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Gain Scheduling Output Feedback Control of Linear Plants with Partial Actuator Saturation ⋆

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Abstract: From a gain-scheduling control perspective, we will study the output feedback control problem for linear systems with some of control channels subject to actuator saturation. This includes the scenario of all actuator saturation as a special case. A feedback controller, expressed in the form of linear fractional transformation, is proposed to guarantee regional stability of the closed-loop system and minimizes disturbance/error effect measured in $\mathcal{L}_2$ gain. The resulting synthesis condition is formulated as linear matrix inequalities (LMIs) and can be solved efficiently. A modified inverted pendulum is utilized to demonstrate the proposed approach.

1. INTRODUCTION

Actuator saturation is a typical nonlinearity and is widely encountered in control engineering. It can significantly deteriorate the performance of a closed-loop system or even render a stable system unstable. Therefore it has attracted a lot of attention from the control community (see, for example, Bernstein et al. [1995], Hu et al. [2001], Lin [1998], Tarbouriech et al. [1997] and the references therein).

Earlier research focuses on stable open-loop systems, for which various control problems have been studied in depth in the global or semi-global framework (see, for example, Lin [1998], Chitour [2001], Lin et al. [1996], Liu et al. [1996], Saberi et al. [1996], Sussmann et al. [1994]). Because many control systems encountered in control engineering are open-loop exponentially unstable, much effort has recently focused onto unstable open-loop systems with actuator saturation. As such systems under actuator saturation are null controllable only in a part of the state space, the objectives are to characterize the null controllable region Hu et al. [2001] and to design feedback controllers that work on a large portion of it or even the entire null controllable region (see, for example, Hindi et al. [1998], Nguyen et al. [1999], Hu et al. [2001], Paim et al. [2002], Hu et al. [2004], Fang et al. [2004]).

So far both state feedback technique and output feedback methods have been proposed through various framework(see, for example Hu et al. [2004], Nguyen et al. [1999], Paim et al. [2002], Scorletti et al. [2002], Wu et al. [2007, 2009], Dai et al. [2009]. More specifically, a saturation control synthesis method has been proposed in Dai et al. [2009] for the construction of output feedback controllers with an internal deadzone loop. It has been shown that the synthesis condition for such a controller can be formulated into the LMI form. In Wu et al. [2009], an output gain-scheduled saturation controller has been developed to attenuate the effect of the disturbances on the system output in addition to achieving local stability. Nevertheless, it has been shown that the control synthesis conditions are in bilinear matrix inequality forms and not easy to solve. Based on the work in Wu et al. [2009], by carefully choosing the auxiliary subspace in representing the saturation/deadzone nonlinearities, the output feedback synthesis condition can be recast into LMIs Wu [2011]. However, the disadvantage is that the form of the controller is somewhat complicated and the synthesis condition is also computationally expensive. It consists of $2^{n_u} + n_u + 1$ LMIs and $2 + n_u + 2 \times 2^{n_u}$ decision variables with n_u representing the number of inputs. Motivated by the systematic gain-scheduling control design techniques developed in Packard [1994], Apkarian [1995], an output feedback controller in the form of linear fractional transformation has been proposed to guarantee regional stability of the closed-loop system and provide disturbance/error attenuation measured in $\mathcal{L}_2$ gain in Ban et al. [2012]. The resulting synthesis condition can be formulated as a set of LMIs and can be solved efficiently. Moreover, it turns out that the controller proposed in Dai et al. [2009] is a special case of the controller proposed in Ban et al. [2012].

While in this research, we will extend the work in Ban et al. [2012] to the linear plants with partial control channels subject to actuator saturation, which includes the plants with all the input channels saturated as a

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special case. The study on the linear plants with partial control channels subject to saturation are important from engineering viewpoint. For multi-input systems, one may wonder which input channel saturation will have a stronger effect on the closed-loop performance. To answer this question, we can calculate the optimal performance of the closed-loop system with just one input channel saturated by using the results developed in this research and repeat the procedure for other input channels. By comparing the results from this trade study, proper selection of actuators can be made to reduce the cost of the control system.

