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Graph Decomposition Based Design and Analysis of Consensus Protocols ⋆

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Abstract: This paper studies the design and analysis problem of nonlinear consensus protocols. First, the communication graph is decomposed into the cascade connection form of the strongly connected components which reveals the agents' different roles in the process of reaching consensus. Then, based on this property, a design and analysis approach is proposed with the aid of the input-to-state stability theory. With this approach, consensus of LTI agents with different linear protocol parameters, and consensus of single-integrator agents with a class of nonlinear protocols are specifically discussed. Finally, applications are given to show the feasibility of the proposed approach.

Keywords: Consensus; Nonlinear protocol; Graph decomposition; Input-to-state stability.

1. INTRODUCTION

In the past decade, the consensus problem of multi-agent systems has been extensively studied due to its close relevance in diverse research topics, such as birds flocking, clock synchronization, and formation of autonomous vehicles, and the related results could be found in the literature, e.g., Olfati-Saber et al. [2007], Arcak [2007], Tian [2012], and Cao et al. [2013].

In the research of consensus, the essential problem is the design and analysis of the consensus protocols. As far as the authors know, the research modes could be generally classified into two types. One is constructing a specific protocol based on the physical knowledge first, and then analyzing its convergence or deriving the condition of the convergence. This research mode is adopted in the most literature, and especially in the case of studying consensus of LTI agents with linear protocols. For the case of nonlinear consensus, a typical example is the well-known Vicsek model which is proposed to describe the phase evolution of particles [Vicsek et al., 1995]. Also using this method, Olfati-Saber and Murray [2003] considered a class of nonlinear protocols satisfying locally passive condition for single-integrator agents, and then proved the effectiveness under the undirected topology. Inspired by the thermodynamic principle, Qing and Haddad [2008] presented another class of nonlinear protocols, and then fulfilled the proof of convergence. In addition, the results involving finite-time convergence [Cortes, 2006] and connectivity preservation [Ji and Egerstedt, 2005] also belong to this research mode. It should be noted that most results of this research mode are for the undirected topology where the Lyapunov function could be relatively easily found.

Differently, the other research mode is describing the required property of the protocol function directly which guarantees the convergence of the closed-loop system, rather than giving a specific form of the protocol. Therefore, the derived protocols using this mode is more of the mathematical meaning, and is abstract to some extent. For discrete-time single-integrator systems, Moreau [2005] presented a convex condition: if the function field of each agent's consensus law is strictly located in the interior of the convex hull formed by the agent's state and its neighbors' states, then consensus of the collective systems could be achieved. Lin et al. [2007] extended the result to the case of the continuous-time single-integrator agents using the notion of the tangent cone condition. It's worth pointing out that this research mode deals with all types of communication topologies identically, not differentiating whether the topology is undirected or directed. Based on the graph decomposition, Xu and Tian [2013] proposed an ISS approach to the design of a class of nonlinear consensus protocols. This paper further studies the problem of analysis and design of linear and nonlinear consensus protocols in the second research mode. Throughout the proposed protocol design and analysis procedure, the graph decomposition and the input-to-state stability theory are two most crucial tools. Moreover, we specifically investigate the problem of designing different linear protocol parameters for consensus of LTI agents, and the problem of analyzing nonlinear consensus of single-integrator agents with a class of nonlinear consensus.

The remainder of the paper is organized as follows. The preliminary and the problem formulation are given in Section 2. The main results are presented in Section 3, and are followed by the application in Section 4. We conclude the paper in Section 5.

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2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Notations

