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FREQUENCY IDENTIFICATION OF HAMMERSTEIN-WIENER SYSTEMS WITH PIECEWISE AFFINE INPUT NONLINEARITY

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Abstract. The problem of identifying Hammerstein-Wiener systems is addressed in the presence of linear subsystem of structure totally unknown and piecewise affine (or hard) input nonlinearity. A frequency identification approach is developed that determines the system frequency response (at a number of frequencies). A three-stage frequency identification method is developed to get estimates of the linear subsystem phase and modulus as well as estimates of the input and output nonlinearities. Finally, all suggested estimators are shown to be consistent.

Keywords: Nonlinear systems, Hammerstein-Wiener models, Frequency identification, Lissajous curves.

1. INTRODUCTION

This paper addresses the problem of identifying Hammerstein-Wiener systems consisting of a linear dynamic subsystem embedded between two nonlinear blocs (Fig. 1). The Hammerstein-Wiener like models are used in a wide range of applications such as chemical processes (Giri and Bai, 2010), ionospheric dynamics (Palanthandalam-Madapusi et al., 2005) and RF power amplifier modelling (Taringou et al., 2010). Different approaches are available that deal with Hammerstein-Wiener system identification. Amongst that are: iterative nonlinear optimization procedures (e.g. Ni et al., 2013; Schoukens, 2012), stochastic methods (e.g. Wang and Ding, 2008), and blind methods (e.g. Giri and Bai, 2010). Most available solutions suggest that, the output nonlinearity is invertible and the linear subsystem is parametric (e.g. Ni et al., 2013; Schoukens et al., 2012; Bai, 2002; Wang et al., 2009). Generally, the iterative methods necessitates a large amount of data, since computation time and memory usage drastically increase, and have local convergence properties which necessitates that a fairly accurate parameter estimates are available to initialize the search process. The stochastic methods are generally relied on specific assumption (e.g. gaussianity, persistent excitation, MA linear subsystems....).

In this paper, a frequency-domain identification scheme is designed for Hammerstein-Wiener systems involving linear subsystem of totally unknown structure, and hard input nonlinearity or piecewise affine with a limited number (q) of segments (Figs. 2a-b). Note that, the proposed frequency identification approach can be applied directly to Hammerstein (with hard nonlinearity) or Wiener models. The identification purpose is to estimate the system nonlinearities and the linear subsystem phase and gain ($\angle G(j\omega_k), |G(j\omega_k)|$), at a number of frequencies ω_k $(k = 1 \ldots m)$.

To close this section, we give an outline of the paper. Section2 formulates the problem and derives some preliminary results. The main results are given in Section 3 along with some remarks and proposition concerning the lissajous curves, useful to estimate the output nonlinearity. The identification scheme is designed and analyzed in Section 4. For space limitation all proofs are removed.

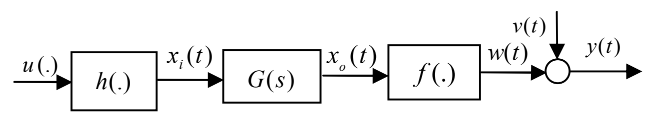

Fig. 1. Hammerstein-Wiener model structure

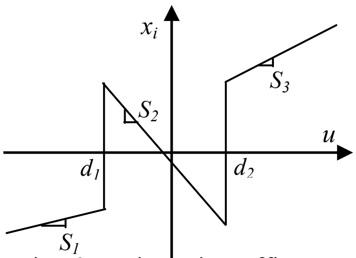

Fig. 2a. Piecewise affine function

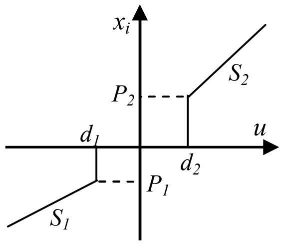

Fig. 2b. Nonlinearity with preload and dead zone

2. IDENTIFICATION PROBLEM STATEMENT

We are considering nonlinear systems that can be described by the Hammerstein-Wiener model (Fig.1), with hard input nonlinearity (h(.) is characterized by a set of straight lines segments). The above model is analytically described by the following equations:

$$x_i(t) = h(u(t)) \tag{1a}$$

$$x_o(t) = g(t) * x_i(t) \tag{1b}$$

$$y(t) = f(x_o(t)) + v(t) \tag{1c}$$

where $g(t) = L^{-1}(G(s))$ is the inverse Laplace transform of $G(s)$; the symbol * refers to the convolution operation; the only measurable signals are the system input $u(t)$ and output $y(t)$. The error $v(t)$ is zero-mean stationary sequence of independent random variables; it accounts for external noise, it is supposed to be ergodic. Apart from stability, no assumption is made on the linear subsystem $G(s)$ which may thus be infinite order. The input nonlinearity is subject to a couple of assumptions that will prove to be useful in the identification process:

A1. (i) $h(.)$ is a hard or piecewise affine (e.g. Figs. 2a-b) with a limited number of segments (e.g. $q = 5$ or 6),

(ii) there exists at least one segment with nonzero slope.