The notations used in this paper are rather standard. $\mathbf{R}$ stands for the set of real numbers and $\mathbf{R}_+$ for the non-negative real numbers. $\mathbf{R}^{m\times n}$ is the set of real $m \times n$ matrices. We use $\mathbf{S}^{n\times n}$ to denote real, symmetric $n \times n$ matrices, and $\mathbf{S}_+^{n\times n}$ for positive-definite matrices. The identity matrix of any dimension is denoted by I. A block-diagonal matrix with matrices $X_1, X_2, \cdots, X_p$ on its main diagonal is denoted as $\mathrm{diag}\{X_1, X_2, \cdots, X_p\}$. In large symmetric matrix expressions, terms denoted as $\star$ will be induced by symmetry. $Co\{S\}$ denotes the convex hull of a set S.

2. PROBLEM FORMULATION

Consider the linear plant subject to partial actuator saturations,

$$\begin{cases} \dot{x} = Ax + B_1 d + B_{2s}\mathrm{sat}(u_s) + B_{2n}u_n \\ e = C_1 x + D_{11} d + D_{12s}\mathrm{sat}(u_s) + D_{12n}u_n \\ y = C_2 x + D_{21} d \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is the plant state; $u_s \in \mathbf{R}^{n_s}$ is the control input subject to saturation; $u_n \in \mathbf{R}^{n_n}$ represent the control inputs which will not be saturated; $d \in \mathbf{R}^{n_d}$ is the exogenous input, possibly including disturbance, measurement noise or reference signals; $y \in \mathbf{R}^{n_y}$ is the measurement output and $e \in \mathbf{R}^{n_e}$ is the performance output. $\mathrm{sat}(\cdot)$ is a vector saturation function with the saturation levels given by a vector $\bar{u} \in \mathbf{R}^{n_s}$, $\bar{u}_i > 0, i = 1, \cdots, n_s$. More specifically,

$$\mathrm{sat}(u_s) = \begin{bmatrix} \sigma(u_1) \\ \vdots \\ \sigma(u_{n_s}) \end{bmatrix},\ \sigma(u_i) = \mathrm{sgn}(u_i)\min\{\bar{u}_i, |u_i|\}.$$

Throughout this article, it is assumed that $(A, [B_{2s}\ B_{2n}])$ is stabilizable and (C_2, A) is detectable.

The deadzone nonlinearity is closely related to saturation function and is defined by $\mathrm{dz}(u_s) = u_s - sat(u_s)$, which could be utilized to obtain more relaxed conditions for regional analysis of unstable plants. The property of the deadzone function is shown in the following lemma.

Lemma 1. (Hu et al. [2006]). Let $h(x) = Hx$ be a linear map and suppose $e_i^T Hx \in [-\bar{u}_i, \bar{u}_i]$, where e_i denotes the ith column of the unity matrix. For any u_i, we have $\sigma(u_i) \in \mathrm{Co}\{u_i, e_i^T Hx\}$ and $\mathrm{dz}(u_i) = \theta_i(u_i - e_i^T Hx)$ for some $\theta_i \in [0, 1]$.

Using the deadzone function, the dynamic equation of the plant can be rewritten as

$$\begin{bmatrix} \dot{x} \\ u_s \\ e \\ y \end{bmatrix} = \begin{bmatrix} A & -B_2E & B_1 & B_2 \\ 0 & 0 & 0 & E^T \\ C_1 & -D_{12}E & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ p \\ d \\ u \end{bmatrix} \tag{2}$$

$$p = \mathrm{dz}(u_s). \tag{3}$$

where $B_2 = [B_{2s}\ B_{2n}], D_{12} = [D_{12s}\ D_{12n}]$ and $u = \begin{bmatrix} u_s \\ u_n \end{bmatrix}, E = \begin{bmatrix} I_{n_s} \\ 0 \end{bmatrix}$.