The directed graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$, is usually used to describe the interaction topology. The vertex set $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \cdots, v_n\}$, represents agent $1, 2, \cdots, n$, respectively, and the edge set $\mathcal{E}(\mathcal{G}) \subset \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$ models the information flow. There exists an edge $(v_i, v_j) \in \mathcal{E}(\mathcal{G})$ if and only if agent j can obtain information from agent i, and agent i is said as a neighbor of agent j. We denote by N_j the set of all neighbors of agent j. A directed path of length k from v_{j_0} to v_{j_k} is an ordered set of distinct nodes$\{v_{j_0}, v_{j_1}, \cdots, v_{j_k}\}$ such that $(v_{j_{i-1}}, v_{j_i}) \in \mathcal{E}(\mathcal{G}), \forall i = 1, 2, \cdots, k$. A directed graph has a spanning tree if there exists at least one node which has directed paths to all other nodes, and such node is called the root node. If each node is a root node, this graph is said to be strongly connected. The graph $\mathcal{G}$ is usually accompanied with a weighted adjacent matrix A (or $A(\mathcal{V})$) which is defined such that $a_{ij} > 0$ if and only if $(v_j, v_i) \in \mathcal{E}(\mathcal{G})$, otherwise $a_{ij} = 0$. Define the Laplacian matrix L (or $L(\mathcal{V})) = [l_{ij}] \in R^{n\times n}$ as $l_{ij} = -a_{ij}$ when $i \neq j$, otherwise $l_{ij} = \sum_{j=1, j\neq i}^{n} a_{ij}$. Therefore, 0 is an eigenvalue of L with the associated eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is the $n \times 1$ column with each element 1. Furthermore, 0 is a simple eigenvalue of L and all other eigenvalues have positive real parts if and only if the directed graph has a spanning tree [Ren and Beard, 2005]. Denote $w = [w_1, \cdots, w_n]$ the left eigenvector of L. If the graph is strongly connected, each component of w is greater than 0 [Berman and Plemmons, 1979].

Other notations used in this paper are quite standard. For a symmetric matrix $Q \in R^{n\times n}$, denote by $\lambda(Q), \lambda_{min}(Q)$ the eigenvalue set and the minimum eigenvalue of Q, respectively. Denote by M_{max} the largest element of an arbitrary real matrix M. For $x \in R^n$, define $f(x) \triangleq [f(x_1), f(x_2), \cdots, f(x_n)]^T$.

2.2 Input-to-state Stability (ISS)

Consider the system

$$\dot{x} = f(t, x, u), \tag{1}$$

where $f : [t_0, \infty) \times R^n \times R^m \to R^n$ is piecewise continuous in t and locally Lipschitz in x and u. The input $u(t)$ is a piecewise continuous, bounded function of t for all $t \geq t_0$.

Definition 1. (Khalil [2002]). The system (1) is said to be input-to-state stable if there exists a class $\mathcal{KL}$ function [1] β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) + \gamma \left(\sup_{t_0 \leq \tau \leq t} ||u(\tau)|| \right). \tag{2}$$

[1] A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Lemma 1. (Khalil [2002]). Let $V : [0, \infty) \times R^n \to R$ be a continuously differentiable function such that

$$\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||), \tag{3}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W_3(x), \quad \forall\, ||x|| \geq \rho(||u||) > 0, \tag{4}$$

$\forall\, (t, x, u) \in [0, \infty) \times R^n \times R^m$, where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, and W_3 is a continuous positive definite function on R^n. Then the system (1) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Consider the cascade system

$$\dot{x}_1 = f_1(t, x_1, x_2), \tag{5}$$

$$\dot{x}_2 = f_2(t, x_2), \tag{6}$$

where $f_1 : [0, \infty) \times R^{n_1} \times R^{n_2} \to R^{n_1}$ and $f_2 : [0, \infty) \times R^{n_2} \to R^{n_2}$ are piecewise continuous in t and locally Lipschitz in $x = [x_1^T, x_2^T]^T$.

Lemma 2. (Khalil [2002]). If the system (5), with x_2 as input, is input-to-state stable and the origin of (6) is globally uniformly asymptotically stable, then the origin of the cascade system (5) and (6) is globally uniformly asymptotically stable.

2.3 Problem Formulation

Consider the agent described by the following equation

$$\dot{X}_i(t) = G(t, X_i, U_i), \tag{7}$$

where $X_i \in R^l$ is the state variable and $U_i \in R^m$ is the control protocol. The paper is aimed to propose an approach to designing a distributed protocol under the directed communication topology,

$$U_i(t) = H_i(t, X_i, X_j)|_{j\in N_i}, \quad i = [1, n]. \tag{8}$$

such that,

$$\lim_{t\to\infty} (X_i - X_j) = 0, \quad i, j = [1, n]. \tag{9}$$

or to analyzing whether (9) holds with the given protocol (8).

