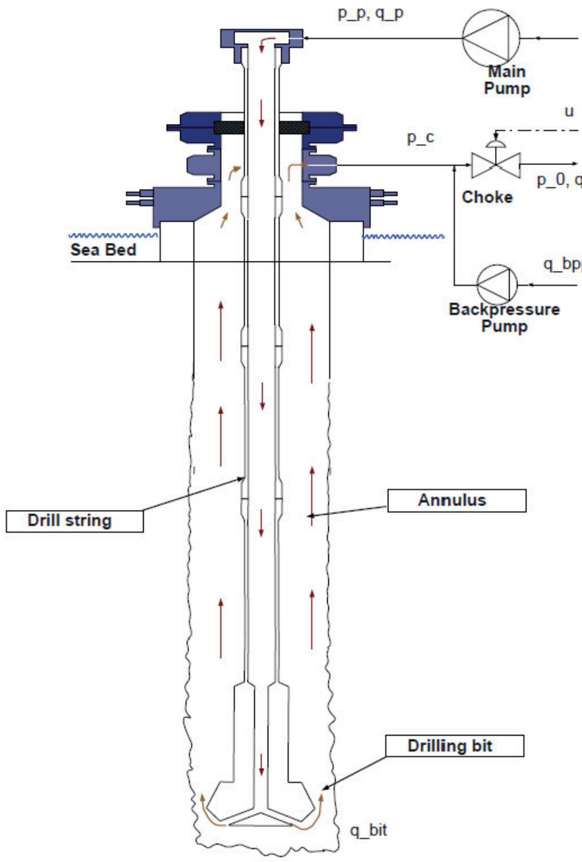

Fig. 1. Schematic of a managed pressure drilling (MPD) system. Courtesy of Statoil ASA.

The paper is organized as follows. In Section II, we pose the disturbance rejection problem for managed pressure drilling. In Section III, we present previous results applicable for a more general class of systems, before a coordinate transformation linking the two systems is presented in Section IV. Simulation results are given in Section V, and concluding remarks are offered in Section VI.

II. Problem Statement

A typical MPD system is depicted in Figure 1. We model the annular pressure and flow in a well of depth l as a linear

2×2 hyperbolic system of PDEs, with the disturbance term assumed to be an autonomously driven harmonic oscillator. The overall model can be stated as

$$p_t(z,t) = -\frac{\mu}{A_1} q_z(z,t) \tag{1a}$$

$$q_t(z,t) = -\frac{A_1}{\rho} p_z(z,t) - \frac{F_1}{\rho} q(z,t) - A_1 g \tag{1b}$$

$$q(0,t) = -A_2 \bar{C} Z(t) \tag{1c}$$

$$p(l,t) = p_l(t) \tag{1d}$$

$$\dot{Z} = \bar{A} Z, \qquad Z(0) = Z_0 \tag{1e}$$

where l is the well depth, $z \in [0, l]$, $t \geq 0$, $p(z,t)$ is the pressure, $q(z,t)$ is the volumetric flow, μ is the mud's bulk modulus, ρ is the mud density, A_1 is the cross sectional area of annulus, A_2 is the cross sectional area of the drill bit, F_1 is the friction factor and g is the gravity constant. $p_l(t)$ is the actuation, and its actuation device is assumed to have significantly faster dynamics than the rest of the system, so that actuator dynamics may be ignored. Also, $q_l(t) = q(l,t)$ is assumed measured. The disturbance term $Z(t)$ is parameterized by a finite set $\{\omega_1, \omega_2, \dots, \omega_n\}$ of known, distinct frequencies and

$$\bar{A} = \text{diag}\left(\begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \omega_n \\ -\omega_n & 0 \end{bmatrix} \right) \tag{2}$$

$$\bar{C} = \begin{bmatrix} 0 & 1 & 0 & 1 & \dots & 0 & 1 \end{bmatrix}. \tag{3}$$

Clearly, the pair $(\bar{A}, \bar{C})$ is observable. The objective is to have a constant pressure at $z = \bar{z} \in (0, l)$, mathematically stated as

$$p(\bar{z}, t) = p_{sp}, \tag{4}$$

where p_{sp} is a desired setpoint for the pressure. The model (1) was originally presented in [1], with the disturbance model (2)–(3) taken from [4].