For a matrix $[H_1\ H_2]$, we will define the set

$$\mathcal{L}([H_1\ H_2]) = \left\{(x, x_k) \in \mathbf{R}^{2n} : \left| e_j^T [H_1\ H_2] \begin{bmatrix} x \\ x_k \end{bmatrix} \right| \le \bar{u}_j, j = 1, \cdots, n_s \right\}.$$

Again e_j denotes the jth column of the identity matrix. x_k is the state of the controller to be designed. Then from Lemma 1, the nonlinear equation (3) can be replaced by

$$p = \Theta(u_s - H_1 x - H_2 x_k) \tag{4}$$

where

$$\Theta = \{\mathrm{diag}\{\theta_1, \cdots, \theta_{n_s}\} :\ 0 \le \theta_j \le 1, j = 1, \cdots, n_s\}.$$

The gain-scheduled controller as described in eqns. (5)-(6) will be synthesized for the plant (1)

$$\begin{bmatrix} \dot{x}_k \\ q_k \\ u \end{bmatrix} = \begin{bmatrix} A_k & B_{k0} & B_{k1} \\ C_{k0} & D_{k00} & D_{k01} \\ C_{k1} & D_{k10} & D_{k11} \end{bmatrix} \begin{bmatrix} x_k \\ p_k \\ y \end{bmatrix} \tag{5}$$

$$p_k = \Theta(q_k - H_{k1}x - H_{k2}x_k), \tag{6}$$

where x_k is the state of the controller. The number of controller states is chosen to be the same as that of plant states n. $p_k, q_k \in \mathbf{R}^{n_s}$ are the pseudo-input and output of the controller, respectively. As a result, the controller parameters will depend on Θ (i.e., the actuator saturation status) in the form of LFT. Note that this form of gain-scheduling controller is similar to the ones shown in Apkarian [1995] and Packard [1994]. But the measurable parameter is restricted to $0 \le \Theta \le I$ here.

It is observed from eqn. (6) that the state vector of the plant x explicitly appears in the controller. If all the states of the plant are measurable, the parameter Θ can be computed in real time and the gain-scheduled controller can be implemented directly. However, for the output feedback control problem, the plant state information is generally not available. In this case we can assume that $H_1 = H_{k1} = 0$. Then we can implement such an output controller. In order to get a complete understanding of the role of H_1 and H_{k1}, however, we still include H_1 and H_{k1} as optimization variables in the following derivation.

In this research, our objective is to synthesize a gain-scheduled output feedback controller to stabilize linear systems with partial actuator saturation (1) and minimize its disturbance effect. For disturbance attenuation, we are mainly concerned with a class of energy-bounded disturbances

$$\mathcal{W}_s = \left\{ d :\ \mathbf{R}_+ \to \mathbf{R}^{n_d}, \int_0^\infty\ d^T(\tau)d(\tau)d\tau \le s^2,\ d \in \mathcal{L}_2 \right\}.$$

By combining the plant and the controller dynamics together, we obtain the closed-loop system as below:

$$\begin{bmatrix} \dot{x}_{cl} \\ q_{cl} \\ e \end{bmatrix} = \begin{bmatrix} A_{cl} & B_{0,cl} & B_{1,cl} \\ C_{0,cl} & D_{00,cl} & D_{01,cl} \\ C_{1,cl} & D_{10,cl} & D_{11,cl} \end{bmatrix} \begin{bmatrix} x_{cl} \\ p_{cl} \\ d \end{bmatrix} \tag{7}$$

$$p_{cl} = \begin{bmatrix} \Theta & 0 \\ 0 & \Theta \end{bmatrix} (q_{cl} - Hx_{cl}), \tag{8}$$

where $x_{cl} = [x \;\; x_k]^T$, $q_{cl} = [u_s \;\; q_k]^T$, $p_{cl} = [p \;\; p_k]^T$, and

$$\begin{bmatrix} A_{cl} & B_{0,cl} & B_{1,cl} \\ C_{0,cl} & D_{00,cl} & D_{01,cl} \\ C_{1,cl} & D_{10,cl} & D_{11,cl} \end{bmatrix} = \left[\begin{array}{cc|cc|c} A & 0 & -B_2E & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_1 & 0 & -D_{12}E & 0 & D_{11} \end{array}\right]$$