3. DESIGN AND ANALYSIS OF CONSENSUS PROTOCOLS

In this section, a design and analysis procedure is formulated based on graph decomposition which could remove the difficulty of finding the Lyapunov function under the directed topology directly. Then with this procedure, different linear protocol parameters are designed for LTI agents. Moreover, consensus of single-integrator agents is analyzed with a class of nonlinear consensus protocols.

3.1 Graph Decomposition Based Design and Analysis Procedure

Lemma 3. ([Fang, 2009]). Based on the notion of the strongly connected component, the vertex set of a directed graph could be classified into several equivalent classes, and each node vertex is located in and only in one of these sets.

The specific procedure was demonstrated by Xu and Tian [2013], and the nodes could be renumbered so that the

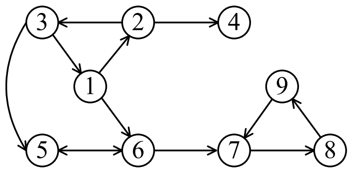

Fig. 1. A communication topology.

adjacent matrix is block lower triangular. The following is an example.

For the directed graph in Fig. 1, the nodes are classified as:

$\mathcal{V}_1 = \{v_1, v_2, v_3\}$;

$\mathcal{V}_2^1 = \{v_4\}, \quad \mathcal{V}_2^2 = \{v_5, v_6\}, \quad \mathcal{V}_2 = \mathcal{V}_2^1 \cup \mathcal{V}_2^2$;

$\mathcal{V}_3^1 = \emptyset, \mathcal{V}_3^2 = \{v_7, v_8, v_9\}, \quad \mathcal{V}_3 = \mathcal{V}_3^1 \cup \mathcal{V}_3^2 = \mathcal{V}_3^2$,

and the adjacent matrix is:

$$A = \begin{pmatrix} 0&0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&0&0 \\ 1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&1 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{pmatrix}.$$

Remark 1. Graph decomposition reveals such a fact that the agents are in block cascade connections. Some agents determine the final consensus state which could be called root agents or group leaders, while other agents assume the task of tracking, either in an individual manner or in a group manner.

Lemma 4. ([Xu and Tian, 2013]). Suppose that the communication topology is static and has a spanning tree. Then, the system (7) with distributed protocol (8) can reach consensus if the following two conditions are satisfied:

(a) the subsystem consisting of all root nodes can reach consensus globally asymptotically,

(b) any other subsystem, with the states of systems which it follows as input, is ISS.

Hence, we propose the following procedure for design and analysis of consensus protocols.

Design and Analysis Procedure

Step 1: check or ensure that the root agent system could reach consensus with the protocol. This is also the necessary condition of reaching consensus.

Step 2: check or ensure that any other tracking agent system, with the states of the agent system it tracks as input, is ISS.

Remark 2. The above procedure indicates that the uniform protocol for all agents is not necessary, and different protocols could be designed for agents belonging to different strongly connected components.

Lin et al. [2007] proposed the tangent cone condition to design the consensus law. In some cases, the results obtained are different by using these two methods. For example, we consider a trivial consensus problem – tracking. The plant studied is the single-integrator system in the space of dimension 2: $\dot{x}_2 = u_2 = f(x_1, x_2)$, where $x_1 \in R^2$ is the state of the static leader. Consider the linear consensus protocol $u_2 = kR(\theta)(x_1 - x_2) = -kR(\theta)x_2 + kR(\theta)x_1$, and $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the rotation matrix. According to Lin et al.'s tangent cone condition, θ should be zero (see Fig. 2). In fact, any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ makes the protocol work which can obtained by our ISS condition (see Fig. 3).

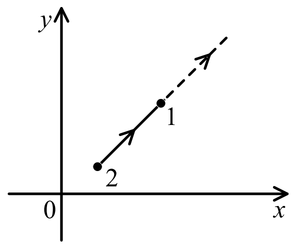

Fig. 2. Illustration of the tangent cone condition.

