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Striped Parameterized Tube Model Predictive Control

Diego Muñoz-Carpintero *,[1] **Basil Kouvaritakis** * **Mark Cannon** *

* *Department of Engineering Science, University of Oxford, Oxford, OX1 3PJ, UK (e-mail: diego.munozcarpintero@eng.ox.ac.uk).*

Abstract: Optimal robust MPC for constrained linear systems that are subject to additive uncertainty requires a closed loop optimization with computation that rises exponentially with the prediction horizon, N. This was overcome by parameterized tube MPC (PTMPC) that used a triangular separable prediction strategy in which predictions are decomposed into a nominal and $N-1$ partial prediction sequences, each concerned with the compensation of a particular future disturbance. Beyond the first N steps, the PTMPC control law can be taken to be a static state feedback. Use of $N-1$ partial prediction sequences causes the number of free variables and constraints to rise quadratically with N. To reduce this to linear, this paper proposes a modification of PTMPC that uses a single partial prediction sequence that is used in connection with all particular future disturbances but allows the compensatory effect of the partial prediction sequences to extend beyond the first N predictions. The prediction structure is of a triangular striped nature and extends over an infinite horizon. As illustrated by a simulation example, the resulting scheme can outperform PTMPC in terms of the size of the domain of attraction and affords a reduction in computation which allows for the use of longer horizon (for the same computational demand).

Keywords: Predictive control; Uncertain linear systems; Bounded disturbances; Robust Control; Constraints

1. INTRODUCTION

Robust model predictive control (RMPC) of linear systems subject to additive uncertainty is an imporant area of research. Optimal RMPC depends on the solution of dynamic programming [Bersekas, 1995] or minimax problems [Scokaert and Mayne, 1998] and is computationally intractable. Thus it is sensible to look for compromises between sub-optimality and complexity. Early RMPC considered open-loop solutions, where a single sequence of inputs is determined for all distubance realizations. This is conservative because it does not acknowledge the fact that future information of the states and uncertainties will be known to, and available for use by the controller and can thus result in poor performances and unfeasibility. Quasi-closed loop formulations on the other hand use a pre-stabilizing law, and optimize perturbations on this law [Lee and Kouvaritakis, 1999], [Langson et al., 2004]. They provide an improvement, but are still conservative because the pre-stabilizing controller is designed offline.

Optimality can be improved through the use of affine-in-the-disturbance MPC (ADMPC) policies [Lofberg, 2003], [Goulart et al., 2006] where, in the near horizon, the predicted control law consists of feedforward plus linear disturbance compensation with a triangular structure. In the far horizon a fixed linear controller is considered. ADMPC has been superseded by parameterized tube MPC (PTMPC) [Raković et al., 2012], which is based on a separable prediction strategy that considers a triangular prediction structure of partial tubes, the first of which describes the nominal dynamics and the rest describe those associated with future disturbances. These are unknown and thus are dealt with in terms of the vertices of the disturbance set. Hence PTMPC deploys a piecewise-affine-in-the-disturbance policy and leads to larger domains of attraction. The number of variables and constraints grows quadratically with the prediction horizon N in both ADMPC and PTMPC which therefore are better suited to low-dimensional systems with short prediction horizons.

Unlike earlier work, this paper presents an RMPC formulation where the degrees of freedom on the predicted control law that are associated with the disturbances (disturbance compensation) affect directly the inputs over the entire prediction horizon. This idea has been explored before but in the context of constructing of parameterized robust control invariant sets [Rakovic and Baric, 2010] or in Stochastic MPC [Kouvaritakis et al., 2013], where an affine-in-the-disturbances control law is used with disturbance compensation extending over the infinite far prediction horizon but with parameters which are designed offline. Here the disturbance compensation going into the far horizon is found online, and leads to a terminal control, more general than linear feedback. This is achieved by a separable prediction scheme [Raković et al., 2012], according to which the predicted states and inputs are separated into the nominal sequence and a single sequence of sets associated with each future disturbance. This gives

[1] Corresponding author

rise to a striped disturbance compensation scheme which is allowed to extend over an infinite prediction horizon and leads to a number of variables and constraints which grows linearly with N (rather than quadratically as for PTMPC). Additionally, allowing for disturbance compensation into the far horizon implies a constraint relaxation. We achieve this at the cost of a weaker stability notion, that of input-to-state stability. Simulations show that this strategy can, for a comparable number of degrees of freedom and constraints, lead to larger domains of attraction.

Section II gives the system description together with a brief review of the separable prediction scheme of [Raković et al., 2012]. Our strategy is presented in Sections III and IV, with the first of these concerning itself with constraint handling and the second analysing the control theoretic properties of our strategy. Section V presents an illustration by simulation of the benefits of the proposed strategy and conclusions are drawn in Section VI.

Notation and Basic Definitions: The sets of non-negative integers, positive integers and positive reals are denoted by $\mathbb{N}$, $\mathbb{N}_+$ and $\mathbb{R}_+$. $\mathbb{N}_{[a,b]}$ denotes the set $\{a, a+1, ..., b\}$. Given two sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^n$ $X \oplus Y = \{x + y : x \in X, y \in Y\}$ denotes the Minkowski addition. $\text{conv}(X)$ denotes the convex hull of the set $X \subset \mathbb{R}^n$. The image of a set $X \subset \mathbb{R}^n$ under matrix $M \in \mathbb{R}^{m\times n}$ is denoted by $MX = \{Mx : x \in X\}$. Given a convex set $X \subset \mathbb{R}^n$ and two matrices $A, B \in \mathbb{R}^{m\times n}$, $(A, B)X$ denotes the set $\text{conv}(\{(Ax, Bx) : x \in X\})$. Given two sets $X = \text{conv}(\{x_1, ..., x_n\})$, $Y = \text{conv}(\{y_1, ..., y_n\})$, then (X, Y) denotes the set $\text{conv}(\{(x_1, y_1), ..., (x_n, y_n)\})$. A polytope is a closed and bounded polyhedron. A continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a $\mathcal{K}$-function if it is continuous, strictly increasing and $\phi(0) = 0$, and it is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and $\phi(s) \to \infty$ as $s \to \infty$. A continuous function $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{KL}$-function if for all $t \in \mathbb{R}_+$, $\phi(\cdot, t)$ is a $\mathcal{K}$-function and for all $s \in \mathbb{R}_+$, and $\phi(s, \cdot)$ is decreasing with $\phi(s, t) \to 0$ as $t \to \infty$.