$$+ \left[\begin{array}{ccc} 0 & 0 & B_2 \\ I & 0 & 0 \\ \hline 0 & 0 & E^T \\ 0 & I & 0 \\ \hline 0 & 0 & D_{12} \end{array}\right] \begin{bmatrix} A_k & B_{k0} & B_{k1} \\ C_{k0} & D_{k00} & D_{k01} \\ C_{k1} & D_{k10} & D_{k11} \end{bmatrix} \left[\begin{array}{cc|cc|c} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ C_2 & 0 & 0 & 0 & D_{21} \end{array}\right]$$

$$:= \begin{bmatrix} \mathcal{A}_0 & \mathcal{B}_{01} & \mathcal{B}_{02} \\ \mathcal{C}_{01} & \mathcal{D}_{011} & \mathcal{D}_{012} \\ \mathcal{C}_{02} & \mathcal{D}_{021} & \mathcal{D}_{022} \end{bmatrix} + \begin{bmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix} \begin{bmatrix} A_k & B_{k0} & B_{k1} \\ C_{k0} & D_{k00} & D_{k01} \\ C_{k1} & D_{k10} & D_{k11} \end{bmatrix} [\mathcal{G}_0 \; \mathcal{G}_1 \; \mathcal{G}_2]$$

$$H = \begin{bmatrix} H_1 & H_2 \\ H_{k1} & H_{k2} \end{bmatrix}.$$

The above linear system description is valid for the partially saturated system when $(x, x_k) \in \mathcal{L}([H_1 \;\; H_2])$.

3. GAIN-SCHEDULING CONTROL WITH PARTIAL ACTUATOR SATURATION

Motivated by the general results in both $\mathcal{H}_\infty$ and gain-scheduling control theories Gahinet et al. [1994], Packard [1994], the main theorem of this research will be presented in the following theorem.

Theorem 2. Given scalars $s > 0$ and $\gamma > 0$, if there exist positive definite matrices $R, S \in \mathbf{S}_+^n$, diagonal matrices $L, J > 0$ and matrices $\hat{H}_1, \hat{H}_2, \hat{H}_{k1} \in R^{n_s \times n}$ such that

$$\begin{bmatrix} \mathcal{N}_{\tilde{\Phi}}^T & 0 \\ 0 & I \end{bmatrix} \times \left[\begin{array}{ccc|c} RA^T + AR & -B_2EL - \hat{H}_2^T & RC_1^T & B_1 \\ -LE^TB_2^T - \hat{H}_2 & -2L & -LE^TD_{12}^T & 0 \\ C_1R & -D_{12}EL & -\gamma^2 I & D_{11} \\ \hline B_1^T & 0 & D_{11}^T & -I \end{array}\right] \times \begin{bmatrix} \mathcal{N}_{\tilde{\Phi}} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} \mathcal{N}_{\Gamma}^T & 0 \\ 0 & I \end{bmatrix} \times \left[\begin{array}{cc|cc} A^TS + SA & SB_1 & -SB_2E - \hat{H}_{k1}^T & C_1^T \\ B_1^TS & -I & 0 & D_{11}^T \\ \hline -E^TB_2^TS - \hat{H}_{k1} & 0 & -2J & -E^TD_{12}^T \\ C_1 & D_{11} & -D_{12}E & -\gamma^2 I \end{array}\right] \times \begin{bmatrix} \mathcal{N}_{\Gamma} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0 \tag{11}$$

$$\begin{bmatrix} L & I \\ I & J \end{bmatrix} > 0 \tag{12}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & e_\ell^T \hat{H}_2 & e_\ell^T \hat{H}_1 \\ \hat{H}_2^T e_\ell & R & I \\ \hat{H}_1^T e_\ell & I & S \end{bmatrix} \geq 0, \quad \forall\, \ell \in \mathbf{I}[1, n_s] \tag{13}$$

where $\mathcal{N}_{\tilde{\Phi}} := \left[\mathcal{N}_{\tilde{\Phi}_1}^T \; \mathcal{N}_{\tilde{\Phi}_2}^T \; \mathcal{N}_{\tilde{\Phi}_3}^T\right]^T = \operatorname{Ker}\left[B_2^T \;\; E \;\; D_{12}^T\right]$ and $\mathcal{N}_\Gamma := \left[\mathcal{N}_{\Gamma_1}^T \; \mathcal{N}_{\Gamma_2}^T\right]^T = \operatorname{Ker}\left[C_2 \;\; D_{21}\right]$ respectively, and

