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Leader-Following Output Consensus in a Network of Linear Agents with Communication Noises ⋆

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Abstract: The leader-following output consensus problem of multi-agent systems (MAS) is studied in this paper. Each agent is modeled by a single-input single-output (SISO) system which can be further described by a controllable and observable linear state space model. An observer is constructed to estimate the agent's state, and the estimated state is shared with neighbor agents via the noisy communication channels. Similar to the previous work, in the proposed protocol a time-varying gain is employed to attenuate the noise's effect. However, in this paper, each agent is allowed to have its own time-varying gain. Some sufficient conditions on the time-varying gain are given for ensuring the consensus in the mean square sense. Finally, a simulation example is presented to verify the theoretical results.

1. INTRODUCTION

The last decade has witnessed a rapid development of studies on the consensus of multi-agent systems (MASs). Solving a consensus problem means to design a distributed control protocol such that all agents' states are convergent to a same value. To this end, agents must share their information through the communication network which is unavoidably corrupted by noises. Recently, a growing number of works have been reported to address the issue of communication noises. To deal with the uncertainties such as communication noises, Hou et al. [2009] and Cheng et al. [2010] proposed some adaptive consensus protocols based on neural networks. Another common way to attenuate noises is to employ a time-varying gain $a(t)$ in the consensus protocol. This idea was first introduced in [Huang et al., 2009] to solve the mean square/almost sure consensus problem of first-order integral MASs. And the time-varying gain should satisfy two conditions: $\int_0^\infty a(t)dt = \infty$ and $\int_0^\infty a^2(t)dt < \infty$, which are called the stochastic-approximation type conditions. These conditions were also proved to be necessary for the mean square consensus in [Li and Zhang, 2009]. Extensions to the mean square consensus of second-order integral MASs and generic linear MASs were made in [Cheng et al., 2011b] and [Cheng et al., 2013a], respectively. The aforementioned papers all consider the consensus of leaderless MASs, while many scholars also paid attention to the consensus of leader-following MASs with communication noises, for example [Ma and Zhang, 2010, Hu et al., 2010, Wang et al., 2013a]. In [Ma and Zhang, 2010], the leader-following mean square consensus problem of first-order integral MASs with communication noises can still be solved by the protocol including the stochastic-approximation type gain. Hu et al. [2010] extended the results to the switching topology case. However, the stochastic-approximation type conditions can only be proved sufficient for ensuring the leader-following consensus. To find the necessary and sufficient conditions, Wang et al. [2013a] studied the continuous-time first-order integral MAS. They proved that if $a(t)$ is a uniformly continuous function, then the necessary and sufficient conditions for the leader-following mean square consensus were: $\int_0^\infty a(t)dt = \infty$ and $\lim_{t\to\infty} a(t) = 0$. The counterpart results in the discrete-time domain were given in [Wang et al., 2013b].

By reviewing the current literature, it can be found that: (1) studies on the consensus with communication noises mainly focus on the first-order integral agent or the second-order integral agent; and (2) the time-varying gain is assumed to be identical for all agents. There are very few papers regarding the generic linear agent except [Cheng et al., 2013a, Wang et al., 2013b], where the agent's full state is assumed to be available for the consensus protocol design. However, the agent's full state may not be available in practical applications. Moreover, it is relatively difficult to perfectly synchronize the time-varying gain among agents in practice. These observations make the motivation of the study in this paper.

The leader-following output consensus of MASs with communication noises is considered in this paper. The dynamical behavior of each agent is modeled by a single-input single-output (SISO) system. The control objective is to design the distributed control protocol for MASs such

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that outputs of following agents can all be convergent to the leader's output in the mean square sense. In addition, only the agent's output information is available for the protocol design. To this end, the agent's SISO dynamics is equivalently written as the controllable and observable linear state space model. And inspired by the previous work [Cheng et al., 2013b], an observer is constructed to estimate each agent's full state. Then agents exchange their estimated states with their neighbor agents in the noisy communication environment. To attenuate the noise effect, the time-varying gain technique is employed in the proposed consensus protocol. However, each agent is allowed to have its own time-varying gain in this paper. It is proved that under the proposed protocol, outputs of following agents can reach a consensus on the leader's output in the mean square sense if the following conditions hold: (1) the communication topology has a spanning tree; (2) $\int_0^\infty \bar{a}(t)dt = \infty$ where $a_i(t)$ is the time-varying gain of agent i and $\bar{a}(t) = \max_{i\in\mathcal{V}_\mathcal{G}}\{a_i(t)\}$; (3) all the time-varying gains $\{a_1(t), \cdots, a_N(t)\}$ are infinitesimal of the same order as time goes to infinity, and for $\forall\beta > 0$, $e^{-\beta t} = o(\bar{a}(t))$; and (4) all roots of a polynomial, whose coefficients are the parameters in the proposed protocol, have negative real parts.

The following notations will be used throughout this paper: $1_n = (1, \cdots, 1) \in \mathbb{R}^n$; $0_n = (0, \cdots, 0) \in \mathbb{R}^n$; I_n denotes the $n \times n$ dimensional identity matrix; $\Theta_{m\times n} \in \mathbb{R}^{n\times m}$ denotes the $m \times n$ dimensional zero matrix (Θ_n denotes the $n\times n$ dimensional zero matrix); $\otimes$ denotes the Kronecker product. For a given matrix X, $\|X\|_2$ denotes its 2-norm. diag$(\cdot)$ denotes a block diagonal matrix formed by its inputs. For the random variable x, $E\{x\}$ denotes its mathematical expectation. For a complex number c, $\Re(c)$ denotes its real part.