2. SYSTEM DESCRIPTION AND SEPARABLE PREDICTION SCHEME

Consider the linear, discrete-time system and constraints

$$x^+ = Ax + Bu + w, \tag{1}$$

$$Fx + Gu \leq 1, \tag{2}$$

where $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}, w \in \mathbb{W} \subset \mathbb{R}^{n_x}$, and $F \in \mathbb{R}^{n_c\times n_x}$, $G \in \mathbb{R}^{n_c\times n_u}$. $\mathbb{W} = \text{conv}\left(\{\tilde{w}_i : i \in \mathbb{N}_{[1,q]}\}\right)$ is a polytopic set that contains the origin. $\mathbb{Y} = \{(x, u) : F_l x + G_l u \leq 1, l \in \mathbb{N}_{[1,n_c]}\}$ is a polytopic set that contains the origin in its interior; the subscript l denotes the l^{th} row.

Assumption 1. (i) The system (A, B) is stabilizable. (ii) The matrix K is such that $A + BK$ is strictly stable and the minimal robust invariant set of system (1) under the control law $u = Kx$, Ω_K^∞, satisfies $(I, K)\Omega_K^\infty \in \text{interior}(\mathbb{Y})$.

Assume a separable prediction scheme [Raković et al., 2012] where $\mathbf{x}_{(0,:)} = \{x_{(0,k)}\}_{k\in\mathbb{N}_+}$ and $\mathbf{u}_{(0,:)} = \{u_{(0,k)}\}_{k\in\mathbb{N}_+}$ are the 0^{th} partial state and control sequences, which account for the nominal dynamics ($w \equiv 0$)

$$x_{(0,k+1)} = Ax_{(0,k)} + Bu_{(0,k)}, \text{ with } x_{(0,0)} = x, \tag{3}$$

and $\mathbf{x}_{(j,:)} = \left\{x_{(j,k)}\right\}_{k\in\mathbb{N}_{[j,\infty]}}$, $\mathbf{u}_{(j,:)} = \left\{u_{(j,k)}\right\}_{k\in\mathbb{N}_{[j,\infty]}}$, for $j \in \mathbb{N}_+$, are the j^{th} partial state and control sequences describing the dynamical contribution of the disturbance acting at the $(j-1)^{th}$ prediction time, w_{j-1}. The full predictions are given by, $\forall k \in \mathbb{N}$

$$x_k = \sum\nolimits_{j=0}^{k} x_{(j,k)}, \quad u_k = \sum\nolimits_{j=0}^{k} u_{(j,k)}, \text{ with} \tag{4a}$$

$$\begin{aligned} x_{(j,k)} \in X_{(j,k)} &= \text{conv}\left(\left\{x_{(i,j,k)} : i \in \mathbb{N}_{[1,q]}\right\}\right), \\ u_{(j,k)} \in U_{(j,k)} &= \text{conv}\left(\left\{u_{(i,j,k)} : i \in \mathbb{N}_{[1,q]}\right\}\right), \end{aligned} \tag{4b}$$

thus defining a triangular prediction structure. From (4b), the j^{th} partial state and control sequences are defined by the j^{th} partial extreme state and control sequences, $\mathbf{x}_{(i,j,:)} = \left\{x_{(i,j,k)}\right\}_{k\in\mathbb{N}_{[j,\infty]}}$ and $\mathbf{u}_{(i,j,:)} = \left\{u_{(i,j,k)}\right\}_{k\in\mathbb{N}_{[j,\infty]}}$, respectively. The dynamics of the j^{th} partial extreme state and control sequences are, for $i \in \mathbb{N}_{[1,q]}$, $j \in \mathbb{N}_+$,

$$x_{(i,j,j)} = \tilde{w}_i, \tag{5a}$$

$$x_{(i,j,k+1)} = Ax_{(i,j,k)} + Bu_{(i,j,k)}, \quad k \in \mathbb{N}_{[j,\infty]}. \tag{5b}$$

In [Raković et al., 2012], $\mathbf{x}_{(i,j,:)}$, $\mathbf{u}_{(i,j,:)}$ are chosen freely for different j, $j \in \mathbb{N}_{[1,N]}$ (while the rest are determined by the state feedback $u = Kx$), but here a striped structure is invoked: for $j \in \mathbb{N}_+$, $k \in \mathbb{R}_{[j,\infty]}$, $i \in \mathbb{R}_{[1,q]}$

$$u_{(i,j,k)} := u_{(i,1,k-j+1)}, \quad x_{(i,j,k)} = x_{(i,1,k-j+1)}, \tag{6}$$

which implies that $X_{(j,k)} = X_{(1,k-j+1)}$ and $U_{(j,k)} = U_{(1,k-j+1)}$. Thus all the j^{th} partial sequences are fully defined by the 1^{st} partial extreme sequences $\mathbf{x}_{(i,1,:)}$ and $\mathbf{u}_{(i,1,:)}$. From (6), the k-step-ahead predictions are bounded by sets given by the Minkowski sum of the nominal k-step-ahead predictions and all the $X_{(1,j)}$ or $U_{(1,j)}$ for $j \in \mathbb{N}_{[1,k]}$. Then, $\forall k \in \mathbb{N}$

$$x_k \in x_{(0,k)} \oplus \bigoplus_{j=1}^{k} X_{(1,j)}, \quad u_k \in u_{(0,k)} \oplus \bigoplus_{j=1}^{k} U_{(1,j)}, \tag{7}$$

where $X_{(1,j)} \in \text{conv}\left(\left\{x_{(i,1,j)} : i \in \mathbb{N}_{[1,q]}\right\}\right)$ and $U_{(1,j)} \in \text{conv}\left(\left\{u_{(i,1,j)} : i \in \mathbb{N}_{[1,q]}\right\}\right)$ as per (4b). It is clear here that all the predicted inputs u_k for $k \in \mathbb{N}_+$ depend on the 1^{st} partial extreme sequences $\mathbf{u}_{(i,1,:)}$; more precisely, the predicted u_k depends on the elements $u_{(i,1,j)}$, $j \in \mathbb{N}_{[1,k]}$. Of these, the elements $u_{(i,1,j)}$, $j \in \mathbb{N}_{[1,N-1]}$ as will be seen next, will be degrees of freedom, which implies that the disturbance compensation moves over the infinite horizon. This setting is different than just reducing the number of decision variables by forcing a striped structured on the parameterization of [Raković et al., 2012], since it only allows compensation for $k \in \mathbb{N}_{[0,N-1]}$.