$$\begin{bmatrix} \hat{H}_1 & \hat{H}_2 \\ \hat{H}_{k1} & \hat{H}_{k2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ J & V \end{bmatrix} \begin{bmatrix} H_1 & H_2 \\ H_{k1} & H_{k2} \end{bmatrix} \begin{bmatrix} I & R \\ 0 & M^T \end{bmatrix} \tag{14}$$

with $MN^T = I - RS, UV^T = I - LJ$, then a nth-order gain-scheduling controller in the form of (5)-(6) will locally asymptotically stabilize the plant (1) and render the $\mathcal{L}_2$ gain of the closed loop system less than γ for any bounded disturbance $d \in \mathcal{W}_s$.

Proof. Use a quadratic Lyapunov function $V(x_{cl}) = x_{cl}^T P x_{cl}$ with $P > 0$ and a positive-definite matrix $\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2 & \Lambda_3 \end{bmatrix} > 0$ commutable with $\operatorname{diag}\{\Theta, \Theta\}$. It is clear that Λ_1, Λ_2 and Λ_3 should be diagonal matrices. Then a regional performance condition

$$\dot{V} + \frac{1}{\gamma^2} e^T e - d^T d + p_{cl}^T \Lambda (q_{cl} - Hx_{cl} - p_{cl}) + (q_{cl} - Hx_{cl} - p_{cl})^T \Lambda p_{cl} < 0 \tag{15}$$

and the set inclusion condition

$$\left\{x_{cl} : \; x_{cl}^T P x_{cl} \leq s^2\right\} \subset \mathcal{L}([H_1 \;\; H_2]) \tag{16}$$

will guarantee the closed-loop stability and $\mathcal{L}_2$ gain performance for partially saturated linear systems. Using the scaled Bounded Real Lemma and exploring the set inclusion relation Dai et al. [2009], we obtain the following two equations for the regional performance condition and the set inclusion condition.

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & \star & \star & \star \\ B_{0,cl}^T P + \Lambda(C_{0,cl} - H) & \Lambda(D_{00,cl} - I) + (D_{00,cl} - I)^T \Lambda & \star & \star \\ B_{1,cl}^T P & D_{01,cl}^T \Lambda & -I & \star \\ C_{1,cl} & D_{10,cl} & D_{11,cl} & -\gamma^2 I \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & e_\ell^T [H_1 \;\; H_2] \\ \begin{bmatrix} H_1^T \\ H_2^T \end{bmatrix} e_\ell & P \end{bmatrix} \geq 0, \qquad \ell \in \mathbf{I}[1, n_s]. \tag{18}$$

In the following, we will show that (9)-(13) is equivalent to the regional performance condition and the set inclusion condition.

Taking eqn. (7) into consideration, the inequality (17) can be rewritten as

$$\Psi + \Gamma^T \Pi^T \Phi + \Phi^T \Pi \Gamma < 0, \tag{19}$$

where

$$\Psi := \begin{bmatrix} \mathcal{A}_0^T P + P\mathcal{A}_0 & \star \\ \mathcal{B}_{01}^T P + \Lambda(\mathcal{C}_{01} - H) & \Lambda(\mathcal{D}_{011} - I) + (\mathcal{D}_{011} - I)^T \Lambda \\ \mathcal{B}_{02}^T P & \mathcal{D}_{012}^T \Lambda \\ \mathcal{C}_{02} & D_{021} \end{bmatrix}$$

$$\begin{bmatrix} \star & \star \\ \star & \star \\ -I & \star \\ \mathcal{D}_{022} & -\gamma^2 I \end{bmatrix}$$

$$\Pi := \begin{bmatrix} A_k & B_{k0} & B_{k1} \\ C_{k0} & D_{k00} & D_{k01} \\ C_{k1} & D_{k10} & D_{k11} \end{bmatrix}$$