2. PROBLEM FORMULATION & PRELIMINARIES

Consider a MAS composed of N agents which are sparsely connected by the communication network. In the literature, this communication network is usually modeled by a digraph $\mathcal{G} = \{\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}, \mathcal{A}_\mathcal{G}\}$, where $\mathcal{V}_\mathcal{G} = \{1, \cdots, N\}$, $\mathcal{E}_\mathcal{G} \subset \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G}$, and $\mathcal{A}_\mathcal{G} = [\alpha_{ij}] \in \mathbb{R}^{N\times N}$ are the node set, edge set and adjacency matrix, respectively. Node i denotes agent i. The directed edge $e_{ij} \in \mathcal{E}_\mathcal{G}$ denotes the communication link from agent j to agent i. And $e_{ij} \in \mathcal{E}_\mathcal{G}$ if and only if there is a communication link from agent j to i. If $e_{ij} \in \mathcal{E}_\mathcal{G}$, the agent j is called the parent of agent i. It is assumed that $e_{ii} \notin \mathcal{E}_\mathcal{G}$. The neighbor set of agent i is defined by $\mathcal{N}_i \triangleq \{j|e_{ij} \in \mathcal{E}_\mathcal{G}\}$. The element α_{ij} of $\mathcal{A}_\mathcal{G}$ represents the communication quality of the communication channel e_{ij}. It is assume that $e_{ij} \in \mathcal{E}_\mathcal{G} \Leftrightarrow \alpha_{ij} > 0$ and $e_{ij} \notin \mathcal{E}_\mathcal{G} \Leftrightarrow \alpha_{ij} = 0$. The Laplacian matrix of $\mathcal{G}$ is defined by $\mathcal{L}_\mathcal{G} = \mathcal{D}_\mathcal{G} - \mathcal{A}_\mathcal{G}$ where $\mathcal{D}_\mathcal{G} = \mathrm{diag}(\deg_1, \cdots, \deg_N)$ and $\deg_i = \sum_{j\in\mathcal{N}_i} \alpha_{ij}$.

A path from node n_0 to n_k is a sequence of end-to-end directed edge $\{e_{n_{i+1}n_i}|i = 0, \cdots, k-1\}$ such that $e_{n_{i+1}n_i} \in \mathcal{E}_\mathcal{G}$ $(i = 0, \cdots, k-1)$. A node is called root, if there is at least one path from this node to any other nodes. A subgraph of the $\mathcal{G}$ is a digraph whose node set is a subset of that of $\mathcal{G}$, and whose edge set is a subset of that of $\mathcal{G}$. A subgraph $\mathcal{G}_s$ of $\mathcal{G}$ is called a spanning subgraph, if $\mathcal{G}_s$ has the same node set as $\mathcal{G}$. A tree is a digraph which has a root, and whose every node, except the root, has exactly one parent node. If a spanning subgraph of $\mathcal{G}$ a tree, then it is called a spanning tree.

An agent is called a "leader" if its neighbor set is null. An agent is called a "follower" if it has at least one parent agent. The MAS considered in this paper is assumed to contain only one leader whose label is 1. Then it is easy to see that $e_{1i} \notin \mathcal{E}_\mathcal{G}$ and $\alpha_{1i} = 0$ $(i \in \mathcal{V}_\mathcal{G})$. Therefore, the Laplacian matrix of $\mathcal{G}$ has the following form

$$\mathcal{L}_\mathcal{G} = \begin{bmatrix} 0 & 0_{N-1}^T \\ L_1 & L_2 \end{bmatrix}. \tag{1}$$

Lemma 1. (Wang et al. [2013a]). If $\mathcal{G}$ has a spanning tree, then all eigenvalues of L_2 have positive real parts and $L_2^{-1}L_1 = -1_{N-1}$.

Lemma 2. For any diagonal matrix $D = \mathrm{diag}(d_1, \cdots, d_N) \in \mathbb{R}^{N\times N}$ with positive diagonal elements, $D\mathcal{L}_\mathcal{G}$ is the Laplacian matrix of a digraph $\hat{\mathcal{G}}$. If $\mathcal{G}$ has a spanning tree, then $\hat{\mathcal{G}}$ also has a spanning tree, and all eigenvalues of D_2L_2 $(D_2 = \mathrm{diag}(d_2, \cdots, d_N))$ have positive real parts.

The ith agent's dynamics is described by the following SISO system

$$y_i^{(n)}(t) + a_{n-1}y_i^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0y(t) = u_i(t), \tag{2}$$

where $u_i(t)$ and $y_i(t)$ are input and output of agent i, respectively.

Since agents exchange information in the noisy communication network, the consensus can not be reached in the deterministic sense. The following definition introduce the concept of mean square leader-following output consensus which is the control objective of this paper.

Definition 1. The control protocol $\{u_1(t), \cdots, u_N(t)\}$ is said to solve the distributed mean square leader-following output consensus problem of the MAS described by (2) if $\lim_{t\to\infty} E\{(y_i(t) - y_1(t))^2\} = 0$, $i = 2, \cdots, N$, and each agent's control $u_i(t)$ only uses the information of its neighbor agents $\mathcal{N}_i$.