3. THE RMPC STRATEGY: PREDICTIONS AND CONSTRAINTS

3.1 Prediction strategy

Let the first N prediction steps, with $N \in \mathbb{N}_+$, be referred to as Mode 1 and the remainder as Mode 2, in which it is usual to deploy a fixed stabilizing terminal control law. Our strategy assumes no fixed terminal control law, but instead limits the degrees of freedom (dof) as follows: (i) the first N inputs of the nominal partial control sequence are free (i.e. dof) and afterwards they are defined by $u = Kx$; and (ii) the first $N-1$ inputs of the j^{th} partial extreme control sequences are dof, and afterwards they are defined by the same static feedback $u = Kx$. Thus, the dynamics for the nominal and uncertain portions are described by

$$x_{(0,0)} = x, \tag{8a}$$
$$x_{(0,k+1)} = Ax_{(0,k)} + Bu_{(0,k)}, \qquad k \in \mathbb{N}_{[0,N-1]}, \tag{8b}$$
$$x_{(0,k+1)} = \Phi x_{(0,k)}, \qquad k \in \mathbb{N}_{[N,\infty]}, \tag{8c}$$

where $\Phi = A + BK$, and x represents the current state

$$x_{(i,1,1)} = \tilde{w}_i, \tag{9a}$$
$$x_{(i,1,k+1)} = Ax_{(i,1,k)} + Bu_{(i,1,k)}, \qquad k \in \mathbb{N}_{[1,N-1]}, \tag{9b}$$
$$x_{(i,1,k+1)} = \Phi x_{(i,1,k)}, \qquad k \in \mathbb{N}_{[N,\infty]}. \tag{9c}$$

The nominal and partial extreme sequences $\mathbf{u}_{(0,N-1)} = \{u_{(0,k)}\}_{k\in\mathbb{N}_{[0,N-1]}}$ and $\mathbf{u}_{(i,1,N-1)} = \{u_{(i,1,k)}\}_{k\in\mathbb{N}_{[1,N-1]}}$ define the dof of the predicted control law, and their number grows linearly with N. The separable scheme of (7),(8),(9) defines the overall mixed state and control tube $\mathrm{Y} = \{Y_k\}_{k\in\mathbb{N}}$

$$Y_k = Y_{(0,k)} \oplus Y_{(1,1)} \oplus ... \oplus Y_{(1,k)}, \tag{10}$$

where the mixed state/control $Y_{(0,k)}$ and $Y_{(1,k)}$ are

$$Y_{(0,k)} = (x_{(0,k)}, u_{(0,k)}), \qquad k \in \mathbb{N}_{[0,N-1]}, \tag{11a}$$
$$Y_{(0,k)} = (x_{(0,k)}, Kx_{(0,k)}), \qquad k \in \mathbb{N}_{[N,\infty]}, \tag{11b}$$

$$Y_{(1,k)} = \mathrm{conv}(\{(x_{(i,1,k)}, u_{(i,1,k)}) : i \in \mathbb{N}_{[1,q]}\}), k \in \mathbb{N}_{[1,N-1]} \tag{12a}$$

$$Y_{(1,k)} = \mathrm{conv}(\{(x_{(i,1,k)}, Kx_{(i,1,k)}) : i \in \mathbb{N}_{[1,q]}\}), k \in \mathbb{N}_{[N,\infty]} \tag{12b}$$

such that $(x_k, u_k) \in Y_k$. Eq. (10) can be rewritten as

$$Y_k = Y_{(0,k)} \oplus \bigoplus_{j=1}^{k} Y_{(1,j)}, \quad k \in \mathbb{N}_{[0,N-1]}, \tag{13a}$$

$$Y_k = (I,K)\,\Phi^{k-N} X_{(0,N)} \oplus \bigoplus_{j=1}^{N-1} Y_{(1,j)} \oplus \bigoplus_{j=N}^{k} (I,K)\,\Phi^{j-N} X_{(1,N)}, \quad k \in \mathbb{N}_{[N,\infty]}, \tag{13b}$$

where $X_{(1,N)} = \mathrm{conv}\left(\{x_{(i,1,N)} : i = 1,..,q\}\right)$, and I is the identity; note that $Y_{(1,j)} = (I,K)X_{(1,j)}$ for $j \in \mathbb{N}_{[N,\infty]}$. This form of writing (10) is more convenient in the sense that it only depends on the dof $\mathbf{u}_{(0,N-1)}$ and $\mathbf{u}_{(i,1,N-1)}$, and the sequences $\mathbf{x}_{(0,N)} = \{x_{(0,k)}\}_{k\in\mathbb{N}_{[0,N]}}$, $\mathbf{x}_{(i,1,N)} = \{x_{(i,1,k)}\}_{k\in\mathbb{N}_{[i,1,N]}}$, which will form the online variables of the online optimization (along with the tightening parameters, which will be introduced below).

3.2 Constraints

The condition for constraint satisfaction is that

$$Y_k \subseteq \mathbb{Y}, \quad \forall k \in \mathbb{N} \tag{14}$$

which implies an infinite number of constraints. Instead, constraints will be enforced explicitly for $k \in \mathbb{N}_{[0,N+N_2-1]}$, with $N_2 \in \mathbb{N}$, whereas for $k \in \mathbb{N}_{[N+N_2,\infty]}$ use will be made of terminal constraints on the nominal and uncertain sequences at the $(N + N_2)^{th}$ prediction instant.