III. Prior Knowledge

In order to achieve (4), we will apply theory originally derived for a more general class of linear 2×2 hyperbolic PDEs. The theory was presented in [5], and applies to systems on the following form

$$u_t(x,t) = -\epsilon_1(x) u_x(x,t) + c_1(x) v(x,t) \tag{5a}$$

$$v_t(x,t) = \epsilon_2(x) v_x(x,t) + c_2(x) u(x,t) \tag{5b}$$

$$u(0,t) = q v(0,t) + CX(t) \tag{5c}$$

$$v(1,t) = U(t) \tag{5d}$$

$$\dot{X}(t) = AX(t) \tag{5e}$$

defined over the domain $x \in [0,1]$ and $t \geq 0$. It is assumed that $\epsilon_1(x), \epsilon_2(x) > 0$ are $\mathcal{C}^1([0,1])$, and $c_1(x), c_2(x)$ are $\mathcal{C}([0,1])$. The disturbance term $X(t) \in \mathbb{R}^n$ is parameterized by $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{1 \times n}$ with the pair (A, C) assumed observable. The parameter $q \neq 0$ is a constant, $U(t)$ is the control input and $u(1,t)$ is assumed measured. The control objective investigated in [5], was the design of $U(t)$ so that

$$u(\bar{x}, t) = r v(\bar{x}, t) \tag{6}$$

is achieved for some given, fixed $\bar{x} \in (0,\ 1)$ and some constant $r \neq 0$.

A. State Feedback

Previous results initially performed a backstepping transformation that essentially decouples (5) and facilitates obtaining closed form solutions. Application of the transformation to obtain stabilizing boundary feedback laws is usually referred to as the backstepping method. The backstepping method for partial differential equations was first developed for parabolic type equations [6]. The first version of the backstepping method in its infinite dimensional form was presented in [7], and has since been extended to hyperbolic PDEs [8], [9], [10], [4]. Theory from the two latter references was essential when the following theorem was proved in [5].

Theorem 1 (State feedback): Suppose $q \neq 0$, $r \neq 0$, $\bar{x} \in (0,1)$. The control law

$$\begin{aligned} U(t) = &\int_0^1 K^{vu}(1,\xi) u(\xi,t) d\xi + \int_0^1 K^{vv}(1,\xi) v(\xi,t) d\xi \\ &+ \frac{1}{r} K_{psf}(\bar{x}) X(t) + \frac{1}{r} \delta(\bar{x}, \bar{x}, t) \\ &+ \frac{1}{r} \int_0^{\bar{x}} L^\alpha(\bar{x}, \xi) \delta(\xi, \bar{x}, t) d\xi \\ &+ \frac{1}{r} \int_0^{\bar{x}} L^\beta(\bar{x}, \xi) \beta(\phi_\beta^{-1}(\kappa_\beta(\xi, \bar{x})), t) d\xi \end{aligned} \tag{7}$$

where $K^{vu}(1,\xi)$, $K^{vv}(1,\xi)$, $K_{psf}(\bar{x})$, $\delta(\xi, x, t)$, $L^\alpha(x,\xi)$, $L^\beta(x,\xi)$, $\beta(x,t)$, $\phi_\beta(x)$ and $\kappa_\beta(\xi, \bar{x})$ are defined in Appendix A, achieves (6) within a finite time given by $t_0 = \phi_\beta(1) - \phi_\beta(\bar{x})$. Moreover, if $|CX(t)|$ is bounded and

$$\left| \frac{q}{r} \right| < e^{-\bar{L} d_r(\bar{x})} \tag{8}$$

where $\bar{L}$ bounds the function $L_o(t, \gamma)$ defined in (53), then $|u(x,t)|$ and $|v(x,t)|$ are bounded.

B. State observer

The control law of Theorem 1 requires full knowledge of the system states $u(x,t)$, $v(x,t)$ as well as the disturbance $X(t)$. In practical problems, measured signals are often limited to sensing at $x = 1$, a fact that requires the control law to be modified. If we assume that estimates of $u(x,t)$, $v(x,t)$ and $X(t)$ are available and rely on the certainty equivalence principle, the controllers can be implemented by replacing states in the controller by their corresponding estimates. The estimates will have to be generated from the only signals available; the sensing at $x = 1$ and the generated controller input $U(t)$. Such an observer estimating the states of the system, was derived in [10], and a proof of exponential convergence was also given. In fact, the estimates were proven to reach their real values in *finite* time.

A modification of the observer from [10] was done in [4] to accommodate the disturbance term entering at the boundary.