Then, the working interval $[u_m \ u_M]$ can be subdivided into q intervals within which $h(.)$ is a linearly curve. One gets:

$$[u_m \ u_M] = [d_0 \ d_1] \cup [d_1 \ d_2] \cup \ldots \cup [d_{q-1} \ d_q] \quad (2)$$

where $d_0 = u_m$ and $d_q = u_M$. First, we assume that the working point $(u(t), x_i(t))$ moves along a single line segment. Analytically, if $u(t) \in [d_{l-1} \ d_l]$ with $1 \le l \le q$, the internal signal $x_i(t)$ can be written as follows (Figs. 2a-b):

$$x_i(t) = (u(t) - D_l)S_l + P_l \quad (3)$$

where (D_l, P_l) is any point belonging to the segment l and S_l its slope. The proposed frequency domain identification method necessitates the application of sine signals:

$$u(t) = V_l + A_l \sin(\omega_k t) \quad (4)$$

for a set of a priori chosen frequencies $(k = 1 \ldots m)$. It follows from (3) and (4) that the internal signal $x_o(t)$, In steady state, is of the form:

$$x_o(t) = A_l S_l |G(j\omega_k)| \sin(\omega_k t - \varphi(\omega_k)) + ((V_l - D_l)S_l + P_l)G(0) \quad (5)$$

with $\varphi(\omega_k) \overset{def}{=} -\angle G(j\omega_k)$. Then, it is supposed that:

A.2. (i) the nonlinearity $f(.)$ verifies $f^{-1}(0) = 0$.

(ii) $f(.)$ is a polynomial of finite order n i.e. $f(x) = \sum_{i=0}^{n} c_i x^i$.

Except for the above assumptions, the system is arbitrary. Presently, we aim at designing an identification scheme that is able to provide a model estimate $(\hat{h}(.), \hat{G}(j\omega_k), \hat{f}(.))$ that represents well the system when this is excited by sinusoidal inputs (4) for a set of frequencies ω_k $(k = 1 \ldots m)$. Since $x_i(t)$ and $x_o(t)$ are not measurable, the system identification should be fully based upon measurements of the input $u(t)$ and the output $y(t)$. Therefore, the considered identification problem does not have a unique solution: if the model $(h(u), G(s), f(x_o))$ represents a solution then, any model of the form $(h(u)/k_1, G(s)/k_2, f(k_1 k_2 x_o))$ is also a solution (where k_1 and k_2 are any nonzero real). This naturally leads to the question: what particular model should we focus on? This question will be answered later.

3. SYSTEM FREQUENCY ANALYSIS

3.1. Basic equations and notations

All along this Section, the identified system is submitted to the sine input (4), where $\omega_k > 0$ is kept constant, by tuning the offset V_l and the amplitude A_l until the input signal moves along a single line segment. This item will be detailed later.

Under these conditions, the internal signal $x_o(t)$ respect the form (5). Let T_k be the corresponding period i.e. $T_k = 2\pi / \omega_k$. The aim of this subsection is to establish key properties characterizing the parameterized curves $(\pm A_l \sin(\omega_k t - \varphi(\omega_k)), w(t))$. Notice that sine signals that oscillate at the same frequency as $\sin(\omega_k t - \varphi(\omega_k))$ and having the amplitude A_l are of the form:

$$z_\delta(t) = A_l \sin(\omega_k t - \delta) \quad (6)$$

where $\delta \in IR$ is arbitrary and IR denotes the set of real numbers. It is readily seen that:

$$A_l \sin(\omega_k t - \varphi(\omega_k)) = z_{\varphi(\omega_k)}(t) \quad (7a)$$

$$-A_l \sin(\omega_k t - \varphi(\omega_k)) = z_{\varphi(\omega_k)+\pi}(t) \quad (7b)$$