Before closing this section, some definitions and results of the Dini derivative are presented. For a continuous function $f(t, x(t))$, its derivative may not exist at some points. But we can always define its upper Dini derivative as follows

$$D^+f(t) = \limsup_{h\to 0^+} \frac{f(t+h, x(t+h)) - f(t, x(t))}{h}.$$

Lemma 3. (Lin et al. [2007]). Let $\mathcal{I}_0 = \{1, 2, \cdots, n\}$ and suppose for $\forall i \in \mathcal{I}_0$, $f_i(t, x(t)) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is derivable and its derivative $\dot{f}_i(t, x(t))$ is a continuous function; let $f(t, x(t)) = \max_{i\in\mathcal{I}_0}\{f_i(t, x(t))\}$; and let $\mathcal{I}(t) = \{i \in \mathcal{I}_0 | f_i(t, x) = f(t, x)\}$. Then,

$$D^+f(t, x(t)) = \max_{i\in\mathcal{I}(t)}\{\dot{f}_i(t, x(t))\}.$$

3. PROTOCOL DESIGN

The SISO system defined by (2) can be further described by the following controllable and observable linear state space model

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \\ y_i(t) = Cx_i(t), \end{cases} \tag{3}$$

where $x_i(t) = (y_i(t), y_i^{(1)}(t), \cdots, y^{(n-1)}(t))^T$ is the state vector;

$$A = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$B = (0, \cdots, 0, 1)^T \in \mathbb{R}^n$, and $C = (1, 0, \cdots, 0) \in \mathbb{R}^{1\times n}$.

Since only the agent's output $y_i(t)$ is accessible to controller design, inspired by the previous work in the noise-free environment [Cheng et at., 2011a, Cheng et al., 2013b], the following observer is constructed to estimate the agent's full state.

$$\dot{\hat{x}}_i(t) = (A + K_3C)\hat{x}_i(t) + Bu_i(t) - K_3y_i(t), \tag{4}$$

where $K_3 \in \mathbb{R}^n$ is selected in such a way that $A + K_3C$ is *Hurwitz* since (A, C) is observable. It is easy to see that $\lim_{t\to\infty} x_i(t) - \hat{x}_i(t) = \lim_{t\to\infty} \exp((A + K_3C)t)(x_i(0) - \hat{x}_i(0)) = 0_n$. Then each agent sends its estimated state $\hat{x}_i(t)$ to neighbor agents through the communication network. However, the communication network is corrupted by additive communication noises. And the real information that agent i receives from its neighbor agent j is assumed to be $\upsilon_{ij}(t) = \hat{x}_j(t) + \rho_{ij}\eta_{ij}(t)$, where $\eta_{ij}(t) = (\eta_{ij1}(t), \cdots, \eta_{ijn}(t))^T$ is the n-dimensional standard white noise, $\rho_{ij} = \mathrm{diag}(\rho_{ij1}, \cdots, \rho_{ijn})$ and $\{\rho_{ijk} > 0|\ k = 1, \cdots, n\}$ are finite noise intensities. It is also assumed that $\{\eta_{ijk}(t)|i, j = 1\cdots, N; k = 1, \cdots, n\}$ are mutually independent.

By the time-varying gain technique, the following protocol is proposed for agent i ($i = 1, \cdots, N$)

$$u_i(t) = K_1\hat{x}_i(t) + a_i(t)\sum_{j\in\mathcal{N}_i} \alpha_{ij}K_2(\upsilon_{ij}(t) - \hat{x}_i(t)), \tag{5}$$

where $a_i(t) > 0$ is a continuous function, $K_1 = (a_0, a_1 - b_1, \cdots, a_{n-1} - b_{n-1})$ and $K_2 = (b_1, \cdots, b_{n-1}, 1)$. And $\{b_1, \cdots, b_{n-1}\}$ are control parameters which are selected in such a way that all roots of the following *parameter polynomial* have negative real parts.

$$\zeta^{n-1} + b_{n-1}\zeta^{n-2} + \cdots + b_2\zeta + b_1 = 0. \tag{6}$$

Remark 1. Employing the time-varying gain is a common way to attenuate the noises' effect in the literature. However, in most exist papers, all agents are assumed to have the same time-varying gain. This is a restrictive assumption because the perfect synchronization of the time-varying gain is hard to be achieved especially in the distributed computation fashion. The protocol defined by (5) allows the agent-dependant gain $a_i(t)$, which is the main contribution of this paper to the current literature.

4. MAIN RESULTS

Substituting (3) and (5) into (4) obtains that the following stochastic system driven by the white noise

$$\begin{aligned} \dot{\hat{X}}(t) = (I_n \otimes (A + BK_1))\hat{X}(t) - (\mathfrak{A}(t)\mathcal{L}_\mathcal{G}) \otimes (BK_2)\hat{X}(t) \\ + (\mathfrak{A}(t) \otimes I_n)\Sigma\eta(t) - (I_N \otimes (K_3C))\Delta(t), \end{aligned} \tag{7}$$

where $\hat{X}(t) = (\hat{x}_1^T(t), \cdots, \hat{x}_n^T(t))^T$, $\mathfrak{A}(t) = \mathrm{diag}(a_1(t), \cdots, a_N(t))$, $\Sigma = \mathrm{diag}(R_1, \cdots, R_N)$, $R_i = BK_2(\alpha_{i1}\rho_{i1}, \cdots, \alpha_{iN}\rho_{iN})$, $\eta(t)$ is an nN^2-dimensional standard white noise vector, and

$$\Delta(t) \triangleq X(t) - \hat{X}(t) = (I_N \otimes \exp((A + K_3C)t))(X(0) - \hat{X}(0)).$$