Proposition 2. Let Ω_0 be an invariant set for the nominal dynamics of (8c) and let Ω_1 be an arbitrary polytopic set that contains the origin. Then conditions

$$x_{(0,N+N_2)} \in \Omega_0, \quad x_{(1,N+N_2)} \in \Omega_1, \tag{15}$$

imply that $\forall k \in \mathbb{N}_{[N+N_2,\infty]}$

$$Y_k \subseteq \bar{Y}_\infty, \quad \text{where,} \tag{16}$$

$\bar{Y}_\infty = (I,K)\Omega_0 \oplus \bigoplus_{j=1}^{N-1} Y_{(1,j)} \oplus \bigoplus_{j=N}^{N+N_2-1} (I,K)\Phi^{j-N}X_{(1,N)}$ $\oplus(\Omega_1^\infty, K\Omega_1^\infty)$ and Ω_1^∞ is the minimal invariant set for the system with dynamics $z^+ = \Phi z + w$, with $w \in \Omega_1$.

Proof. This follows from: (i) $Y_{(0,k)} = (x_{(0,k)}, Kx_{(0,k)}) \subseteq (\Omega_0, K\Omega_0)$, $k \in \mathbb{N}_{[N+N_2,\infty]}$, provided that $x_{(0,N+N_2)} \in \Omega_0$ and that Ω_0 is invariant for the dynamics of (8c); and (ii) $\bigoplus_{j=N+N_2}^{k} \Phi^{k-(N+N_2)} X_{(1,N+N_2)} \subseteq \Omega_1^\infty$ and therefore $\bigoplus_{j=N+N_2}^{k} (I,K)\Phi^{k-(N+N_2)} X_{(1,N+N_2)} \subseteq (\Omega_1^\infty, K\Omega_1^\infty)$.

Having a set that contains the state and input predictions $\forall k \in \mathbb{N}_{[N+N_2,\infty]}$, enables the use of a finite number of constraints to ensure constraint satisfaction.

Corollary 3. Satisfaction of (14) is guaranteed if the predicted nominal and uncertain states satisfy (15) and

$$Y_k \subseteq \mathbb{Y}, \quad k \in \mathbb{N}_{[0,N+N_2-1]} \tag{17}$$
$$\bar{Y}_\infty \subseteq \mathbb{Y} \tag{18}$$

Proof. Satisfaction of (14) for $k \in \mathbb{N}_{[0,N+N_2-1]}$ is guaranteed by (17), and satisfaction of (14) for $k \in \mathbb{N}_{[N+N_2,\infty]}$ is guaranteed by (18) following Proposition 2.

Eq. (17) can be written as linear constraints through the use of slack variables that account for the worst case partial extreme state and control sequences, as follows:

For $l \in \mathbb{N}_{[1,n_c]}, k \in \mathbb{N}_{[0,N-1]}$ and $i \in \mathbb{N}_{[1,q]}$

$$F_l x_{(0,k)} + G_l u_{(0,k)} + \sum_{j=1}^{k} f_{(l,1,j)} \leq 1, \tag{19a}$$

$$f_{(l,1,j)} \geq F_l x_{(i,1,j)} + G_l u_{(i,1,j)}, \quad j \in \mathbb{N}_{[1,N-1]} \tag{19b}$$

and for $l \in \mathbb{N}_{[1,n_c]}, k \in \mathbb{N}_{[N,N+N_2-1]}$ and $i \in \mathbb{N}_{[1,q]}$,

$$(F_l + G_l K)\Phi^{k-N} x_{(0,N)} + \sum_{j=1}^{k} f_{(l,1,j)} \leq 1, \tag{19c}$$

$$f_{(l,1,j)} \geq (F_l + G_l K)\Phi^{j-N} x_{(i,1,N)}, \quad j \in \mathbb{N}_{[N,N+N_2-1]}. \tag{19d}$$

(18) can also be treated in the same way:

$$f_{(l,0,\infty)} + \sum_{j=1}^{N+N_2-1} f_{(l,1,j)} + f_{(l,1,\infty)} \leq 1 \tag{20}$$

where $f_{(l,0,\infty)} = \max_{x\in\Omega_0}(F_l + G_l K)x$, $f_{(l,1,\infty)} = \max_{x\in\Omega_1^\infty}(F_l + G_l K)x$. If $\Omega_0 = \{x : H_l x \leq 1, l \in \mathbb{N}_{[1,n_{f0}]}\}$ and $\Omega_1 = \{x : L_l x \leq 1, l \in \mathbb{N}_{[1,n_{f1}]}\}$, then (15) can be expressed as

$$H_l x_{(0,N+N_2)} \leq 1, l \in \mathbb{N}_{[1,n_{f0}]}, \tag{21a}$$
$$L_l x_{(i,1,N+N_2)} \leq 1, l \in \mathbb{N}_{[1,n_{f1}]}, i \in \mathbb{N}_{[1,q]} \tag{21b}$$

Assumption 4. Ω_1 is defined by

$$\Omega_1 = \Phi^{N+N_2-1}\mathbb{W}, \tag{22}$$

and Ω_0 is a robust invariant set for $z^+ = \Phi z + w,\ w \in \Phi\Omega_1$.

Remark 5. The assumption above allows to satisfy the conditions on Ω_0 and Ω_1 in Proposition 2 and allows for recursive feasibility, which as shown in the proof of Theorem 9, requires that Ω_0 is a robust invariant set for $z^+ = \Phi z + w,\ w \in \Phi\Omega_1$.