We repeat here the observer equations from [4]

$$\hat{u}_t(x,t) = -\epsilon_1(x)\hat{u}_x(x,t) + c_1(x)\hat{v}(x,t) + p_1(x)(Y(t) - \hat{u}(1,t)) \quad (9a)$$

$$\hat{v}_t(x,t) = \epsilon_2(x)\hat{v}_x(x,t) + c_2(x)\hat{u}(x,t) + p_2(x)(Y(t) - \hat{u}(1,t)) \quad (9b)$$

$$\hat{u}(0,t) = q\hat{v}(0,t) + C\hat{X}(t) \quad (9c)$$

$$\hat{v}(1,t) = U(t) \quad (9d)$$

$$\dot{\hat{X}} = A\hat{X} + e^{A\phi_\alpha(0)}L(Y(t) - \hat{u}(1,t)) \quad (9e)$$

where

$$Y(t) = u(1,t) \quad (10)$$

is the measurement. The matrix L is a gain matrix chosen such that $(A - LC)$ is Hurwitz. The functions $p_1(x)$ and $p_2(x)$ are injection gains defined in Appendix B.

IV. Application to MPD

System (1) will have to be mapped to the form (5) in order to use the theory presented in the previous section.

Lemma 2 (Modified from Lemma 10 in [4]): Assume $\bar{z} \in (0,l)$ and p_{sp} are given. Let

$$\bar{x} = \frac{\bar{z}}{l}, \quad (11)$$

then the transformation

$$u(x,t) = \frac{1}{2}\left[q(xl,t) + \frac{A_1}{\sqrt{\mu\rho}}\left(p(xl,t) - p_{sp} + \rho g l(x-\bar{x})\right)\right] \times e^{\frac{lF_1}{2\sqrt{\mu\rho}}(x-\bar{x})} \quad (12a)$$

$$v(x,t) = \frac{1}{2}\left[q(xl,t) - \frac{A_1}{\sqrt{\mu\rho}}\left(p(xl,t) - p_{sp} + \rho g l(x-\bar{x})\right)\right] \times e^{-\frac{lF_1}{2\sqrt{\mu\rho}}(x-\bar{x})} \quad (12b)$$

maps the system (1) to the form (5) with

$$X(t) = Z(t) \quad (13)$$

$$U(t) = \frac{1}{2}(q_l(t) - \frac{A_1}{\sqrt{\mu\rho}}(p_l(t) - p_{sp} + \rho g l(1-\bar{x}))) \times e^{-\frac{lF_1}{2\sqrt{\mu\rho}}(1-\bar{x})} \quad (14)$$

$$\epsilon_1(x) = \epsilon_2(x) = \epsilon,\ c_1(x) = a_0 e^{\gamma x},\ c_2(x) = b_0 e^{-\gamma x} \quad (15)$$

$$q = -e^{-\gamma\bar{x}} \quad (16)$$

$$A = \bar{A}, \quad C = -e^{\frac{\gamma}{2}\bar{x}}A_2\bar{C} \quad (17)$$

where

$$\epsilon = \frac{1}{l}\sqrt{\frac{\mu}{\rho}}, \quad \gamma = \frac{lF_1}{\sqrt{\mu\rho}} \quad (18)$$

$$a_0 = c_0 e^{-\gamma\bar{x}}, \quad b_0 = c_0 e^{\gamma\bar{x}}, \quad c_0 = -\frac{1}{2}\frac{F_1}{\rho}. \quad (19)$$

Moreover, the control objective (4) is transformed to (6) with $r = 1$.

Proof: We remove the constant term and shift the origin by defining

$$\bar{p}(z,t) = p(z,t) - p_{sp} + \rho g(z - \bar{z}) \quad (20)$$

from which we find

$$\bar{p}_z(z,t) = p_z(z,t) + \rho g \quad (21)$$

$$\bar{p}_t(z,t) = p_t(z,t). \quad (22)$$

This yields the following modified system

$$\bar{p}_t(z,t) = -\frac{\mu}{A_1}q_z(z,t) \quad (23a)$$

$$q_t(z,t) = -\frac{A_1}{\rho}\bar{p}_z(z,t) - \frac{F_1}{\rho}q(z,t) \quad (23b)$$

$$\bar{p}(l,t) = p_l(t) - p_{sp} + \rho g(l - \bar{z}). \quad (23c)$$

Consider now the diagonalizing change of variables

$$\bar{u}(z,t) = \frac{1}{2}\left(q(z,t) + \frac{A_1}{\sqrt{\mu\rho}}\bar{p}(z,t)\right) \quad (24a)$$

$$\bar{v}(z,t) = \frac{1}{2}\left(q(z,t) - \frac{A_1}{\sqrt{\mu\rho}}\bar{p}(z,t)\right) \quad (24b)$$

from which we find

$$\bar{u}_t(z,t) = -\sqrt{\frac{\mu}{\rho}}\bar{u}_z(z,t) - \frac{1}{2}\frac{F_1}{\rho}(\bar{u}(z,t) + \bar{v}(z,t)) \quad (25a)$$

$$\bar{v}_t(z,t) = \sqrt{\frac{\mu}{\rho}}\bar{v}_z(z,t) - \frac{1}{2}\frac{F_1}{\rho}(\bar{u}(z,t) + \bar{v}(z,t)). \quad (25b)$$