Let $\xi_i(t) = K_2\hat{x}_i(t)$ and $\Xi(t) = (\xi_1(t), \cdots, \xi_N(t))^T$, then we have the following *auxiliary system*

$$\begin{aligned} \dot{\Xi}(t) = -\mathfrak{A}(t)\mathcal{L}_\mathcal{G}\Xi(t) - (I_N \otimes (K_2K_3C))\Delta(t) \\ + \mathfrak{A}(t)\bar{\Sigma}\eta(t), \end{aligned} \tag{8}$$

where $\bar{\Sigma} = \mathrm{diag}(\bar{R}_1, \cdots, \bar{R}_N)$ and $\bar{R}_i = K_2R_i$. By the knowledge of stochastic process, this stochastic system is equivalent to the following Itô stochastic differential equation:

$$\begin{aligned} d\Xi(t) = -\mathfrak{A}(t)\mathcal{L}_\mathcal{G}\Xi(t)dt - (I_N \otimes (K_2K_3C))\Delta(t)dt \\ + \mathfrak{A}(t)\bar{\Sigma}dW(t). \end{aligned} \tag{9}$$

where $W(t)$ is the nN^2-dimensional standard Brownian motion.

In the rest of this section, we first studies the convergence of the auxiliary system (8). Then, based on the convergence of (8), some sufficient conditions are given for ensuring the mean square leader-following output consensus.

Before further discussion, the following four conditions are presented.

(C1). $\mathcal{G}$ has a spanning tree.
(C2). $\int_0^\infty \bar{a}(t)dt = \infty$ where $\bar{a}(t) = \max_{i=1,\cdots,N}\{a_i(t)\}$.
(C3). All the time-varying gains $\{a_1(t), \cdots, a_N(t)\}$ are infinitesimal of the same order as time goes to infinity [Canuto and Tabacco , 2008]. And for $\forall\beta > 0$, $e^{-\beta t} = o(\bar{a}(t))$.
(C4). All roots of the parameter polynomial defined by (6) have the negative real parts.

Lemma 4. If Condition (C3) holds, then for any $i \in \mathcal{V}_\mathcal{G}$, there exists a positive constant $c_i^* < \infty$ such that $\lim_{t\to\infty} a_i(t)/\bar{a}(t) = c_i^*$.

Proof. By Condition (C3), we know that $\forall\ i, j \in \mathcal{V}_\mathcal{G}$, there exists a positive constant $c_{ij} < \infty$ such that $\lim_{t\to\infty} a_i(t)/a_j(t) = c_{ij}$. Therefore, $\forall i \in \mathcal{V}_\mathcal{G}$

$$\begin{aligned} \lim_{t\to\infty} \frac{a_i(t)}{\bar{a}(t)} &= \lim_{t\to\infty} \min_{j\in\mathcal{V}_\mathcal{G}} \left\{\frac{a_i(t)}{a_j(t)}\right\} \\ &= \min_{j\in\mathcal{V}_\mathcal{G}} \lim_{t\to\infty} \left\{\frac{a_i(t)}{a_j(t)}\right\} = \min_{j\in\mathcal{V}_\mathcal{G}}\{c_{ij}\} \triangleq c_i > 0. \end{aligned}$$

4.1 Convergence of the Auxiliary System (8)

The main result in this subsection is based on the following two Lemmas.

Lemma 5. If Condition (C1) and (C2) hold, and $\|\Xi(0)\|_2$ is bounded, then $|E\{\xi_i(t)\}|$ ($i = 1, \cdots, N$) and $E\{\|\Xi(t)\|_2^2\}$ are bounded.

Proof. See Appendix A.

Lemma 6. (Wang et al. [2013a]). Consider the differential equation

$$\dot{\zeta}(t) = -a(t)J_r(\lambda)\zeta(t),$$

where $\zeta(t) = (\zeta_1(t), \cdots, \zeta_r(t))^T \in \mathbb{R}^r$, $a(t) > 0$, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, and $J_r(\lambda) \in \mathbb{R}^{r\times r}$ is a Jordan block with diagonal element λ. Its state transition matrix is

$$\Phi_\lambda(t, t_0) = \begin{bmatrix} P_0^\lambda(t, t_0) & P_1^\lambda(t, t_0) & \cdots & P_r^\lambda(t, t_0) \\ 0 & P_0^\lambda(t, t_0) & \cdots & P_{r-1}^\lambda(t, t_0) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_0^\lambda(t, t_0) \end{bmatrix},$$

where $P_0^\lambda(t, t_0) = \exp(-\lambda \int_{t_0}^t a(\tau)d\tau)$ and $P_i^\lambda(t, t_0) = -\int_{t_0}^t a(\tau)P_0^\lambda(t, \tau)P_{i-1}^\lambda(\tau, t_0)d\tau$, $i = 1, 2, \cdots, r-1$. Moreover, if $\int_0^\infty a(t)dt = 0$, then $\lim_{t\to\infty} \Phi_\lambda(t, t_0) = \Theta_r$.

Theorem 7. If Conditions (C1), (C2) and (C3) hold, there exists a deterministic value ξ^* such that $\lim_{t\to\infty} \xi_1(t) = \xi^*$ and $\lim_{t\to\infty} E\{(\xi_i(t) - \xi^*)^2\} = 0$, $i = 2, \cdots, N$.