$$\Gamma := \begin{bmatrix} \mathcal{G}_0 & \mathcal{G}_1 & \mathcal{G}_2 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc|c|c} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ C_2 & 0 & 0 & 0 & D_{21} & 0 \end{array}\right]$$

$$\Phi := \begin{bmatrix} \mathcal{E}_0^T P & \mathcal{E}_1^T \Lambda & 0 & \mathcal{E}_2^T \end{bmatrix} = \left[\begin{array}{cc|cc|c|c} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ B_2^T & 0 & E & 0 & 0 & D_{12}^T \end{array}\right] \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Applying Elimination Lemma Packard [1994], and partitioning Lyapunov matrix P and scaling matrix Λ according to plant and controller states as

$$P = \begin{bmatrix} S & N \\ N^T & \star_1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} R & M \\ M^T & \star_2 \end{bmatrix},$$
$$\Lambda = \begin{bmatrix} J & V \\ V^T & \star_3 \end{bmatrix}, \qquad \Lambda^{-1} = \begin{bmatrix} L & U \\ U^T & \star_4 \end{bmatrix},$$

we obtain the equations (9) and (10) from (19).

As for the eqns. (11)-(12), according to Packard [1994], $P > 0$, $P^{-1} > 0$ if and only if eqn. (11) holds. Similarly, $\Lambda > 0$, and $\Lambda^{-1} > 0$ is equivalent to condition (12).

To verify eqn. (13), we choose the invertible matrix $Z = \text{diag}\left\{1, \begin{bmatrix} R & M \\ I & 0 \end{bmatrix}\right\}$ and multiply condition (18) by Z from the left and Z^T from the right side. Using the fact $\begin{bmatrix} S & N \\ N^T & \star_1 \end{bmatrix} \begin{bmatrix} R & M \\ M^T & \star_2 \end{bmatrix} = I$, then condition (13) is confirmed through variable changes (14), which completes the proof.

Remark 3. Note that Theorem 2 provides a solvability condition for the existence of the gain-scheduled controller in the form of (5)-(6) which depends on both x and x_k. To get a practical output feedback controller, we need to set $\hat{H}_1 = \hat{H}_{k1} = 0$ in the control synthesis conditions (9)-(13), which corresponds to $H_1 = H_{k1} = 0$ as mentioned before.

Remark 4. The synthesis conditions (9)-(13) are in the form of LMIs and can be solved efficiently using interior point optimization algorithms. Moreover, the feasibility condition (9)-(13) can be formulated as the following optimization problem

$$\min_{R,S,L,J,\hat{H}_1,\hat{H}_2,\hat{H}_{k1}} \gamma^2$$
$$\text{subj. to } (9) - (13).$$

to minimize the closed-loop performance γ.

As a special case, if the open-loop system is asymptotically stable, the deadzone nonlinearity will be captured globally by the condition $\text{dz}(u) = \Theta u, 0 \le \Theta \le I$. Then the equivalent open-loop system becomes

$$\begin{bmatrix} \dot{x} \\ u_s \\ e \\ y \end{bmatrix} = \begin{bmatrix} A & -B_2 E & B_1 & B_2 \\ 0 & 0 & 0 & E^T \\ C_1 & -D_{12} E & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ p \\ d \\ u \end{bmatrix} \tag{20}$$

$$p = \Theta u_s. \tag{21}$$

In other words, auxiliary linear subspace is not required in the equivalent LFT description. Then the gain-scheduling control synthesis condition would degenerate to eqns. (9)-(12) with $\hat{H}_1 = \hat{H}_2 = 0$ and $\hat{H}_{k1} = 0$.

4. CONTROLLER RECONSTRUCTION

Given any feasible solution to the above LMI constraints, the parameters of a corresponding controller can be determined via a constructive procedure as follows:

Step 0 Set $\hat{H}_1 = 0$ and $\hat{H}_{k1} = 0$, by solving the LMIs (9)-(13), we obtain R, S, L, J, and $\hat{H}_2$.
Step 1 Choosing M, N and U, V matrices such that $MN^T = I - RS$ and $UV^T = I - LJ$.
Step 2 Compute $H_2 = \hat{H}_2 M^{-T}$.
Step 3 Calculate H_{k2} and Π by (19) as an LMI feasibility problem, then we obtain the controller gains.