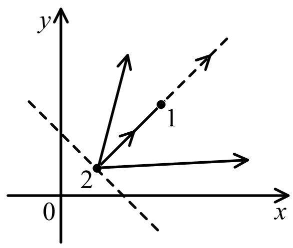

Fig. 3. Illustration of the ISS condition.

3.2 Linear Consensus Protocol with Different Parameters

The proposed procedure could also be used to discuss the linear consensus problem. Consider the LTI multi-agent systems

$$\dot{X}_i = AX_i + BU_i, \tag{10}$$

where $X_i \in R^n, U_i \in R^m$ are the state and the input, and $i \in [1, n]$. In the present literature, all agents choose the same linear protocol parameters. Here we assume that different protocol parameters may be chosen for different agent groups, that is

$$U_i = K_i \sum_{j=1}^{N} a_{ij}(X_j - X_i). \tag{11}$$

As introduced in Subsection 3.1, the Laplacian matrix could be written as

$$L = \begin{bmatrix} L_{11} & 0 & 0 & 0 & 0 & 0 \\ L_{21}^1 & L_{22}^1 & 0 & 0 & 0 & 0 \\ L_{21}^2 & 0 & L_{22}^2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ L_{P1}^1 & L_{P2}^1 & \cdots & \cdots & L_{PP}^1 & 0 \\ L_{P1}^2 & L_{P2}^2 & \cdots & \cdots & 0 & L_{PP}^2 \end{bmatrix}.$$

For expression brevity, we assume the dimensions of the matrix L_{11}, L_{pp}^1, L_{pp}^2 are $(m_1-1)\times(m_1-1)$, 1×1, $m_{p2}\times m_{p2}$, $p \in [2, P]$, respectively. Moreover, we unfold L_{11} as $L_{11} = \begin{bmatrix} l_{11} & \alpha^T \\ \beta & \tilde{L}_{11} \end{bmatrix}$. Denote the Laplacian matrix of the induced subgraph $\mathcal{G}_{22} = (\mathcal{V}_{22}, \mathcal{E}_{22})$ by L_{22}, and define $\text{diag}\{B\} = \sum_{j=1}^{m_1-1} \text{diag}\{b_j\}$ where $B = [b_1, \cdots, b_{m_1-1}] \in R^{m_{22}\times(m_1-1)}$, then we have $L_{22}^2 = L_{22} - \text{diag}\{L_{21}^2\}$. Moreover, define the consensus error $X_{i1} = X_i - X_1, i \in [2, n]$, and $X_{[2,n]1} = [X_{21}^T, \cdots, X_{n1}^T]^T$.

Firstly, consider the agents in the node set $\mathcal{V}_1$. Choose the protocol parameter $K_{\mathcal{V}_1}$, then the closed-loop error system could be written as

$$\dot{X}_{[2,m_1-1]1} = [I_{m_1-2} \otimes A - (\tilde{L}_{11} + \mathbf{1}_{m_1-2}\alpha^T) \otimes BK_{\mathcal{V}_1}]X_{[2,m_1-1]1}. \tag{12}$$

It has been proved in [Seo et al., 2009] that the system (12) is asymptotically stable if and only if the parameter $K_{\mathcal{V}_1}$ stabilizes $m_1 - 2$ subsystems $\dot{\tilde{X}} = (A - \lambda_i BK_{\mathcal{V}_1})\tilde{X}, i = [1, m_1 - 2]$ where λ_i is the eigenvalue of the matrix $\tilde{L}_{11} + 1_{m-1}\alpha^T$, and $\lambda\{\tilde{L}_{11} + I_{m-1}\alpha^T\} = \lambda\{L_{11}\}\backslash\{0\}$. Therefore, the choice of $K_{\mathcal{V}_1}$ is only relevant with L_{11} apart from A and B.