We scale the domain into $[0,1]$ by using $x = z/l$ and get rid of the terms $\bar{u}$ in (25a) and $\bar{v}$ in (25b) by defining

$$u(x,t) = \bar{u}(xl,t)e^{\frac{lF_1}{2\sqrt{\mu\rho}}(x-\bar{x})} \quad (26a)$$

$$v(x,t) = \bar{v}(xl,t)e^{-\frac{lF_1}{2\sqrt{\mu\rho}}(x-\bar{x})}, \quad (26b)$$

where (11) has been used. From (26) and (25), we obtain

$$u_t(x,t) = -\frac{1}{l}\sqrt{\frac{\mu}{\rho}}u_x(x,t) - \frac{1}{2}\frac{F_1}{\rho}v(z,t)e^{\frac{lF_1}{\sqrt{\mu\rho}}(x-\bar{x})} \quad (27a)$$

$$v_t(x,t) = \frac{1}{l}\sqrt{\frac{\mu}{\rho}}v_x(x,t) - \frac{1}{2}\frac{F_1}{\rho}u(z,t)e^{-\frac{lF_1}{\sqrt{\mu\rho}}(x-\bar{x})} \quad (27b)$$

which is on the form (5) with the coefficients given by (15) and (18)–(19). Composing the transformations (26), (24) and (20), we find (12).

The connection between $p_l(t)$ and $U(t)$ in (14) is verified by inserting $x = 1$ in (12b) and using (5d). The parameters in the boundary condition (5c) can be expressed by forming

$$u(0,t) + v(0,t)e^{-\frac{lF_1}{\sqrt{\mu\rho}}\bar{x}} = q(0,t)e^{-\frac{lF_1}{2\sqrt{\mu\rho}}\bar{x}} \quad (28)$$

and defining q, C as in (17) and (16), respectively. Lastly, by inserting $x = \bar{x} = \bar{z}/l$ into (12a) and (12b), we obtain

$$u(\bar{x},t) = \frac{1}{2}\left[q(\bar{x}l,t) + \frac{A_1}{\sqrt{\mu\rho}}\left(p(\bar{x}l,t) - p_{sp}\right)\right] \quad (29a)$$

$$v(\bar{x},t) = \frac{1}{2}\left[q(\bar{x}l,t) - \frac{A_1}{\sqrt{\mu\rho}}\left(p(\bar{x}l,t) - p_{sp}\right)\right]. \quad (29b)$$

Hence, the controller objective (4) is achieved if

$$u(\bar{x},t) = v(\bar{x},t), \quad (30)$$

thus, $r = 1$ in (6). ■

Having established that the drilling model admits the form (5a)-(5c), we can apply the results from Section III.

Theorem 3: Consider the MPD system (1). Given a desired setpoint p_{sp} and a chosen coordinate $\bar{z} \in (0, l)$ for pressure attenuation, and let

$$p_l(t) = \frac{\sqrt{\mu\rho}}{A_1}\left(q_l(t) - 2U(t)e^{\frac{\gamma}{2}(1-\bar{x})}\right) + p_{sp} - \rho g l(1-\bar{x}) \tag{31}$$

where $\bar{x} = \bar{z}/l$, $\gamma = \frac{lF_1}{\sqrt{\mu\rho}}$ and $U(t)$ is given by the control law of Theorem 1 with $u(x,t)$ and $v(x,t)$ needed by the control law acquired from $q(z,t)$ and $p(z,t)$ by means of the transformation (12). Then (4) is achieved for

$$t \geq \sqrt{\frac{\rho}{\mu}}(l - \bar{z}). \tag{32}$$

Proof: As the system (1) admits the form (5) following the results of Lemma 2, it will suffice to show that the actuation $p_l(t)$ in (1) relates to the actuation $U(t)$ in (5) according to (31), and that the given time constraint corresponds to the time constraint of Theorem 1. The expression (31) follows trivially from (14) by solving (14) for $p_l(t)$. The time constraint of Theorem 1 is $t \geq \phi_\beta(1) - \phi_\beta(\bar{x})$. For $\epsilon_1(x)$ and $\epsilon_2(x)$ constant and equal as in (18) and using (51b), we find

$$\phi_\beta(1) - \phi_\beta(\bar{x}) = \int_0^1 \frac{d\gamma}{\epsilon} - \int_0^{\bar{x}} \frac{d\gamma}{\epsilon} = \frac{1}{\epsilon}(1-\bar{x}). \tag{33}$$

Inserting for the ϵ in (18) and using (11) we find the desired result. ■

The observer of Section III-B can be implemented for the drilling case simply by defining the measurement as

$$Y(t) = \frac{1}{2}\left[q_l(t) + \frac{A_1}{\sqrt{\mu\rho}}\left(p_l(t) - p_{sp} + \rho g l(1-\bar{x})\right)\right] \times e^{\frac{lF_1}{2\sqrt{\mu\rho}}(1-\bar{x})} \tag{34}$$

which is found by inserting $x = 1$ into (12a).