Let $C_\delta^{\omega_k, A_l, V_l}$ be the parameterized locus constituted of all points of coordinates $(z_\delta(t), w(t))$ $(t \ge 0)$. The dependence of that curve on ω_k, V_l and A_l comes from the fact that $w(t)$ depends on ω_k, V_l and A_l. As $z_\delta(t)$ and $w(t)$ are periodical, with the same period $(2\pi / \omega_k)$, the curve $C_\delta^{\omega_k, A_l, V_l}$ turns out to be an oriented closed-locus. The orientation sense indicates the increasing time. The $C_\delta^{\omega_k, A_l, V_l}$ are viewed as a generalization of the Lissajous curves used in linear system frequency analysis (Rochdi et al., 2010). Then, the system input and output are both sine signals with identical frequency leading to a closed-locus in the form of an ellipse. Presently, the system output $w(t)$ is not necessarily sinusoidal and, consequently, the locus $C_\delta^{\omega_k, A_l, V_l}$ is not necessarily an ellipse. Therefore, the $C_\delta^{\omega_k, A_l, V_l}$'s will be referred too Lissajous-like curves. Let us define the variables:

$$\varphi^+(\omega_k) = \varphi(\omega_k) = -\angle G(j\omega_k) \quad (8a)$$

$$\varphi^-(\omega_k) = -\angle G^-(j\omega_k) = -\angle G(j\omega_k) + \pi \quad (8b)$$

$$X_l = ((V_l - D_l)S_l + P_l)G(0) \quad (8c)$$

Proposition 1. Consider the Hammerstein-Wiener system described by equations (1a-c) and excited by the input (4), with V_l and A_l are judiciously chosen so that the input signal moves along a single linear segment of $h(.)$. Then, one has:

1) If $\delta = \varphi(\omega_k)$ (modulo π), the oriented locus $C_\delta^{\omega_k, A_l, V_l}$ is static. Furthermore, $C_\delta^{\omega_k, A_l, V_l}$ and $C_{\delta+\pi}^{\omega_k, A_l, V_l}$ are symmetric, with respect to the axis $x = X_l$, where X_l is constant.

2) If $\delta \ne \varphi(\omega_k)$ (modulo π), the curve $C_\delta^{\omega_k, A_l, V_l}$ is not static (the area of the curve $C_\delta^{\omega_k, A_l, V_l}$ is non-null).

3) If the output nonlinearity $f(.)$ is a polynomial function, the locus $C_\delta^{\omega_k, A_l, V_l}$ is polynomial if and only if $\delta = \varphi(\omega_k)$ (modulo π). □

3.2. Estimation of the Parameterized Curves $C_\delta^{\omega_k, A_l, V_l}$

Propositions 1 is quite important because it shows that $\varphi(\omega_k) = -\angle G(j\omega_k)$ can be recovered (modulo π) by just tuning the parameter δ until the closed-locus $C_{\delta}^{\omega_k,A_l,V_l}$ displays a static curve. The point is that the locus $C_{\delta}^{\omega_k,A_l,V_l}$ depends on the signal $w(t)$ which is not accessible to measurement. This is presently coped with making full use of the information at hand, namely the periodicity (with period $2\pi/\omega_k$) of both $z_{\delta}(t)$ and $w(t)$ and the ergodicity of the noise $v(t)$. Bearing these in mind, the relation $y(t)=w(t)+v(t)$ suggests the following estimator:

$$\hat{w}(t,N)\overset{def}{=}\frac{1}{N}\sum_{k=1}^{N}y(t+kT);\quad t\in[0,T) \tag{9a}$$

$$\hat{w}(t+kT,N)\overset{def}{=}\hat{w}(t,N)\quad \text{for any integer } k>0 \tag{9b}$$

where $T=2\pi/\omega_k$ and N is a sufficiently large integer. Specifically, for a fixed time instant t, the quantity $\hat{w}(t,N)$ turns out to be the mean value of the (measured) sequence $\{y(t+kT);k=0\ 1...\}$. Then, an estimate $\hat{C}_{\delta,N}^{\omega_k,A_l,V_l}$ of $C_{\delta}^{\omega_k,A_l,V_l}$ is simply obtained substituting $\hat{w}(t,N)$ to $w(t)$ when constructing $C_{\delta}^{\omega_k,A_l,V_l}$. Accordingly, $\hat{C}_{\delta,N}^{\omega_k,A_l,V_l}$ turns out to be the parameterized locus including all points $(z_{\delta}(t),\hat{w}(t,N))$ $(t\geq 0)$. These remarks lead to the following proposition:

Proposition 2. Consider the problem statement of Proposition 1. Then, one has:

1) $\hat{w}(t,N)$ Converges in probability to $w(t)$ (as $N\to\infty$).