Then, consider the agents in the node set $\mathcal{V}_{21}$. Choose the protocol parameter $K_{\mathcal{V}_{21}}$, then we have

$$\dot{X}_{m_11} = \left(A - BK_{\mathcal{V}_{21}}\sum_{j=1}^{m_1} a_{m_1j}\right)X_{m_11} + BK_{\mathcal{V}_{21}} \cdot \sum_{j=1}^{m_1} a_{m_1j}X_{j1} - BK_{\mathcal{V}_1}\sum_{j=1}^{m_1-1} a_{1j}X_{j1}. \tag{13}$$

The ISS condition of the system (13) is equivalent to the Hurwitz condition of the matrix $A - BK_{\mathcal{V}_{21}}\sum_{j=1}^{m_1} a_{m_1j}$. Therefore, the choice of $K_{\mathcal{V}_{21}}$ is only relevant with L_{21}^1, and has no relationship with L_{11}. In other words, the design of the parameters $K_{\mathcal{V}_1}$ and $K_{\mathcal{V}_{21}}$ is independent.

Next, consider the agents in the node set $\mathcal{V}_{22}$. Choose the protocol parameter $K_{\mathcal{V}_{22}}$, then

$$\begin{aligned}&\dot{X}_{[m_1+1,m_1+m_{22}]1}\\ &= \left(I_{m_{22}} \otimes A - L_{22}^2 \otimes BK_{\mathcal{V}_{22}}\right)X_{[m_1+1,m_1+m_{22}]1}\\ &\quad - \left(L_{21}^2 \otimes BK_{\mathcal{V}_{22}}\right)X_{[1,m_1-1]1}\\ &\quad - \left(1_{m_{22}} \otimes \alpha^T \otimes BK_{\mathcal{V}_1}\right)X_{[2,m_1-1]1}.\end{aligned} \tag{14}$$

The ISS condition of the system (14) is equivalent to the Hurwitz condition of the matrix $I_{m_2} \otimes A - L_{22}^2 \otimes K_{\mathcal{V}_{22}}$. This requires the parameter $K_{\mathcal{V}_{22}}$ to stabilize simultaneously m_{22} subsystems $\dot{\tilde{X}} = (A - \lambda_i BK_{\mathcal{V}_{22}})\tilde{X}$. Here λ_i, $i \in [1, m_{22}]$ is the eigenvalue of the matrix $L_{22}^2 = L_{22} - \text{diag}\{L_{21}^2\}$. Therefore, the choice of $K_{\mathcal{V}_{22}}$ is only relevant with L_{21}^2 and L_{22}, and has no relationship with L_{11}. In other words, the design of the parameters $K_{\mathcal{V}_1}$ and $K_{\mathcal{V}_{22}}$ is independent.

Same conclusions could be drawn on the design of protocol parameters $K_{\mathcal{V}_{31}}$, $K_{\mathcal{V}_{32}}, \cdots, K_{\mathcal{V}_{P1}}$, $K_{\mathcal{V}_{P2}}$. Therefore, it's not necessary to select a uniform protocol parameter which was widely adopted in the literature, and the design of different parameters may result in a less conservative result. Readers could refer to Ma and Zhang [2010] for an explicit selection procedure of the parameters $K_{\mathcal{V}_{i1}}$, $K_{\mathcal{V}_{i2}}$ which was based on the Riccati equation method.

3.3 Nonlinear Consensus for Single-integrator Systems

According to sequence in which the nonlinear function applies, there are three typical nonlinear consensus protocols for single-integrator systems, namely,

$$u_i(t) = \sum_{j=1}^{N} a_{ij}f(x_j(t) - x_i(t)), \tag{15}$$

$$u_i(t) = f\left(\sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t))\right), \tag{16}$$

$$u_i(t) = \sum_{j=1}^{N} a_{ij}[f(x_j(t)) - f(x_i(t))]. \tag{17}$$

The protocol (15) and (16) have been studied by Olfati-Saber and Murray [2003], and Xu and Tian [2013], respectively. Here we discuss the protocol (17).

Assumption 1: The function $f : R \longrightarrow R$ is an increasing and odd function.

For $x^* \in R$ and f satisfying Assumption 1, define a new function $g : R \longrightarrow R, g(x) = f(x + x^*) - f(x^*)$. Moreover, define the mappings $[0, \infty) \longrightarrow R : \rho_1(r) = \min\{-g(-r), g(r)\}, \rho_2(r) = \max\{-g(-r), g(r)\}, \rho_3(r) = \rho_1^{-1}[k\rho_2(r)]$ where $k > 0$. Then we have the following result,

Lemma 5. (1) $g(r)$ is strictly increasing, and $g(0) = 0$;
(2) $\rho_1(r), \rho_2(r), \rho_3(r)$ are $\mathcal{K}$ functions.