Proof. By (1), the Itô stochastic differential equation (9) can be rewritten as

$$d\xi_1(t) = -K_2K_3C\Delta_1(t)dt, \tag{10a}$$

$$d\Xi_2(t) = -\mathfrak{A}_2(t)L_1\xi_1(t) - \mathfrak{A}_2(t)L_2\Xi_2(t) + \mathfrak{A}_2(t)\bar{\Sigma}_2dW_2(t) - (I_{N-1} \otimes (K_2K_3C))\Delta_2(t)dt, \tag{10b}$$

where $\Delta_1(t) = x_1(t) - \hat{x}_1(t)$, $\Xi_2(t) = (\xi_2(t), \cdots, \xi_N(t))^T$, $\mathfrak{A}_2(t) = \mathrm{diag}(a_2(t), \cdots, a_N(t))$, $\bar{\Sigma}_2 = \mathrm{diag}(\bar{R}_2, \cdots, \bar{R}_N)$, $\Delta_2(t) = (x_2^T(t) - \hat{x}_2^T(t), \cdots, x_N^T(t) - \hat{x}_N^T(t))^T$, and $W_2(t)$ is the $nN(N-1)$-dimensional standard Brownian motion.

By (10a), it follows that $\xi_1(t)$ is convergent to $\xi^* = \xi_1(0) + \int_0^\infty K_2K_3C\exp((A + K_3C)t)(x_1(0) - \hat{x}_1(0))dt$. Since $A + K_3C$ is *Hurwitz*, it is easy to see that $|\xi^*| < \infty$.

Let $\hat{\Xi}_2(t) = \Xi_2(t) - 1_{N-1}\xi_1(t)$ and $c_i(t) = a_i(t)/\bar{a}(t)$ ($i = 1, \cdots, N$), it can be obtained by (10b) and Lemma 1 that

$$d\hat{\Xi}_2 = -\bar{a}(t)\mathfrak{C}^*L_2\hat{\Xi}_2(t) + \bar{a}(t)(\mathfrak{C}^* - \mathfrak{C}_2(t))L_2\hat{\Xi}_2(t) + \hat{\Delta}(t)dt + \mathfrak{A}_2(t)\bar{\Sigma}_2dW_2(t), \tag{11}$$

where $\mathfrak{C}_2(t) = \mathrm{diag}(c_2(t), \cdots, c_N(t))$; $\mathfrak{C}_2^* = \mathrm{diag}(c_2^*, \cdots, c_N^*)$; and $\hat{\Delta}(t) = (I_{N-1} \otimes K_2K_3C)\Delta_2(t) - 1_{N-1}K_2K_3C\Delta_1(t)$.

By Itô formula, the solution to stochastic differential equation (11) is

$$\hat{\Xi}_2(t) = \Pi_1(t) + \Pi_2(t) + \Pi_3(t) + \Pi_4(t).$$

where

$$\Pi_1(t) = \Phi(t, 0)\hat{\Xi}_2(0);$$
$$\Pi_2(t) = \int_0^t \Phi(t, \tau)\bar{a}(\tau)(\mathfrak{C}_2^* - \mathfrak{C}_2(\tau))L_2\hat{\Xi}_2(\tau)d\tau;$$
$$\Pi_3(t) = \int_0^t \Phi(t, \tau)\hat{\Delta}(\tau)d\tau;$$
$$\Pi_4(t) = \int_0^t \Phi(t, \tau)\mathfrak{A}_2(\tau)\bar{\Sigma}_2dW_2(\tau);$$

and $\Phi(\cdot, \cdot)$ is the state transition matrix of $dX(t) = -\bar{a}(t)\mathfrak{C}_2^*L_2X(t)dt$. In the rest of this proof, the convergence of $\Pi_1(t)$, $\Pi_2(t)$, $\Pi_3(t)$, and $\Pi_4(t)$ are analyzed, respectively.

Firstly, the convergence of $\Pi_1(t)$ is studied. Let T be a nonsingular matrix such that $T^{-1}\mathfrak{C}_2^*L_2T = \mathrm{diag}(J_{r_1}(\lambda_1), \cdots, J_{r_s}(\lambda_s))$ where $J_{r_i}(\lambda_i) \in \mathbb{R}^{r_i\times r_i}$ is Jordan block whose diagonal element is λ_i, and $\{\lambda_1, \cdots, \lambda_s\}$ are the eigenvalues of $\mathfrak{C}_2^*L_2$. By Lemma 2 and Condition (C1), $\Re(\lambda_i) > 0$ ($i = 1, \cdots, s$). Therefore,

$$\Phi(t, t_0) = T\mathrm{diag}(\Phi_{\lambda_1}(t, t_0), \cdots, \Phi_{\lambda_s}(t, t_0))T^{-1}, \tag{12}$$

where $\Phi_{\lambda_i}(t, t_0)$ ($i = 1, \cdots, s$) are defined in Lemma 6 with $a(t) = \bar{a}(t)$, $r = r_i$, and $\lambda = \lambda_i$. By Lemma 6 and Condition (C2), it is obtained that $\lim_{t\to\infty} \Phi(t, t_0) = \Theta_{N-1}$ which indicates $\lim_{t\to\infty} \Pi_1(t) = 0_{N-1}$.

Secondly, the convergence of $\Pi_2(t)$ is analyzed. By (12) and Lemma 6, the elements of $\Pi_2(t)$ are linear combinations of following terms:

$$I_{ijkl}(t) = \int_0^t \bar{a}(t)P_i^{\lambda_j}(t, \tau)(c_k^* - c_k(\tau))\xi_l(\tau)d\tau,$$

where $0 \le i \le N-1$, $1 \le j \le s$, $2 \le k \le N$, $1 \le l \le N$ and $P_j^{\lambda_j}(t, t_0)$ is defined in Lemma 6 with $a(t) = \bar{a}(t)$. By L'Hôspital rule and the knowledge of mean square integral, it follows that

$$\lim_{t\to\infty} E^{\frac{1}{2}}\{|I_{ijkl}(t)|^2\} \le \frac{M}{(\Re(\lambda_j))^{i+1}} \lim_{t\to\infty}(c_k^* - c_k(t)),$$

where $M = \max_{i\in\mathcal{V}_\mathcal{G}} \sup_{t\ge 0} E^{\frac{1}{2}}\{|\xi_i(t)|^2\}$, which is bounded by Lemma 5. Hence, $\lim_{t\to\infty} E\{|I_{ijkl}(t)|^2\} = 0$, which implies that $\lim_{t\to\infty} E\{\|\Pi_2(t)\|_2^2\} = 0$.