Remark 6. A possible choice for Ω_0 is given by

$$\Omega_0 = \alpha\bar{\Omega}_K \oplus \tilde{\Omega}^\infty_{\Phi\Omega_1}, \quad (23)$$

where $0 < \alpha \leq 1$ is a scalar, $\bar{\Omega}_K$ is the maximal invariant set for $x^+ = \Phi x$ s.t. (2) under $u = Kx$, and $\tilde{\Omega}^\infty_{\Phi\Omega_1}$ is an invariant approximation of the minimal robust invariant set [Raković et al., 2005] for $z^+ = \Phi z + w,\ w \in \Phi\Omega_1$. Ω_0 is clearly robustly invariant for $z^+ = \Phi z + w,\ w \in \Phi\Omega_1$. The motivation for this construction is as follows. In the absence of disturbances Ω_0 would have to be chosen to be $\bar{\Omega}_K$ so as to relax the LHS of (15). However, since (15) is applied at $k = N + N_2$, Ω_0 might be too large thereby causing (18) to be too tight. To avoid this we introduce the scaling factor α whose size should be commensurate with the contraction provided by Φ^{N_2}. In the presence of disturbances however, recursive feasibility, as indicated in Remark 5, requires that Ω_0 accounts for these disturbances which justifies the inclusion of the second term in the LHS of (23). This term has to be made to be as small as possible in order to relax (18), hence it is taken to be the minimal robust invariant set. Since the minimal robust invariant set however may not have a finite description it has been replaced in (23) by a robustly invariant approximation $\tilde{\Omega}^\infty_{\Phi\Omega_1}$. Similarly, to reduce computational complexity $\bar{\Omega}_K$ can be replaced by an invariant approximation.

4. THE RMPC STRATEGY AND ITS PROPERTIES

Let θ be the vector of all the decision variables in the formulation which will be used in the online optimization: $\mathbf{x}_{(0,N)}$, $\mathbf{x}_{(i,1,N)}$, $\mathbf{u}_{(0,N-1)}$, $\mathbf{u}_{(i,1,N-1)}$, and $\mathbf{f}_{(l,1,N+N_2-1)} = \{f_{(l,1,k)}\}_{k\in\mathbb{N}_{[1,N+N_2-1]}}$. The set of admissible variables is

$$\Theta(x) = \{\theta : (8a),(8b),(9a),(9b),(19),(20),(21)\}, \quad (24)$$

and the domain of attraction is

$$\mathcal{X} = \{x : \Theta(x) \neq \emptyset\} \quad (25)$$

Assumption 7. N, N_2 and Ω_0 are such that $\mathcal{X} \neq \emptyset$ and $\mathcal{X}$ contains the origin in its interior.

Remark 8. This assumption can always be satisfied provided that Assumption 1 is satisfied. Set all the predicted inputs to be given by $u = Kx$, then $\bar{Y}_\infty = (I,K)\Omega_0 \oplus \bigoplus_{j=1}^{N+N_2-1}(I,K)\Phi^{j-1}\mathbb{W} \oplus (\Omega_1^\infty, K\Omega_1^\infty) = (I,K)\Omega_0 \oplus \bigoplus_{j=0}^{\infty}(I,K)\Phi^j\mathbb{W}$, and $\bar{Y}_\infty = (I,K)\Omega_K^\infty$ if $\Omega_0 = \{0\}$. But $(I,K)\Omega_K^\infty \in \text{interior}(\mathbb{Y})$, therefore there will always exist an $\bar{N}$ such that for all $N + N_2 \geq \bar{N}$, $\Omega_1 = \Phi^{N+N_2-1}\mathbb{W}$ is small enough so that Ω_0 can be chosen to be small enough such that $(I,K)\Omega_0 \bigoplus_{j=1}^{N+N_2-1}(I,K)\Phi^{j-1}\mathbb{W}\oplus (\Omega_1^\infty, K\Omega_1^\infty) \subset \mathbb{Y}$. This guarantees that for any $x \in \Omega_0$ and N, N_2 such that $N + N_2 \geq \bar{N}_2$, the solution generated by setting $u = Kx$ over the entire prediction will be feasible, which guarantees that $\Omega_0 \subseteq \mathcal{X}$. Ω_0 contains the origin in its interior since it is an invariant set for a contractive system, which implies that $\mathcal{X}$ contains the origin in its interior, and then Assumption 7 is satisfied.

Let $J(\theta)$ be the predicted cost which only considers nominal state and input sequences:

$$J(\theta) = \sum_{k=0}^{N-1} x^T_{(0,k)}Qx_{(0,k)} + u^T_{(0,k)}Ru_{(0,k)} + x^T_{(0,N)}Px_{(0,N)} \quad (26)$$

where $Q, P \succeq 0$, $R \succ 0$, satisfy the Lyapunov condition $\Phi^TP\Phi - P \leq -(Q + K^TRK)$.

The optimal control problem $\mathcal{P}(x)$ is defined by

$$V^*(x) = \min_\theta J(\theta), \text{s.t.} \quad \theta \in \Theta(x). \quad (27a)$$

$$\theta^*(x) = \arg\min_\theta J(\theta), \text{s.t.} \quad \theta \in \Theta(x). \quad (27b)$$

where, $V^*(x)$ is the value function and $\theta^*(x)$ is the optimal arguments function. The control law is then given by

$$\kappa^*(x) = u^*_{(0,0)}(x) \quad (28)$$

Theorem 9. For system (1) under the control law of (28) the optimal control problem of (27) is recursively feasible and guarantees satisfaction of (2). $\mathcal{X}$ is invariant for system (1) under the control law of (28).

Proof. The extension (the "tail") of optimal solutions at the current time (indicated by $(\ )^*$) to the next time, where the state is $x^+ = Ax + B\kappa^*(x) + w$, is

$$\begin{aligned}&\tilde{x}_{(0,k)} = x^*_{(0,k+1)}(x) + x_{(1,k+1)}, \quad k \in \mathbb{N}_{[0,N-1]},\\ &\tilde{x}_{(0,N)} = \Phi\tilde{x}_{(0,N-1)},\end{aligned} \quad (29a)$$

$$\begin{aligned}&\tilde{u}_{(0,k)} = u^*_{(0,k+1)}(x) + u_{(1,k+1)}, \quad k \in \mathbb{N}_{[0,N-2]},\\ &\tilde{u}_{(0,N-1)} = K\tilde{x}_{(0,N-1)},\end{aligned} \quad (29b)$$

$$\tilde{x}_{(i,1,k)} = x^*_{(i,1,k)}(x), \quad k \in \mathbb{N}_{[1,N]}, i \in \mathbb{N}_{[1,q]}, \quad (29c)$$