Theorem 6. consider the single-integrator system

$$\dot{x}_i = u_i, i \in [1, n] \tag{18}$$

and the nonlinear protocol

$$u_i(t) = \sum_{j=1}^{n} a_{ij}[f(x_j(t)) - f(x_i(t))]. \tag{19}$$

If the communication topology has a spanning tree, and the nonlinear function f satisfies Assumption 1, then consensus can be reached among the multi-agent systems (18)-(19).

Proof. The proof of the theorem consists of three steps.

Step 1: consider the subsystems corresponding to the nodes in $\mathcal{V}_1$. Choose the Lyapunov function $V_1 = \sum_{i=1}^{m_1-1} w_i \int_0^{x_i} f(\tau)\,d\tau$, then

$$\begin{aligned}\dot{V}_1 &= \sum_{i=1}^{m_1-1} w_i f(x_i)\dot{x}_i\\ &= \sum_{i=1}^{m_1-1} w_i f(x_i)\sum_{j=1}^{m_1-1} a_{ij}(f(x_j) - f(x_i))\\ &= -f^T(x)\,\text{diag}\{w\}L(\mathcal{V}_1)f(x)\\ &= -\frac{1}{2}f^T(x)[\,L^T(\mathcal{V}_1)\text{diag}\{w\} + \text{diag}\{w\}L(\mathcal{V}_1)]f(x)\end{aligned}$$

Since the topology composed of the root nodes is strongly connected, the matrix $L^T(\mathcal{V}_1)\text{diag}\{w\} + \text{diag}\{w\}L(\mathcal{V})$ corresponds to a connected undirected topology. Therefore $\dot{V}_1 \le 0$, and $\dot{V} \equiv 0 \Rightarrow f(x) = c\mathbf{1} \Rightarrow x = f^{-1}(c)\mathbf{1} \triangleq x^*\mathbf{1}$.

By LaSalle's Invariance Principle, the subsystems will converge to the largest invariant set $M = \{x|\dot{V}_1(x) \equiv 0\} = \{x|x = x^*\mathbf{1}, x^* \in R\}$. Furthermore, it is easy to check that x^* is a constant. This indicates that root agents reach the static consensus.

Step 2: consider the subsystems corresponding to the nodes in $\mathcal{V}_2$. Define $y_i = x_i - x^*, i \in [1, m_1 + m_{22}]$, and denote $y(\mathcal{V}_1) = [y_1, \cdots, y_{m_1-1}], y(\mathcal{V}_{22}) = [y_{m_1+1}, \cdots, y_{m_1+m_{22}}]^T$, then

$$\begin{aligned}\dot{y}_i &= \sum_{j=1}^{m_1+m_{22}} a_{ij}[f(x_j) - f(x^*) - f(x_i) + f(x^*)] \\ &= \sum_{j=1}^{m_1+m_{22}} a_{ij}[g(y_j) - g(y_i)].\end{aligned}$$

Firstly, consider the subsystems corresponding to the nodes in $\mathcal{V}_{21}$. Choose the Lyapunov function $V_{21} = \int_0^{y_{m_1}} g(\tau)\, d\tau$, then we have

$$\begin{aligned}\dot{V}_{21} &= g(y_{m_1})\dot{y}_{m_1} \\ &= g(y_{m_1}) \sum_{j=1}^{m_1} a_{m_1 j}[g(y_j) - g(y_{m_1})] \\ &= -\sum_{j=1}^{m_1} a_{m_1 j} g(y_{m_1})^2 + g(y_{m_1}) \sum_{j=1}^{m_1} a_{m_1 j} g(y_j) \\ &\le -(1-\delta)\sum_{j=1}^{m_1} a_{m_1 j} g(y_{m_1})^2 - \delta \sum_{j=1}^{m_1} a_{m_1 j} \cdot g(y_{m_1})^2 \\ &\quad + m_1 a_{max}\, |g(y_{m_1})| \cdot ||g(y(\mathcal{V}_1))||_\infty,\end{aligned}$$