Thirdly, the convergence of $\Pi_3(t)$ is studied. Since $A + K_3C$ is *Hurwitz*, there must exist two positive constants $L \le \infty$ and γ such that $\|\hat{\Delta}(t)\|_2 < Le^{-\gamma t}$. Then, it can be proved that

$$\lim_{t\to\infty} \left\| \Pi_3(t) - \int_0^t \Phi_\infty\hat{\Delta}(\tau)d\tau \right\|_2 = 0,$$

where $\Phi_\infty = \lim_{t\to\infty} \Phi(t, t_0)$. By the analysis in the first step, $\lim_{t\to\infty} \Phi(t, t_0) = \Theta_{N-1}$ which means that $\lim_{t\to\infty} \Pi_3(t) = 0_{N-1}$.

Finally, the convergence of $\Pi_4(t)$ is analyzed. It is easy to see that the elements of $E\{\Pi_4(t)\Pi_4^T(t)\}$ are linear combinations of the following terms:

$$H_{ijkl}(\lambda_p, \lambda_q, t) = \int_0^t a_i(\tau)a_j(\tau)P_k^{\lambda_p}(t, \tau)P_l^{\lambda_q}(t, \tau)d\tau,$$

where $2 \le i, j \le N$; $0 \le k, l \le N-1$; $1 \le p, q \le s$. Then by L'Hôspital rule and mathematical induction, it can be proved that $\lim_{t\to\infty} H_{ijkl}(\lambda_p, \lambda_q, t) = 0$ which indicates that $\Pi_4(t)$ is convergent in mean square to 0_{N-1}.

By the above analysis, it can be obtained that $\hat{\Xi}_2(t)$ is convergent in mean square to 0_{N-1}. Because $\xi_1(t)$ is convergent to ξ^* and $\hat{\Xi}_2(t) = \Xi_2 - 1_{N-1}\xi_1(t)$, it is proved that $\lim_{t\to\infty} E\{(\xi_i(t) - \xi_i^*)^2\} = 0$, ($i = 2, \cdots, N$).

4.2 Mean Square Leader-Following Output Consensus

Lemma 8. Consider the following differential equation

$$\frac{d\varphi(t)}{dt} = \alpha\varphi(t) + \psi(t),$$

where $\Re(a) < 0$. If $\lim_{t\to\infty} \psi(t) = \psi^*$, then $\lim_{t\to\infty} \varphi(t) = -\psi^*/\alpha$. If $\psi(t)$ is a mean square continuous random process and convergent in mean square to ψ^*, then $\varphi(t)$ is convergent in mean square to $-\psi^*/a$.

Proof. It can proved easily by the knowledge of stochastic differential equation, which is omitted here due to the page limit.

Theorem 9. If Conditions (C1), (C2), (C3) and (C4) hold, then the mean square leader-following output consensus problem can be solved by the proposed protocol defined by (5).

Proof. Let $\{r_1, \cdots, r_{n-1}\}$ denote the roots of *parameter polynomial* defined by (6), then by the definition of $\xi_i(t)$, it follows that $\prod_{j=1}^{n-1}(\boldsymbol{D} - r_j)y_i(t) = \xi_i(t)$ $(i = 1, \cdots, N)$, where $\boldsymbol{D}$ is the differential operator, i.e., $\boldsymbol{D}^k y_i(t) = y_i^{(k)}(t)$.

By Theorem 7, if Conditions (C1), (C2), and (C3) hold, then $\xi_1(t)$ is convergent to a deterministic value ξ^*. This together with Lemma 8 and Condition (C4) implies that $\prod_{i=2}^{n-1}(\boldsymbol{D} - r_i)y_1(t)$ is convergent to $-\xi^*/r_1$. By repeating this procedure $n-1$ times, we have that $y_1(t)$ is convergent to $\xi^*/(\prod_{i=1}^{n-1}(-r_i)) = \xi^*/b_1$.

By the similar procedure, it can be proved that $y_i(t)$ $(i = 2, \cdots, N)$ are convergent in mean square to ξ^*/b_1. Therefore, $\lim_{t\to\infty} E\{(y_i(t) - y_1(t))^2\} = 0$ $(i = 2, \cdots, N)$, which closes the proof.

5. SIMULATION

Consider a MAS composed of five agents whose communication network is shown in Fig. 1. From Fig. 1, we know that the communication topology graph has a spanning tree. And agent 1 is the leader and the others are followers. The agent's dynamics is modeled by the following SISO system $y_i^{(4)}(t) + 2y_i^{(3)}(t) - y_i^{(1)}(t) + y_i(t) = u_i(t)$.