For open-loop stable systems, the construction of output feedback saturation control follows the same procedure by setting $\hat{H}_1 = \hat{H}_2 = 0$ and $\hat{H}_{k1} = \hat{H}_{k2} = 0$.

5. NUMERICAL EXAMPLES

In this section, a modified inverted pendulum will be used to demonstrate the proposed gain-scheduling saturation control approach (see Fig. 1). One motor is used to drive the cart of the inverted pendulum. Different from a typical setup, another motor is installed on the moving cart to control the pendulum bar directly. This will provide a two-input controlled pendulum.

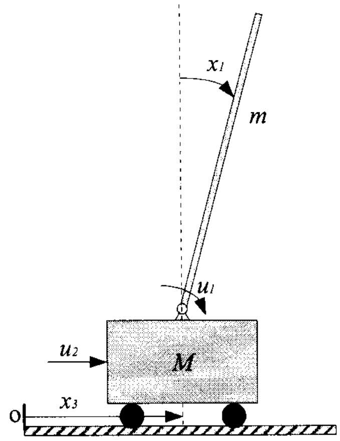


Fig. 1. Schematic drawing of the inverted pendulum.

The linearized equations of motion for the inverted pendulum at its equilibrium $(0, 0, 0, 0)$ are given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ \frac{amgl}{c} & -\frac{af}{c} & 0 & \frac{\mu ml}{c}\\ 0 & 0 & 0 & 1\\ -\frac{m^2l^2g}{c} & \frac{mlf}{c} & 0 & -\frac{b\mu}{c}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0\\ \frac{a}{c} & \frac{ml}{c} & \frac{a}{c} & -\frac{ml}{c}\\ 0 & 0 & 0 & 0\\ \frac{ml}{c} & \frac{b}{c} & -\frac{ml}{c} & \frac{b}{c}\end{bmatrix}\begin{bmatrix}d_1\\ d_2\\ \mathrm{sat}(u_1)\\ u_2\end{bmatrix} \tag{22}$$

$$\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\\ y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0.1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0.1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ \mathrm{sat}(u_1)\\ u_2\end{bmatrix}. \tag{23}$$

where $c = (M+m)(J+ml^2) - m^2l^2$, $a = M+m$, $b = J+ml^2$. M and m represent the mass of the cart and the bar respectively; l is the half length of the bar; J is bar's moment of inertia about its mass center; μ and f are friction coefficients of the bar and the cart, respectively. x_1, x_2, x_3, and x_4 denote the angle of the bar from vertical axis, its angular velocity, the position of the cart along the linear track and its velocity. d_1 is the disturbance torque applied to the bar, and d_2 represents the disturbance force on the cart. u_1, u_2 represents the motor torque applied to the bar and the force exerted on the cart. It is assumed that control input u_1 is subject to saturation with its magnitude $\bar{u}_1 = 1Nm$. On the other hand, no saturation limit is imposed on the control input u_2. n_1, n_2 are the measurement noises. Our design objective is to stabilize the inverted pendulum and optimize the controlled performance with input saturation on u_1. The values of pendulum parameters are listed in Table 1. It is easy to verify that the open-loop system is

Table 1. Parameters of the inverted pendulum.

parameter	value (units)
M	$1kg$
m	$0.1kg$
g	$9.8m/sec^2$
l	$0.4m$
f	0.01
μ	0.01

unstable because it contains one positive pole at 4.38. By solving the synthesis condition (9)-(13), we determine a suboptimal performance level of $\gamma = 6.0519$ for $s = 0.1$. The following controller matrices are also obtained using controller construction algorithm.