V. Simulations

We will test the controller of Theorem 3 on the system (1). The system parameters used in the subsequent simulations are

$$\begin{aligned} \mu &= 7317 \cdot 10^5 Pa,\ A_1 = 0.024\ m^2,\ \rho = 1250\ kg/m^3 \\ F_1 &= 10\ kg/m^3 s,\ g = 9.81\ m/s^2,\ A_2 = 0.02\ m^2 \\ l &= 3000\ m,\ p_{sp} = 350 \cdot 10^5\ Pa \\ \omega_1 &= \frac{2\pi}{12}. \end{aligned} \tag{35}$$

Thus, the disturbance is a single harmonic of period 12 seconds, a typical dominant wave period in the North Sea. The chosen depth for pressure rejection is 2000 m which corresponds to $\bar{z} = 1000\ m$. The observer poles were placed at $-0.30 \pm 0.04j$.

To better see the effect of the controller, the system is initially left in open loop with the controller and observer switched on at $t = 20$. By inserting the numerical values into

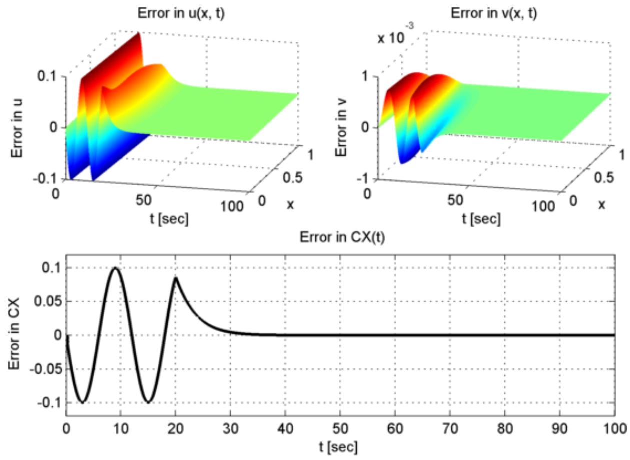

Fig. 2. Observer errors.

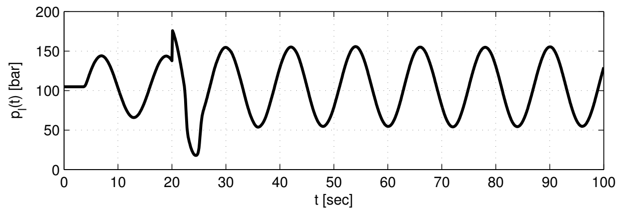

Fig. 3. Controller output for the state feedback implementation.

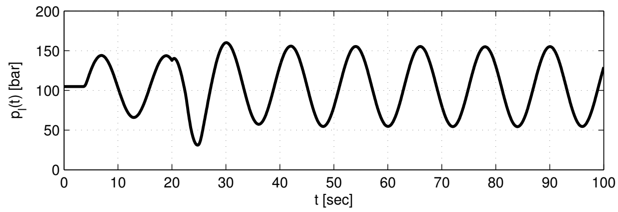

Fig. 4. Controller output for the output feedback implementation.

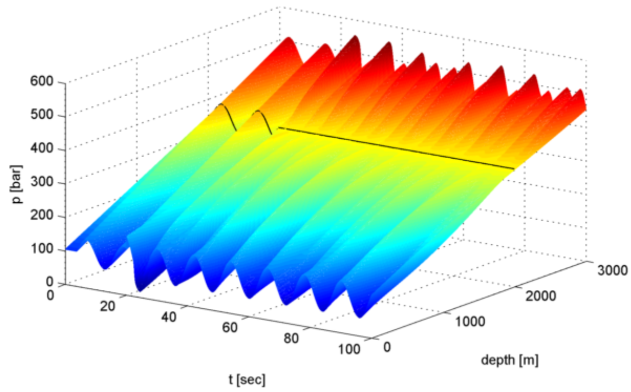

Fig. 5. Pressure distribution throughout the well for the state feedback implementation.