According to Section 3 and Theorem 9, the parameter vectors K_1, K_2, and K_3 are set to be $K_1 = (1, -2, -4, -3)$, $K_2 = (1, 3, 3, 1)$, and $K_3 = (-17, -97, -196, -44)^T$. The initial states of agents are set to be $x_1(0) = (10, 0, 0, 0)^T$, $x_2(0) = (5, 0, 0, 0)^T$, $x_3(0) = (12, 0, 0, 0)^T$, $x_4(0) = (16, 0, 0, 0)^T$ and $x_5(0) = (6, 0, 0, 0)$. The initial estimated states are set to be $\hat{x}_1(0) = (9, 0, 0, 0)^T$, $\hat{x}_2(0) = (7, 0, 0, 0)^T$, $\hat{x}_3(0) = (10, 0, 0, 0)^T$, $\hat{x}_4(0) = (12, 0, 0, 0)^T$ and $\hat{x}_4(0) = (8, 0, 0, 0)$, The time-varying gain of each agent is given in Table 5. It is easy to see that

$$\bar{a}(t) = \max_{i=1,\cdots,5}\{a_i(t)\} = \begin{cases} \dfrac{5}{3t+1}; & 0 \le t < \dfrac{1}{7} \\ \dfrac{4}{t+1}; & t \ge \dfrac{1}{7} \end{cases}.$$

Therefore, Conditions (C2) and (C3) hold. And it can calculated that $c_1^* = 1/8$, $c_2^* = 1$, $c_3^* = 5/12$, $c_4^* = 1/2$ and $c_5^* = 5/16$ in Lemma 4.

The simulation result is shown in Fig. 2. We can see that the leader's output is convergent to $y^* = 6.7619$ and all followers' outputs are convergent to y^* in the mean square sense. Therefore the proposed protocol defined by (5) is capable of solving the mean square leader-following output consensus of the MAS defined by (2).

6. CONCLUSION

This paper studies the leader-following output consensus of linear MASs. The contribution of this paper can be summarized as follows: (1) the agent's dynamics is modeled by an SISO system and only the agent's output information is available for the controller design; (2) the communication network among agents is assumed to be corrupted by the additive noises; (3) the time-varying gain is employed to attenuate the noise's effect and each agent is allowed to have its own gain; and (4) some sufficient conditions are given to ensure the mean square leader-following output consensus. Finally, the theoretical results are verified by the illustrative example.

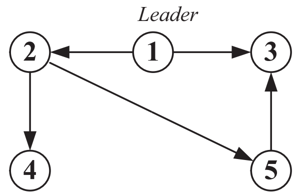

Fig. 1. The communication topology of the MAS composed of five agents.

Table 1. The time-varying gain $a_i(t)$ used in the protocol defined by (5).

$a_1(t)$	$a_2(t)$	$a_3(t)$	$a_4(t)$	$a_5(t)$
$\frac{0.5}{t+1}$	$\frac{4}{t+1}$	$\frac{5}{3t+1}$	$\frac{2}{t+2}$	$\frac{5}{4t+1}$

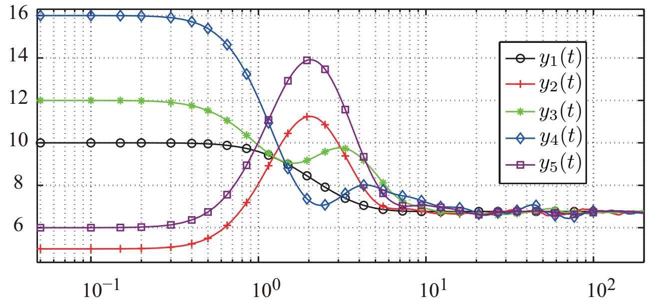

Fig. 2. The trajectories of five agents' outputs.

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Appendix A. THE PROOF OF LEMMA 5

Firstly, it is proved that $|E\{\xi_i(t)\}|$ $(i = 1, \cdots, N)$ are bounded. Because $A + K_3C$ is *Hurwitz*, there must exist two positive numbers $M_1, \beta < \infty$ such that $\forall i = 1, \cdots, N$,

$$|K_2K_3C(x_i(t)-\hat{x}_i(t))| = |K_2K_3Ce^{(A+K_3C)t}(x_i(0)-\hat{x}_i(0))| \leq M_1e^{-\beta t}. \quad \text{(A.1)}$$

By (1) and (9), it is obtained that $d\xi_1(t) = -K_2K_3C(x_1(t)-\hat{x}_1(t)dt$, which together with (A.1) leads to that

$$|\xi_1(t)| \leq |\xi_1(0)| + M_1 \int_0^t e^{-\beta\tau}d\tau \leq M_2 < \infty, \quad \text{(A.2)}$$

where $M_2 = |\xi_1(0)| + M_1 \int_0^\infty e^{-\beta t}dt$.

Let $\xi_{\max}(t) = \max_{i=1,\cdots,N}\{E\{\xi_i(t)\}\}$, then it follows by (9) and Lemma 3, that

$$D^+\xi_{\max}(t) \leq \max_{i=1,\cdots,N}(-K_2K_3C(x_i(t) - \hat{x}_i(t))) \leq M_1e^{-\beta t}.$$

This together with [Lakshmikantham and Leela , 1969] implies that $\xi_{\max}(t) \leq M_1 \int_0^\infty e^{-\beta\tau}d\tau \triangleq M_3 < \infty$.

$$\xi_{\max}(t) \leq M_1 \int_0^\infty e^{-\beta\tau}d\tau \triangleq M_3 < \infty.$$

Similarly, it can be obtained that $\xi_{\min}(t) \geq -M_3$ where $\xi_{\min}(t) = \min_{i=1,\cdots,N}\{E\{\xi_i(t)\}\}$. Hence $|E\{\xi_i(t)\}| \leq M_3,\ i = 1, \cdots, N$.