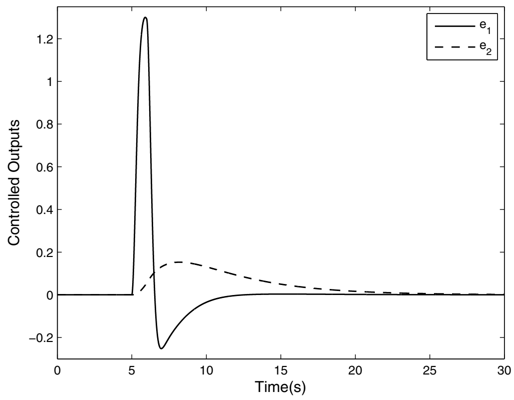
$$\Pi = \left[\begin{array}{cccc|c|cc} -14.99 & 1.14 & -0.56 & -0.18 & 15.25 & 0.56 & -0.00\\ -10.23 & -7.36 & -0.15 & -1.55 & -10.01 & -0.03 & -0.01\\ -0.54 & 0.01 & -1.94 & 0.99 & 0.58 & 1.94 & -0.00\\ -3.46 & 0.38 & -1.95 & -0.46 & 4.68 & 1.89 & -0.00\\ \hline 1.21 & -0.18 & 0.10 & -0.12 & -2.14 & -0.12 & 1.00\\ \hline 1.11 & -0.05 & 0.15 & -0.83 & -1.15 & -0.29 & -0.00\\ -0.71 & 0.55 & -0.10 & 0.05 & -2.14 & -0.12 & 1.00 \end{array}\right],$$

$H_2 = [-0.67\ -0.10\ -0.03\ -0.18]$,

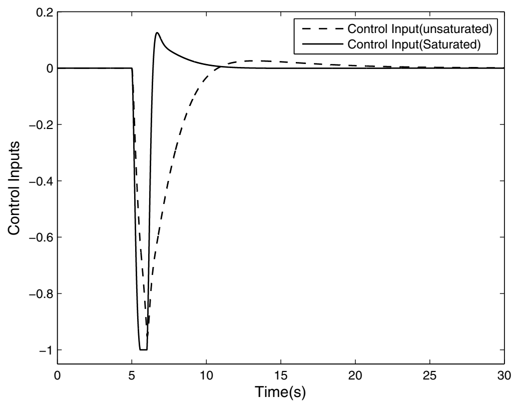
$H_{k2} = [-2.59\ 0.63\ -0.23\ -0.01]$,

with $H_1 = H_{k1} = 0$.

In our simulation, the disturbance d_1 is chosen as a pulse force of magnitude $0.42Nm$ starting at $5sec$ and ending at $6sec$, and other disturbances are all set to 0 for convenience. The response of the closed-loop system for $s = 0.1$ case is shown in Fig. 2. As suggested by the



(a) output e



(b) control u

Fig. 2. Response of the inverted pendulum: $s = 0.1$ case.

main theorem, one can obtain controllers that tolerate a larger class of disturbances with higher energy level by increasing the value of s. Nevertheless, the performance of the closed-loop system will degrade as seen from Table 2. Note that the condition number of the resulting controllers is not necessarily increasing monotonically.

Table 2. Relation between disturbance level s and performance γ.

disturbance s	performance γ	condition number
0.001	4.6055	4.9262e+004
0.01	4.6111	2.2854e+003
0.1	6.0519	593.1303
0.2	47.7137	386.7040
0.4	63.5936	628.4768
0.8	70.1903	1.5092e+003
2.0	74.3208	634.8559

When the value of s increases to 2, the relaxed performance level grows to $\gamma = 74.3208$. This controller can tolerate much larger disturbance than the one for $s = 0.1$.

Using the controller designed for $s = 2$, the closed-loop system remains stable for the disturbance up to $4.2Nm$. Nevertheless, for $s = 0.1$, the closed-loop system becomes unstable when the magnitude of d_1 increases to $0.46Nm$.

6. CONCLUSION

In this article, a gain-scheduling control approach is proposed to design output feedback controllers for linear plants with partial input channels saturated. Within this framework, the control design methods for linear systems and saturated linear systems are unified. The control synthesis condition is formulated as a convex optimization in terms of LMIs. For unstable plants, if the solution to the LMIs exists, a gain-scheduled output feedback controller can be constructed that guarantees regional stability of the closed-loop system and achieves prescribed disturbance/error attenuation performance. As for stable plants, a globally stabilizing output feedback control could be obtained. The gain-scheduling control design provides an alternative approach to the existing actuator saturation control methods.

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