$$\tilde{u}_{(i,1,k)} = u^*_{(i,1,k)}(x), \quad k \in \mathbb{N}_{[1,N-1]}, i \in \mathbb{N}_{[1,q]}, \quad (29d)$$

where $(x_{(1,k)}, u_{(1,k)}) = \sum_{i=1}^q \lambda^*_{(i,1,1)}(x_{(i,1,k)}(x), u_{(i,1,k)}(x))$ such that the $\lambda^*_{(i,1,1)}$ are the least squares (or any other convenient selection criterion) convex interpolation parameters that satisfy $w = \sum_{i=1}^q \lambda^*_{(i,1,1)}\tilde{w}_i$, $\sum_{i=1}^q \lambda^*_{(i,1,1)} = 1$ and $\lambda^*_{(i,1,1)} \geq 0$, for all $i \in \mathbb{N}_{[1,q]}$. This implies that $(x_{(1,k)}, u_{(1,k)}) \in Y^*_{(1,k)}$. This construction implies that $\tilde{x}_{(0,k)}, \tilde{u}_{(0,k)}, \tilde{x}_{(i,1,k)}$ and $\tilde{u}_{(i,1,k)}$ satisfy (8a),(8b),(9a), (9b), and that the tubes defined by the tail, $\tilde{Y}_k$, and the tubes defined by the current optimal solution, Y^*_k, satisfy $\tilde{Y}_k \subseteq Y^*_{k+1}$, $k \in \mathbb{N}$. But a feasible current solution satisfies (17),(18), which guarantees $Y^*_k \subseteq \mathbb{Y}$, $k \in \mathbb{N}$, so then (17) is guaranteed for $\tilde{Y}_k$. Since $\tilde{Y}_{(1,j)} = Y^*_{(1,j)}$ and Ω_0 and Ω_1 are fixed, (18) and the RHS of (15) are satisfied. The satisfaction of the LHS of (15) follows from $\tilde{x}_{(0,N+N_2)} = \Phi x^*_{(0,N+N_2)} + \Phi x_{(1,N+N_2)}$, $x_{(1,N+N_2)} \in \Omega_1$ and that Ω_0 is invariant for $z^+ = \Phi z + w$, where $w \in \Phi\Omega_1$. Then if $\mathcal{P}(x)$ is feasible, it will be feasible for the next, and all future instants. Invariance of $\mathcal{X}$ follows directly. Satisfaction of (2) follows trivially from the fact that $Y^*_0 \subseteq \mathbb{Y}$.

Remark 10. The treatment of recursive feasibility, as shown above, is more involved than in usual RMPC formulations, where a terminal set for x_N that is robustly invariant for the fixed terminal control law is enough. This is because letting the disturbance compensation to extend over an infinite horizon implies that the terminal control law is not fixed. In this different setting separate terminal sets for the nominal and the partial extreme predictions are used, Ω_0 and Ω_1, and as shown in the proof above it is required that Ω_0 is robustly invariant for $z^+ = \Phi z + w$, with $w \in \Phi\Omega_1$, thus justifying Assumption 4.

In [Raković et al., 2012] the stability result is expressed in terms of exponential convergence to the minimal invariant set Ω_K^∞ and the control law $u = Kx$. However, here we do not use $u = Kx$ as the terminal control and so the stability notions of [Raković et al., 2012] do not apply. Instead we consider Input-to-State stability [Jiang and Wang, 1998]. Assume time-invariant nonlinear dynamics

$$x^+ = f(x, w) \tag{30}$$

where $x \in \mathbb{R}^{n_x}$ is the state, w is a disturbance that lies in a compact set $\mathbb{W} \subset \mathbb{R}^{n_w}$ and $f(\cdot,\cdot)$ is a continuous function such that $f(0,0) = 0$.

Definition 11. For system (30), the origin is input-to-state stable (ISS) in $\mathcal{E} \in \mathbb{R}^n$, which contains the origin in its interior, if there exist a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that, for all $x_0 \in \mathcal{E}$ and all admissible disturbances $w_t \in \mathbb{W}$, the evolution of (30) satisfies $\forall t \in \mathbb{N}$

$$|x_t| \leq \beta(||x_0||, t) + \gamma \sup(\{||w_k|| : k \in \mathbb{N}_{[0,t-1]}\}) \tag{31}$$

Remark 12. ISS implies that: (i) the origin is asymptotically stable for $x^+ = f(x, 0)$ with domain of attraction $\mathcal{E}$; (ii) all state trajectories are bounded for bounded w_t; and (iii) as $t \to \infty$ all trajectories go to the origin if $w_t \to 0$.

The following results give a characterization of systems (30) that are ISS, state relevant properties of $\mathcal{X}$, $V^*(x)$, $\kappa^*(x)$, which allow to conclude the ISS property of system (1) under the control law of (28). The proofs of these results are omitted for brevity; the interested reader is referred to [Goulart et al., 2006] where analogous ISS results are discussed.

Lemma 13. Let $\mathcal{E} \subseteq \mathbb{R}^n$ contain the origin in its interior and be a robust invariant set for (30). Furthermore, let there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ and a function $V : \mathcal{E} \to \mathbb{R}_+$ that is Lipschitz continuous on $\mathcal{E}$ such that for all $x \in \mathcal{E}$,

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||) \tag{32a}$$

$$V(f(x, 0)) - V(x) \leq -\alpha_3(||x||) \tag{32b}$$

$V(\cdot)$ is a ISS-Lyapunov function and the origin is ISS for system (30) with domain of attraction $\mathcal{E}$ if either: (i) $f(\cdot,\cdot)$ is Lipschitz continuous on $\mathcal{E} \times \mathbb{W}$, or (ii) $f(x, w) := g(w) + w$, where $g : \mathcal{E} \to \mathbb{R}^n$ is continuous on $\mathcal{E}$.

Proposition 14. (i) $\kappa^*(x)$ is a unique Lipschitz continuous function $\forall x \in \mathcal{X}$. (ii) $V^*(x)$ is strictly convex and Lipschitz continuous $\forall x \in \mathcal{X}$. (iii) $\mathcal{X}$ is a polytopic set that contains the origin in its interior. (iv) $V^*(0) = 0$ and $\kappa^*(0) = 0$.