Secondly, it is proved that $E\{\|\Xi(t)\|_2^2\}$ is bounded. This part of the proof is motivated by the proof of Lemma 3 in [Wang et al., 2014]. Let $V_i(t) = E\{\xi_i^2(t)\}$ and $V(t) = E\{\|\Xi(t)\|_2^2\}$. By Itô formula, it is obtained that

$$\begin{aligned} dV_i(t) &= E\{2(d\xi_i(t)\xi_i(t) + (d\xi_i(t))^2\} \\ &\leq a_i(t) \sum_{j\in\mathcal{N}_i} \alpha_{ij}(V_j(t) - V_i(t))dt + M_1M_3e^{-\beta t}dt \\ &\quad + a_i^2(t) \sum_{j\in\mathcal{N}_i} \alpha_{ij}^2 K_2\rho_{ij}^2K_2^T dt + o(dt). \end{aligned} \quad \text{(A.3)}$$

By (9) and (A.3), it can be proved that $V_i(t)$ and $\dot{V}_i(t)$ $(i = 1, \cdots, N)$ are continuous functions.

Assume that $V(t)$ is unbounded. Then, for $\forall G > \max_{i=1,\cdots,N}\{V_i(0), M_2^2\}$, there must exist $t_0, \Delta t_0 > 0$ and $p_0 \in \mathcal{V}_\mathcal{G}$ such that: for $\forall t \leq t_0$, $V_i(t) \leq G$ $(i = 1, \cdots, N)$; and for $\forall t \in (t_0, t_0 + \Delta t_0)$, $V_{p_0}(t) > G$, $\dot{V}_{p_0}(t) > 0$, and $V_{p_0}(t) \geq V_i(t)$ $(i = 1, \cdots, N)$. If $\mathcal{N}_{p_0} = \emptyset$, then agent p_0 is the leader. However, by (A.2), $V_1(t) \leq M_2^2 < G$ which causes a contradiction. The proof is closed. If $\mathcal{N}_{p_0} \neq \emptyset$, it follows by (A.3) that for $\forall t \in (t_0, t_0 + \Delta t_0)$ and any $p_1 \in \mathcal{N}_{p_0}$

$$\begin{aligned} V_{p_1}(t) > V_{p_0}(t) - a_{p_0}(t) \sum_{j\in\mathcal{N}_{p_0}} \alpha_{p_0j}^2 K_2\rho_{p_0j}^2K_2^T/\alpha_{p_0p_1} \\ - \frac{M_1M_3e^{-\beta t}}{a_i(t)} \geq G - C_0. \end{aligned} \quad \text{(A.4)}$$

where $C_0 = \max_{i=1,\cdots,N}\{M_1M_3e^{-\beta t}/a_i(t)\} < \infty + \max_{t\geq 0; i=1,\cdots,N; k\in\mathcal{N}_i}\{a_i(t) \sum_{j\in\mathcal{N}_i} \alpha_{ij}^2 K_2\rho_{ij}^2K_2^T/\alpha_{ik}\}$.

Because G can be arbitrarily large, we can assume that $G - C_0 > \max\{V_{p_1}(0), M_2^2\}$. There must exist $t_1 \in (0, t_0)$ and $\Delta t_1 > 0$ such that $t_1 + \Delta t_1 \leq t_0$; $\forall t \leq t_1$, $V_{p_1}(t) \leq G - C_0$; and $\forall t \in (t_1, t_1 + \Delta t_1)$, $V_{p_1}(t) > G - C_0$ and $\dot{V}_{p_1}(t) > 0$. If $\mathcal{N}_{p_1} = \emptyset$, then agent p_1 is the leader. However, by (A.2), $V_1(t) \leq M_2^2 < G - C_0$ which causes a contradiction. The proof is closed. If $\mathcal{N}_{p_1} \neq \emptyset$, it follows by (A.3) that for $\forall t \in (t_1, t_1 + \Delta t_1)$ and any $p_2 \in \mathcal{N}_{p_1}$

$$\begin{aligned} V_{p_2}(t) > &V_{p_1}(t) - a_{p_1}(t) \sum\nolimits_{j\in\mathcal{N}_{p_1}} \alpha_{p_1j}^2 K_2\rho_{p_1j}^2K_2^T/\alpha_{p_1p_2} \\ &- \sum\nolimits_{j\in\mathcal{N}_{p_1}, j\neq p_2} \alpha_{p_1j}(V_j(t) - V_{p_1}(t))/(\alpha_{p_1p_2}) \\ &- M_1M_3e^{-\beta t}/a_i(t) \geq G - C_1, \end{aligned}$$

where $C_1 = 2C_0 + C_0 \max_{i=1,\cdots,N; k\in\mathcal{N}_i}\{\sum_{j\in\mathcal{N}_i} \alpha_{ij}/\alpha_{ik}\} < \infty$.

Because Condition (C1) holds and there are finite agents in the system, by repeating the above procedure at most $N-1$ times, we can find the agent p_{N-1} is the leader (i.e., $p_{N-1} = 1$) and there exist $t_{n-2} \in (0, t_0)$ and $\Delta t_{n-2} > 0$ such that $t_{n-2} + \Delta t_{n-2} < t_0$; and $\forall t \in (t_{n-2}, t_{n-2} + \Delta t_{n-2})$, $V_1(t) > G - C_{n-2}$, where $C_{n-2} = C_0 + C_{n-3}(1 + \max_{i=1,\cdots,N; k\in\mathcal{N}_i}\{\sum_{j\in\mathcal{N}_i} \alpha_{ij}/\alpha_{ik}\}) < \infty$. Because of the arbitrariness of G, we know $V_1(t)$ can be arbitrarily large which contradicts (A.2). Therefore, $V(t)$ is bounded.