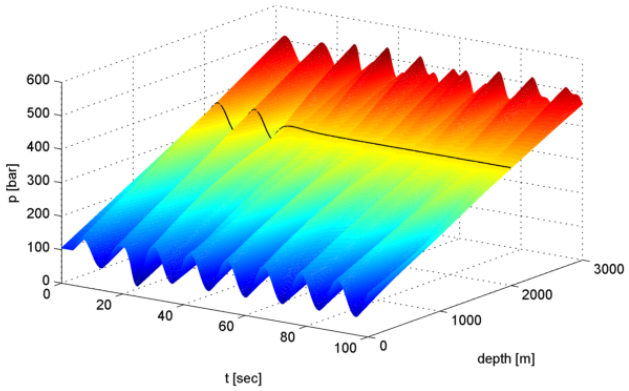

Fig. 6. Pressure distribution throughout the well for the output feedback implementation.

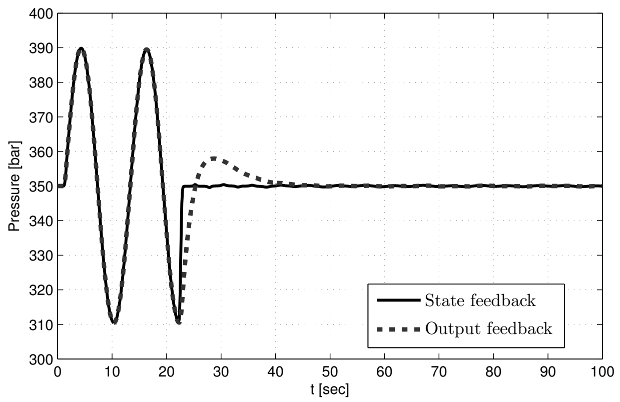

Fig. 7. Pressure distribution at $z = \bar{z}$.

(32), we evaluate the right hand side to be approximately 2.61 seconds.

From Figure 2 it is seen that the observer states start to exponentially converge to their true values as soon as the observer is switched on at $t = 20$ seconds. Figures 3 and 4 show the control signal calculated using Theorem 3 for the state feedback and the output feedback implementations, respectively. Prior to $t = 20$ seconds, $U(t) = 0$, which corresponds to leaving the choke opening at a constant value. The resulting pressure distributions in the well are shown in Figures 5 and 6, respectively. The pressure at $z = \bar{z}$ is highlighted in black in both of these figures. Figure 7 shows the pressure at $z = \bar{z}$ for both the state feedback and the output feedback implementation. A closer look from $t = 30$ seconds is offered in Figure 8.

It is clearly seen that the pressure fluctuations are strongly attenuated at the designated depth. The fluctuations are rejected after approximately 2.6 seconds for the state feedback case, as predicted, while asymptotic convergence is observed for the output feedback case. The fluctuations are reduced from approximately ± 40 bar to less than ± 0.2 bar; a factor of 200. The small fluctuations still present in Figure 8, are due to numerical inaccuracies from the discretization method used in the simulations. Admittedly, this is the ideal case of no noise or modelling errors.

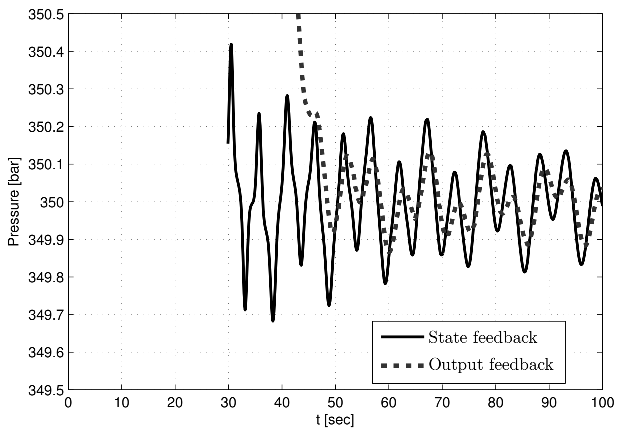

Fig. 8. Pressure distribution at $z = \bar{z}$, zoomed.

VI. Conclusions

We have applied theory previously derived for systems modelled as linear 2×2 partial differential equations of the hyperbolic type to design a control law for rejection of heave-induced pressure fluctuations in managed pressure drilling. The control law was also combined with an observer generating full state estimates from topside measurements. The control law was tested through simulations, and showed significant rejection properties. It remains to investigate robustness to noise and modelling errors.