Theorem 15. The origin is ISS for system (1) under the control law of (28) with domain of attraction $\mathcal{X}$.

Remark 16. If Ω_0 satisfies

$$\Omega_0 \oplus \Omega_K^\infty \subseteq \Omega_K, \tag{33}$$

where Ω_K is the maximal invariant set for (1) s.t. (2) under $u = Kx$, then it can be proved that for $x \in \Omega_K$, the solution generated by setting $u = Kx$ over the entire prediction horizon will be feasible, and if K is the LQR optimal gain of cost of (26) then that solution will be the optimal. Therefore, if during the closed-loop execution the state enters Ω_K, the control law $\kappa^*(x)$ will revert to $u = Kx$. Since $(I, K)\Omega_K^\infty \in \text{interior}(\mathbb{Y})$ from Assumption 1 and following the same reasoning as in Remark 8, there will always exist an N_2 large enough such that Ω_0 can be chosen to be small enough such that (33) is met.

Finally, the dependence of the domain of attraction $\mathcal{X}$ on the horizons N and N_2, is studied; this dependence is now made explicit by writing $\mathcal{X}_{N,N_2}$. Ω_1 also depends on N and N_2, so the dependence is also made explicit by writing $\Omega_1(N, N_2)$. The desired properties are $\mathcal{X}_{N,N_2} \subseteq \mathcal{X}_{N,N_2+1}$ and $\mathcal{X}_{N,N_2} \subseteq \mathcal{X}_{N+1,N_2}$. The following theorem presents an alternative way to construct Ω_0 such that the desired nestedness properties are satisfied.

Theorem 17. Let Ω_0 be a robust invariant set for system $z^+ = \Phi z + w$, but with $w \in \mathcal{W}$, where $\mathcal{W}$ is any outer approximation of $\text{conv}(\Phi^{N_3}\mathbb{W} \cup \Phi^{N_3+1}\mathbb{W} \cup \Phi^{N_3+2}\mathbb{W}\ldots)$, with $N_3 \in \mathbb{N}_+$. Then the domain of attraction of system (1) controlled by (28) satisfies the following properties for all N, N_2, such that $N+N_2 \geq N_3$: (i) $\mathcal{X}_{N,N_2} \subseteq \mathcal{X}_{N+1,N_2-1}$; (ii) $\mathcal{X}_{N,N_2} \subseteq \mathcal{X}_{N,N_2+1}$; and (iii) $\mathcal{X}_{N,N_2} \subseteq \mathcal{X}_{N+1,N_2}$.

Proof. (i) Note that the optimal control problem $\mathcal{P}(x)$ when using N and N_2 is the same one as when using $\tilde{N} = N + 1$ and $\tilde{N}_2 = N_2 - 1$, except that in the former case $u_{(0,N)}$ and $u_{(i,1,N)}$ are fixed to be $u_{(0,N)} = Kx_{(0,N)}$ and $u_{(i,1,N)} = Kx_{(i,1,N)}$, but in the latter these are dof. In the latter case $u_{(0,N)}$ and $u_{(i,1,N)}$ can be chosen to be $u_{(0,N)} = Kx_{(0,N)}$ and $u_{(i,1,N)} = Kx_{(i,1,N)}$, and then the problem for $\tilde{N}$, $\tilde{N}_2$ is feasible when it is feasible for N, N_2.

(ii) If $x \in \mathcal{X}_{N,N_2}$, then there exists θ that includes $\mathbf{x}_{(0,N)}, \mathbf{u}_{(0,N-1)}, \mathbf{x}_{(i,1,N)}, \mathbf{u}_{(i,1,N-1)}$, and $\mathbf{f}_{(l,1,N+N_2-1)}$ such that (8a),(8b),(9a),(9b),(19),(20),(21) are satisfied. Then $\tilde{\theta}$ (that is composed of the variables in θ and $f_{(l,1,N+N_2)} = \max_{i\in\mathbb{N}_{[1,q]}}\{(F_l + G_l K)\Phi^{N_2}x_{(i,1,N)}\}$, $l \in \mathbb{N}_{[1,n_c]}$) trivially satisfies (8a), (8b), (9a), (9b), (19) for $\tilde{N} = N$ and $\tilde{N}_2 = N_2 + 1$. The definition of Ω_1 implies that $\Omega_1(N, N_2 + 1) = \Phi\Omega_1(N, N_2)$. Then, since $\Phi^{N_2}x_{(i,1,N)} \subseteq \Omega_1(N, N_2)$ from (21b) we obtain that $\Phi^{N_2+1}x_{(i,1,N)} \subseteq \Omega_1(N, N_2+1)$ and so (21b) is satisfied. Ω_0 is defined such that it will not change for different values of N, N_2, s.t. $N + N_2 \geq N_3$, and that it satisfies $\Phi\Omega_0 \subseteq \Omega_0$. This implies that if $x_{(0,N+N_2)} \in \Omega_0$ then $x_{(0,N+N_2+1)} = \Phi x_{(0,N+N_2)} \in \Omega_0$, and thus (21a) is satisfied. Finally, satisfaction of (20) for N, N_2 implies that $\bar{Y}_\infty \in \mathbb{Y}$. Eq. (22) implies that $\Omega_1^\infty(N, N_2) = \Omega_1^\infty(N, N_2 + 1) \oplus \Phi^{N_2}X_{(1,N)}$, then $\tilde{Y}_\infty = (I, K)\Omega_0 \oplus \bigoplus_{j=1}^{N-1} Y_{(1,j)} \oplus \bigoplus_{j=N}^{N+N_2}(I, K)\Phi^{j-N}X_{(1,N)} \oplus (I, K)\Omega_1^\infty(N, N_2+1) = \bar{Y}_\infty \subseteq \mathbb{Y}$, and therefore (20) is satisfied for $\tilde{N}$, $\tilde{N}_2$. All this proves that $x \in \mathcal{X}_{N,N_2+1}$, which yields the desired result.