where $0 < \delta < 1$. If

$$|y_{m_1}| \ge \rho_1^{-1}\left[\frac{m_1 a_{max}}{\delta \sum_{j=1}^{m_1} a_{m_1 j}} \rho_2(||y(\mathcal{V}_1)||_\infty)\right] \triangleq \rho(||y(\mathcal{V}_1)||_\infty),$$

then we have

$$|g(y_{m_1})| \ge \frac{m_1 a_{max}}{\delta \sum_{j=1}^{m_1} a_{m_1 j}} ||g(y(\mathcal{V}_1))||_\infty,$$

and therefore,

$$\dot{V}_{21} \le -(1-\delta)\sum_{j=1}^{m_1} a_{m_1 j} g(y_{m_1})^2.$$

By Lemma 1, the subsystem is ISS. Moreover, since the $y(\mathcal{V}_1)$ is globally asymptotically stable, the cascade systems $y(\mathcal{V}_1)$ and y_{m_1} are also globally asymptotically stable according to Lemma 2. This indicates that consensus could be reached among agent systems corresponding to the nodes in $\mathcal{V}_1 \cup \mathcal{V}_{21}$.

Next, consider the subsystems corresponding to the nodes in $\mathcal{V}_{22}$. Choose the Lyapunov function $V_{22} = \sum_{i=m_1+1}^{m_1+m_{22}} \int_0^{y_i} g(\tau)\, d\tau$, then

$$\begin{aligned}\dot{V}_{22} &= \sum_{i=m_1+1}^{m_1+m_{22}} w_i g(y_i)\dot{y}_i \\ &= \sum_{i=m_1+1}^{m_1+m_{22}} w_i g(y_i)\left(\sum_{j=1}^{m_1+m_{22}} a_{ij}[g(y_j) - g(y_i)]\right) \\ &= \sum_{i=m_1+1}^{m_1+m_{22}} \left[w_i g(y_i)\left(\sum_{j=m_1+1}^{m_1+m_{22}} a_{ij}[g(y_j) - g(y_i)] \right.\right. \\ &\quad \left.\left. - \sum_{j=1}^{m_1-1} a_{ij} g(y_i)\right) + w_i g(y_i) \sum_{j=1}^{m_1-1} a_{ij} g(y_j)\right] \\ &= -g^T(y(\mathcal{V}_{22}))\left(\frac{1}{2}L(\mathcal{V}_{22})^T \mathrm{diag}\{w\} + \frac{1}{2}\mathrm{diag}\{w\}L(\mathcal{V}_{22}) \right. \\ &\quad \left. + \mathrm{diag}\{w\} \cdot \mathrm{diag}\{-L_{21}^2\}\right) g(y(\mathcal{V}_{22})) \\ &\quad + \sum_{i=m_1+1}^{m_1+m_{22}} \left(w_i g(y_i) \sum_{j=1}^{m_1-1} a_{ij} g(y_j)\right) \\ &\le -\lambda_{min}(Q)(1-\delta)k_0^2 ||g(y(\mathcal{V}_{22})\,||_\infty^2 \\ &\quad - \lambda_{min}(Q)\delta\, k_0^2 ||g(y(\mathcal{V}_{22}))||_\infty^2 + \\ &\quad (m_1 - 1)m_{22} a_{max} w_{max} ||g(y(\mathcal{V}_{22}))||_\infty ||g(y(\mathcal{V}_1))||_\infty,\end{aligned}$$

where $0 < \delta < 1$, and $Q = \frac{1}{2}L(\mathcal{V}_{22})^T \mathrm{diag}\{w\} + \frac{1}{2}\mathrm{diag}\{w\}L(\mathcal{V}_{22}) + \mathrm{diag}\{w\} \cdot \mathrm{diag}\{-L_{21}^2\}$. We utilize the norm equivalence property to obtain the last inequality, and $k_0 > 0$ is the corresponding coefficient. If