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Appendix

A. Expressions used in Theorem 1

The backstepping transformation used in [5], was

$$\gamma(x,t) = w(x,t) - \int_0^x K(x,\xi) w(\xi,t) d\xi, \qquad (36)$$

$$w(x,t) = \begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}, \; \gamma(x,t) = \begin{bmatrix} \alpha(x,t) \\ \beta(x,t) \end{bmatrix} \tag{37}$$

where the kernel in (36)

$$K(x,\xi) = \begin{bmatrix} K^{uu}(x,\xi) & K^{uv}(x,\xi) \\ K^{vu}(x,\xi) & K^{vv}(x,\xi) \end{bmatrix} \tag{38}$$

is the solution to

$$0 = \Sigma(x)K_x(x,\xi) + K_\xi(x,\xi)\Sigma(\xi) + K(x,\xi)\Sigma'(\xi) - K(x,\xi)\Pi(\xi) \tag{39a}$$

$$0 = \Sigma(x)K(x,x) - K(x,x)\Sigma(x) + \Pi(x) \tag{39b}$$

$$0 = K(x,0)\Sigma(0)Q_0 \tag{39c}$$

defined over the triangular domain

$$\mathcal{T} = \{(x,\xi) \; : \; 0 \le \xi \le x \le 1\}, \tag{40}$$

with

$$\Sigma(x) = \begin{bmatrix} -\epsilon_1(x) & 0 \\ 0 & \epsilon_2(x) \end{bmatrix}, \; \Pi(x) = \begin{bmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{bmatrix}, \tag{41}$$

$$Q_0 = \begin{bmatrix} 0 & q \\ 0 & 1 \end{bmatrix}. \tag{42}$$

The inverse of (36) is

$$w(x,t) = \gamma(x,t) - \int_0^x L(x,\xi)\gamma(\xi,t)d\xi, \tag{43}$$

where

$$L(x,\xi) = \begin{bmatrix} L^{\alpha\alpha}(x,\xi) & L^{\alpha\beta}(x,\xi) \\ L^{\beta\alpha}(x,\xi) & L^{\beta\beta}(x,\xi) \end{bmatrix} \tag{44}$$

is the solution to

$$0 = \Sigma(x)L_x(x,\xi) + L_\xi(x,\xi)\Sigma(\xi) + L(x,\xi)\Sigma'(\xi) + \Pi(\xi)L(x,\xi) \tag{45a}$$

$$0 = \Sigma(x)L(x,x) - L(x,x)\Sigma(x) + \Pi(x) \tag{45b}$$

$$0 = L(x,0)\Sigma(0)Q_0 \tag{45c}$$

defined over $\mathcal{T}$. Proofs of existence and uniqueness for solutions of (39) and (45) were given in [10], and it was also proved that the solutions are continuous over $\mathcal{T}$.

Further expressions used in the Theorem are

$$K_{psf}(\bar{x}) = \Omega_\alpha(\bar{x},\bar{x}) - r\Phi_\beta(\bar{x},1)e^{A\phi_\beta(1)} + \int_0^{\bar{x}} L^\alpha(\bar{x},\xi)\Omega_\alpha(\xi,\bar{x})d\xi + \int_0^{\bar{x}} L^\beta(\bar{x},\xi)\Omega_\beta(\xi,\bar{x})d\xi \tag{46}$$

$$\delta(\xi,\bar{x},t) = \begin{cases} \delta_\alpha(\xi,\bar{x},t) & \text{if } \kappa_\alpha(\xi,\bar{x}) \le \phi_\alpha(0) \\ \delta_\beta(\xi,\bar{x},t) & \text{otherwise} \end{cases} \tag{47a}$$

$$\delta_\alpha(\xi,\bar{x},t) = \alpha(\phi_\alpha^{-1}(\kappa_\alpha(\xi,\bar{x})),t) \tag{47b}$$

$$\delta_\beta(\xi,\bar{x},t) = q\beta(\phi_\beta^{-1}(\kappa_\alpha(\xi,\bar{x}) - \phi_\alpha(0)),t) \tag{47c}$$

$$\Omega_\alpha(\xi,\bar{x}) = \begin{cases} \Omega_{\alpha\alpha}(\xi,\bar{x}) & \text{if } \kappa_\alpha(\xi,\bar{x}) \le \phi_\alpha(0) \\ \Omega_{\alpha\beta}(\xi,\bar{x}) & \text{otherwise} \end{cases} \tag{48a}$$

$$\Omega_{\alpha\alpha}(\xi,\bar{x}) = \Phi_\alpha(\xi,\phi_\alpha^{-1}(\kappa_\alpha(\xi,\bar{x})))e^{A\kappa_\alpha(\xi,\bar{x})} \tag{48b}$$

$$\Omega_{\alpha\beta}(\xi,\bar{x}) = (q\Phi_\beta(0,\phi_\beta^{-1}(\kappa_\alpha(\xi,\bar{x}) - \phi_\alpha(0))) + C) \times e^{A(\kappa_\alpha(\xi,\bar{x}) - \phi_\alpha(0))} + \Phi_\alpha(\xi,0)e^{A\kappa_\alpha(\xi,\bar{x})} \tag{48c}$$