(iii) This follows directly from combining (i) and (ii).

Remark 18. The condition that Ω_0 is invariant for $z^+ = \Phi z + w$, with $w \in \mathcal{W}$, $\mathcal{W}$ being any outer approximation of $\text{conv}(\{\Phi^{N_3}\mathbb{W} \cup \Phi^{N_3+1}\mathbb{W} \cup \Phi^{N_3+2}\mathbb{W}\ldots\})$, in addition to ensure that Ω_0 is the same for different values of N, N_2, guarantees that Ω_0 satisfies Assumption 4 for all N, N_2 such that $N + N_2 \geq N_3$. Then recursive feasibility is guaranteed by Theorem 9 whenever $N + N_2 \geq N_3$. Thus, N_3 should be chosen small enough as dictated by practical values of N, N_2, but large enough to maintain Ω_0 small.

Remark 19. The construction of Remark 6 and (23) is designed so that Ω_0 satisfies Assumption 4. However, Theorem 17 establishes a stronger condition on Ω_0. A modification of (23) in order to satisfy these new conditions is given by $\Omega_0 = \alpha\bar{\Omega}_K \oplus \tilde{\Omega}_{\mathcal{W}}^\infty$, where $\tilde{\Omega}_{\mathcal{W}}^\infty$ is an invariant approximation of the minimal invariant set for $z^+ = \Phi z + w$, $w \in \mathcal{W}$.

Remark 20. $\mathcal{W}$ is defined as any outer approximation of (instead of being equal to) $\text{conv}(\Phi^{N_3}\mathbb{W} \cup \Phi^{N_3+1}\mathbb{W} \cup \Phi^{N_3+2}\mathbb{W}\ldots)$, in order to reduce the number of vertices or facets needed to define $\mathcal{W}$ and hence Ω_0.

5. ILLUSTRATIVE EXAMPLE

The benefits of the proposed strategy are illustrated by a simulation example. Consider a system defined by $A = \begin{bmatrix} 0.787 & -0.933 \\ 1.015 & 1.033 \end{bmatrix}$; $B = \begin{bmatrix} 0.331 \\ -1.006 \end{bmatrix}$, $\mathbb{W} = \text{conv}\,(\{\pm[0.1, 0.1]^T, \pm[0.1, -0.1]^T\})$, $\mathbb{Y} = \{(x, u) : \pm[-0.044, 0.092]x \leq 1, \pm[0.009, 0.093]x \leq 1, u \leq 1, -u \leq 1\}$. Ω_0 is constructed according to Remark 19 with $N_3 = 6$, $\alpha = 0.01$ and to ensure satisfaction of (33) we set $N_2 = 5$. The area of the domains of attraction is computed using SPTMPC (proposed in this paper) and PTMPC (proposed in [Raković et al., 2012]) and the results are recorded, respectively, under columns M1 and M2, along with the number of variables, equality, and inequality constraints involved in the online optimization.

Table 1: Domain of Attraction and Computational Aspects

N	A_N		# vars.		# ineqs.		# eqs.	
	M1	M2	M1	M2	M1	M2	M1	M2
1	2.38	2.38	43	24	196	61	12	10
2	2.87	3.01	64	71	226	158	22	28
3	3.57	3.66	85	144	256	303	32	54
4	4.52	4.19	106	243	286	496	42	88
5	5.14	4.59	127	368	316	737	52	130
6	5.39	4.88	148	519	346	1026	62	180
7	5.83	5.13	169	696	376	1363	72	238
8	5.86	5.34	190	899	406	1748	82	304
9	5.95	5.48	211	1128	436	2181	92	378
10	5.95	5.59	232	1383	466	2662	102	460

Thus for the same N and $N \geq 4$ SPTMPC yields larger domains of attraction than PTMPC. The improvements achieved by SPTMPC are due mainly to two reasons: (i) that in SPTMPC the predicted state is not constrained to be inside Ω_K at any instant; and (ii) that disturbance compensation is not restricted to the first N prediction steps, but is instead allowed to enter into Mode 2, thereby relaxing the terminal constraints. On the other hand, PTMPC has a "full" triangular prediction structure, which is more general than the striped structured of the proposed strategy and therefore, depending on the parameters of the model, has the potential to outperform SPTMPC. However, the price of the "full" triangular structure is that it implies a computational load that grows quadratically with N, whereas the online computation of SPTMPC grows only linearly with N. Thus with SPTMPC one can use longer horizons, thereby enlarging the size of the domain of attraction, at a computational cost which is still less than that required by PTMPC. For example, while for $N = 1$ SPTMPC has more variables and constraints, it can be seen that for $N \geq 4$ SPTMPC leads to larger domains of attraction while using fewer variables and constraints. Even more, due to the quadratic increase (in number of variables and constraints) in PTMPC versus the linear increase in SPTMPC, larger values of N in SPTMPC can be used, thus obtaining even larger domains of attractions and still using fewer variables and constraints (see for instance SPTMPC with $N = 10$ and PTMPC with $N = 5$).

Although SPTMPC requires the solution of a quadratic program (QP) and PTMPC a linear program (LP), the comparison above is still valid because the computational cost involved in LP and QP are comparable. SPTMPC can be reformulated using a different cost function that leads to an LP formulation which, as is the case with PTMPC, would result in a modest increase in the number of variables and constraints, so that the results would be very similar to those presented here.

Table 1 does not attempt a comparison of performance because the predicted costs used by PTMPC and SPTMPC are different (even in an LP formulation of SPTMPC, because it considers only nominal behaviour).

6. CONCLUSIONS

By considering a predicted control law where the degrees-of-freedom affect the inputs over the entire prediction horizon, a RMPC strategy is proposed that has a number of variables and constraints that grow only linearly with the prediction horizon, and yet have the potential to lead to domains of attraction which are larger than those possible through the use of PTMPC. This benefits come at the cost of a weaker notion of stability, namely ISS, as opposed to guaranteeing convergence to the static feedback $u = Kx$ for any realization of the disturbances. This aspect forms a topic for further research in order to obtain similar benefits but retaining the stronger stability properties of PTMPC.

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