$$||y(\mathcal{V}_{22})||_\infty \ge \rho_1^{-1}\left[\frac{(m_1-1)m_{22} a_{max} w_{max}}{\lambda_{min}(Q)\delta\, k_0^2} \rho_2(||y(\mathcal{V}_1)||_\infty)\right],$$

then we have

$$||g(y(\mathcal{V}_{22}))||_\infty \ge \frac{(m_1-1)m_{22} a_{max} w_{max}}{\lambda_{min}(Q)\delta\, k_0^2} ||g(y(\mathcal{V}_1))||_\infty,$$

and therefore,

$$\dot{V}_{22} \le -\lambda_{min}(Q)(1-\delta)k_0^2 ||g(y(\mathcal{V}_{22}))||_\infty^2,$$

By Lemma 1, the subsystem $y(\mathcal{V}_{22})$ is ISS. Moreover, since the subsystems $y(\mathcal{V}_1)$ is globally asymptotically stable, the cascade systems $y(\mathcal{V}_1)$ and $y(\mathcal{V}_{22})$ are also globally asymptotically stable according to Lemma 2. This indicates that consensus could be reached among agent systems corresponding to nodes in the set $\mathcal{V}_1 \cup \mathcal{V}_{22}$.

Combining the above results, we can conclude that consensus could be reached among agent systems corresponding to nodes in the set $\mathcal{V}_1 \cup \mathcal{V}_2$.

Step 3: by induction, we can get the conclusion that the whole multi-agent systems reach consensus. This completes the proof of the theorem.

4. APPLICATIONS

In this section, we provide two numerical examples to demonstrate the applications of our proposed approach.

4.1 Different Protocol Parameters for double-integrator systems

Consider the linear protocol

$$u_i = \sum_{j=1}^{N} a_{ij}[k_1(x_j - x_i) + k_2(\dot{x}_j - \dot{x}_i)],$$

for the double-integrator systems

$$\ddot{x}_i = u_i,$$

and the interaction graph is shown in Fig. 1. Let $k_1 = 1$, then the feasible range of the parameter k_2 could be calculated using the method in Ma and Zhang [2010] and given in Table 1 . This example shows that it's not

Table 1. Feasible range of k_2.

	uniform k_2	different k_2
Agent 1-3	$k_2 > 0.4082$	$k_2 > 0.4082$
Agent 4		$k_2 > 0$
Agent 5-6		$k_2 > 0$
Agent 7-9		$k_2 > 0.2691$

necessary to choose the uniform parameter for all agents, and less conservative results may be obtained by designing different protocol parameters.

4.2 Application of Nonlinear Consensus Protocol

We consider the consensus protocol (17) with the nonlinear function $f(x) = \tanh(x)$. The communication topology is shown in Fig. 1. Fig. 4 and Fig. 5 demonstrate the time response of the state and the control, respectively. Actually, if the agent amount is known, the nonlinear consensus protocol (17) with $f(x) = \tanh(x)/2n$ could bound the magnitude of the control input less than 1.

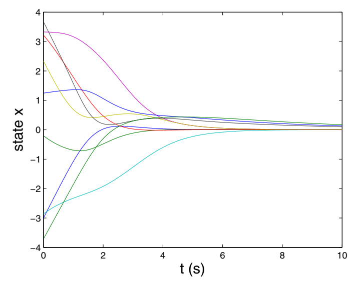

Fig. 4. Time response of the state.

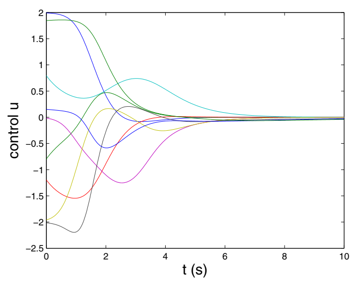

Fig. 5. Time response of the control.

5. CONCLUSION

In this paper, a hierarchical design and analysis approach for consensus of multi-agent systems was proposed with the aid of the graph decomposition and ISS theory. Using this approach, we specifically studied the design of different linear protocol parameters for LTI agents, and analyzed consensus of single-integrator agents with a class of nonlinear protocols.

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