$$\Omega_\beta(\xi,\bar{x}) = \Phi_\beta(\xi,\phi_\beta^{-1}(\kappa_\beta(\xi,\bar{x})))e^{A\kappa_\beta(\xi,\bar{x})} \tag{48d}$$

$$\kappa_\alpha(\xi,\bar{x}) = \phi_\alpha(\xi) + \phi_\beta(1) - \phi_\beta(\bar{x}) \tag{49a}$$

$$\kappa_\beta(\xi,\bar{x}) = \phi_\beta(\xi) + \phi_\beta(1) - \phi_\beta(\bar{x}) \tag{49b}$$

$$\Phi_\alpha(y,z) = -\epsilon_1(0)\int_{\phi_\alpha(y)}^{\phi_\alpha(z)} K^{uu}(\phi_\alpha^{-1}(\tau),0)Ce^{-A\tau}d\tau \tag{50a}$$

$$\Phi_\beta(y,z) = -\epsilon_1(0)\int_{\phi_\beta(y)}^{\phi_\beta(z)} K^{vu}(\phi_\beta^{-1}(\tau),0)Ce^{-A\tau}d\tau \tag{50b}$$

$$\phi_\alpha(z) = \int_z^1 \frac{d\gamma}{\epsilon_1(\gamma)} \tag{51a}$$

$$\phi_\beta(z) = \int_0^z \frac{d\gamma}{\epsilon_2(\gamma)} \tag{51b}$$

$$L^\alpha(\bar{x},\xi) = L^{\alpha\alpha}(\bar{x},\xi) - rL^{\beta\alpha}(\bar{x},\xi) \tag{52a}$$

$$L^\beta(\bar{x},\xi) = L^{\alpha\beta}(\bar{x},\xi) - rL^{\beta\beta}(\bar{x},\xi). \tag{52b}$$

$$L_o(t,\gamma) = \begin{cases} \frac{1}{q}L^\beta\left(\bar{x},\phi_\beta^{-1}(\gamma - t + d_2)\right)\epsilon_2\left(\phi_\beta^{-1}(\gamma - t + d_2)\right) \\ \qquad \text{for } \gamma \in (t - d_2, t], \\ L^\alpha\left(\bar{x},\phi_a^{-1}(\gamma - t + d_1)\right)\epsilon_1\left(\phi_a^{-1}(\gamma - t + d_1)\right) \\ \qquad \text{for } \gamma \in (t - d_3, t - d_2], \end{cases} \tag{53}$$

$$d_1 = \phi_\alpha(0) - \phi_\beta(\bar{x}), \; d_2 = \phi_\beta(\bar{x}), \tag{54}$$

$$d_3 = \phi_\alpha(0) - \phi_\alpha(\bar{x}) + \phi_\beta(\bar{x}) \tag{55}$$

B. Expressions used in the observer equations

The observer gains are given as

$$p_1(x) = Ce^{A\phi_\alpha(x)}L - \epsilon_1(1)P^{uu}(x,1) - \int_x^1 P^{uu}(x,1)Ce^{A\phi_\alpha(\xi)}Ld\xi \tag{56a}$$

$$p_2(x) = -\epsilon_1(1)P^{vu}(x,1) - \int_x^1 P^{vu}(x,1)Ce^{A\phi_\alpha(\xi)}Ld\xi \tag{56b}$$

where the kernels are the solution to[1]

$$\epsilon_1(x)P_x^{uu}(x,\xi) + \epsilon_1(\xi)P_\xi^{uu}(x,\xi) = -\epsilon_1'(\xi)P^{uu}(x,\xi) + c_1(x)P^{vu}(x,\xi) \tag{57a}$$

$$\epsilon_2(x)P_x^{vu}(x,\xi) - \epsilon_1(\xi)P_\xi^{vu}(x,\xi) = \epsilon_1'(\xi)P^{vu}(x,\xi) - c_2(x)P^{uu}(x,\xi) \tag{57b}$$

with boundary conditions

$$P^{uu}(0,\xi) = qP^{vu}(0,\xi) \tag{58a}$$

$$P^{vu}(x,x) = -\frac{c_2(x)}{\epsilon_1(x) + \epsilon_2(x)} \tag{58b}$$

defined over the triangular domain

$$\mathcal{T}_0 = \{(x,\xi) \; : \; 0 \le x \le \xi \le 1\}. \tag{59}$$

It was in [10] proved that there exists a unique solution to (57)-(58), and that the solution is continuous over $\mathcal{T}_0$.

[1] Apparently, the original kernel equations stated in [10] contained some typos. The kernel equations stated here are the ones found in [4].