2) $\hat{C}_{\delta,N}^{\omega_k,A_l,V_l}$ Converges in probability to $C_{\delta}^{\omega_k,A_l,V_l}$ (as $N\to\infty$) *i.e.* for all $t\geq 0$:

$$\lim_{N\to\infty}(z_{\delta}(t),\hat{w}(t,N))=(z_{\delta}(t),w(t))\quad (w.p.1)$$

3) Consequently, $\lim_{N\to\infty}\hat{C}_{\delta,N}^{\omega_k,A_l,V_l}$ is static curve (*w.p.1*) if and only if $\delta=\varphi(\omega_k)$ (modulo π).

4) Suppose that $\lim_{N\to\infty}\hat{C}_{\delta,N}^{\omega_k,A_l,V_l}$ is static curve for some δ. Then, one of the following statements holds *w.p.1*:

a) $\delta=\varphi(\omega_k)$ (modulo 2π) and the $\lim_{N\to\infty}\hat{C}_{\delta,N}^{\omega_k,A_l,V_l}$-shape coincides with that of output nonlinearity $f(S_l|G(j\omega_k)|z_{\varphi(\omega_k)}+X_l)$ where X_l is defined by (8c).

b) $\delta=\varphi(\omega_k)+\pi$ (modulo 2π) and the $\lim_{N\to\infty}\hat{C}_{\delta,N}^{\omega_k,A_l,V_l}$-shape coincides with that of $f(-S_l|G(j\omega_k)|z_{\varphi(\omega_k)+\pi}+X_l)$. □

On the other hand, define:

$$\tilde{f}(z_{\varphi(\omega_k)}(t))=\bar{f}(z_{\varphi(\omega_k)}(t)-X_l)=f(S_l|G(j\omega_k)|z_{\varphi(\omega_k)}) \tag{10}$$

Then, the curve $\tilde{f}(z_{\varphi(\omega_k)}(t))$ is, more or less, spread version of the output nonlinearity $f(z_{\varphi(\omega_k)}(t))$, depending on the value of $S_l|G(j\omega_k)|$ (Fig. 3). Under the above conditions, $\bar{f}(.)$ is a variant more or less spread (Giri et al., 2013) of $f(.)$ and shifted by the value X_l with respect to the origin of the x-axis (Fig. 3). It is clear that the nonlinearities $\bar{f}(.)$ and $\tilde{f}(.)$ are also polynomials. Let introduce the parameters vectors, associated respectively to $\bar{f}(z_{\varphi(\omega_k)}(t))$ and $\tilde{f}(z_{\varphi(\omega_k)}(t))$:

$$\bar{C}(\omega_k)=[\bar{c}_0(\omega_k)\ldots\bar{c}_n(\omega_k)]^T;\tilde{C}(\omega_k)=[\tilde{c}_0(\omega_k)\ldots\tilde{c}_n(\omega_k)]^T$$
(11)

Accordingly, using (10) and part 2 of hypothesis A2, the coefficients $\tilde{c}_i(\omega_k)$ and $c_i\,(i=0\ldots n)$ are related by the following relation:

$$\tilde{c}_i(\omega_k)=c_i\,(S_l|G(j\omega_k)|)^i;\ \text{for all}\ \omega_k\in\{\omega_1,\omega_2,\ldots,\omega_m\} \tag{12}$$

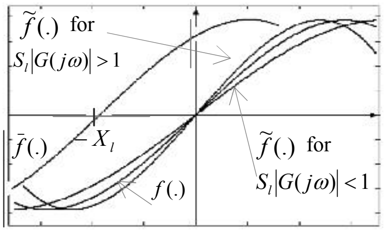

Fig. 3. $\tilde{f}(.)$ is a more or less spread version of $f(.)$

4. FREQUENCY IDENTIFICATION METHOD

4.1 Phase Estimation

The results observed in Section 3 show that the phase of the linear subsystem can be discerned by exciting the system with a sine wave moving along a single line segment of *h*(.). The knowledge of just one segment is sufficient to begin the procedure for phase identification. To this end, the system is excited by the signal (4) throughout this part, the choice of the parameters V_l and A_l may be experimentally. To facilitate the adjustment of parameters V_l and A_l, the signal (4) will be replaced by the following excitation:

$$u(t)=d_{l-1}+\varepsilon_l(1+\sin(\omega_k t))=u_m+\varepsilon_l(1+\sin(\omega_k t))\ (l=1) \tag{13}$$

where $V_1=d_0+\varepsilon_l=u_m+\varepsilon_l$ and ε_l is an arbitrary small value and the frequency ω_k is constant. If the resulting system steady-state response *y*(.) turns out to be constant (up to noise), then the segment in question has a zero slope. Consequently, exciting in another segment i.e. V_l should be increased until the output changes. Afterwards, establish the lissajous curves $C_{\delta}^{\omega_k,A_l,V_l}$, by tuning the angle δ in [0 π[. If $C_{\delta}^{\omega_k,A_l,V_l}$ becomes static for any value of δ, the value of ε_l can be adapted to expand the obtained static curve $f(S_l|G(j\omega_k)|z_{\varphi(\omega_k)}+X_l)$, change A_l if necessary. Let $\bar{l}$ denotes the first having a nonzero slope. Note $A_{\bar{l}}=\varepsilon_{\bar{l}}$ and $V_{\bar{l}}$ the chosen final values of A_l and V_l, respectively. Generally, a preliminary selection of the segment can be done practically by exciting with a set of constant values and observing the output response $y(t)$ (in the steady-state).

Remark 1. It is interesting to note that the first part of the assumption A1 greatly simplifies the search for a linear segment with a slope non-zero. □

Finally, the propositions 1 and 2 suggest the following phase estimator:

Table 1. Phase Estimator (PE)

Step 1: Let $\omega_k \in \{\omega_1, \omega_2, \ldots, \omega_m\}$. The system (1a-c) is excited by the sine input (4). For any small amplitude A_l and any offset $V_l \in [u_m \; u_M]$, change V_l if $y(t)$ turns out to be constant (up to noise), otherwise construct the curve $(z_\delta(t), y(t))$, where $z_\delta(t)$ is defined by (6). Change V_l, if $(z_\delta(t), y(t))$ never turns out to be a static curve (up to noise) for any $\delta \in [0 \;\; \pi)$. Else, adjust the value of A_l to zoom the obtained static curve (if needed change V_l). Let $l = \bar{l}$.

Step 2: Take a record of the output $y(t)$ over the interval $[0 \;\; N2\pi / \omega_k)$ for some $N >> 1$.

Step 3: Compute the filtered output $\hat{w}(t, N)$ applying (9a-b).

Step 4: Plot $\hat{C}_{\delta,N}^{\omega_k, A_{\bar{l}}, V_{\bar{l}}} = \{(A_{\bar{l}} \sin(\omega_k t + \delta), \hat{w}(t, N)), \; 0 \le t \le T\}$, for different values of δ until $\hat{C}_{\delta,N}^{\omega_k, A_{\bar{l}}, V_{\bar{l}}}$ becomes static. Let δ^* denotes the first value of δ such that $\hat{C}_{\delta,N}^{\omega_k, A_{\bar{l}}, V_{\bar{l}}}$ is static curve. Then, take $\hat{\phi}_N(\omega_k) = \delta^*$.

Step 5: Repeat the above steps for all $\omega_k \in \{\omega_1, \omega_2, \ldots, \omega_m\}$. Check that the lissajous curves obtained at a given step is similarly static as all those obtained in previous steps.

Remark 2. It is crucial to emphasize that the phase estimate $\hat{\phi}_N(\omega_k)$ obtained at step 4, for a frequency ω_k and a sufficiently large N, could be either $\varphi(\omega_k) = -\angle G(j\omega_k)$ or $\varphi(\omega_k) + \pi$. It is impossible to know which one of the previous values is actually determined. This uncertainty is not an issue as long as phase estimation is performed at a single frequency. But, when the phase is to be computed for a set of frequencies $\{\omega_1, \omega_2, \ldots, \omega_m\}$, the corresponding estimates $\hat{\phi}_N(\omega_k)$ must be coherent in the sense that either $\hat{\phi}_N(\omega_k)$ corresponds to $\varphi(\omega_k) = -\angle G(j\omega_k)$, for all ω_k's, or corresponds to $\varphi(\omega_k) + \pi$. It does not matter to know which case is being actually focused on. This coherency issue is coped to the previous estimator. □

Theorem 1. Consider the problem statement of Proposition 1. The phase estimator $\hat{\phi}_N(\omega_k)$, described in Table 1, is consistent in the sense that one of the following two statements does hold w.p.1:

(i) $\lim_{N \to \infty} \hat{\phi}_N(\omega_k) = \varphi(\omega_k)$ (for all $\omega_k \in \{\omega_1, \omega_2, \ldots, \omega_m\}$)

(ii) $\lim_{N \to \infty} \hat{\phi}_N(\omega_k) = \varphi(\omega_k) + \pi$ (for all $\omega_k \in \{\omega_1, \omega_2, \ldots, \omega_m\}$) □

4.2 Estimation of the output nonlinearity and Gain Modulus

The phase Estimator provides, for each frequency ω_k, an estimate of the phase $\varphi(\omega_k) = -\angle G(j\omega_k)$. For reasons of simplification, it is assumed that the phase identification has been correctly carried, i.e. for all $\omega_k \in \{\omega_1, \ldots, \omega_m\}$: $\hat{\phi}_N(\omega_k) = \varphi(\omega_k)$ or $\hat{\phi}_N(\omega_k) = \varphi(\omega_k) + \pi$. Then, all curves of the family $\{\hat{C}_{\hat{\phi}_N(\omega_k),N}^{\omega_k, A_{\bar{l}}, V_{\bar{l}}}\}$ coincide with $\bar{f}^+(z_{\varphi(\omega_k)}(t))$ for all $\omega_k \in \{\omega_1, \ldots, \omega_m\}$ or with $\bar{f}^-(z_{\varphi(\omega_k)+\pi}(t))$ for all $\omega_k \in \{\omega_1, \ldots, \omega_m\}$. The resulting nonlinearity and phase will be denoted respectively later $\bar{f}(z_{\phi(\omega_k)}(t))$ and $\phi(\omega_k)$. On the other hand, since $f(.)$ is a polynomial function, note that the identification of $f(.)$ can be done using only the data of Table1. For each frequency ω_k, the vector $\overline{C}(\omega_k)$ can be determined minimizing the error:

$$J(\omega_k, N) = \int_0^T \left(\hat{w}(t, N) - \sum_{i=0}^{n} \hat{\tilde{c}}_i(\omega_k, N) \left(A_l \sin(\omega_k t - \phi(\omega_k)) \right)^i \right)^2 dt \tag{14}$$

Then, the estimate $\hat{\tilde{C}}(\omega_k, N) = [\hat{\tilde{c}}_0(\omega_k, N) \ldots \hat{\tilde{c}}_n(\omega_k, N)]^T$ of the vector $\tilde{C}(\omega_k)$ can easily be discerned (part 1 of hypothesis A2). Specifically, for each ω_k, horizontally moving the estimate nonlinearity of the value $X_{\bar{l}}$ (relocate the non-linearity at the origin), where $X_{\bar{l}}$ (defined by (8c)) can be determined easily i.e. the intersection of $\bar{f}(.)$ with the x-axis (Fig. 3)). The coefficients $\tilde{c}_i \, (i = 1 \ldots n)$ are not uniquely determined, because the quantities $|G(j\omega_k)| \; (k = 1 \ldots m)$ are also unknown. This is a direct consequence of the problem of multiplicity of solutions discussed in Section 2. Therefore, the models $(h(u), G(s), f(x_o))$ and $(h(u)/k_1, G(s)/k_2, f(k_1 k_2 x_o))$ are solutions of the above identification problem whatever $k_1 \ne 0$ and $k_2 \ne 0$. To solve this problem, it is suggested the following choice of the scaling factor:

$$k_1 = S_{\bar{l}} \quad \text{and} \quad k_2 = \left(\sum_{i=1}^{n} |c_i|^{n/i} \right)^{-1/n} \tag{15}$$

This model is the only that checks the property:

$$S_{\bar{l}} = 1 \quad \text{and} \quad \sum_{i=1}^{n} |c_i|^{n/i} = 1 \tag{16}$$

Using (15), it readily follows from (12) that for $i = 1 \ldots n$:

$$|\tilde{c}_i(\omega_k)|^{n/i} = |c_i|^{n/i} \left(S_{\bar{l}} \, |G(j\omega_k)| \right)^n \text{ for all } \omega_k \in \{\omega_1, \ldots, \omega_m\} \tag{17}$$

Adding the both sides of (17) over $i = 1 \ldots n$ and using (16):

$$\sum_{i=1}^{n} |\tilde{c}_i(\omega_k)|^{n/i} = \sum_{i=1}^{n} |c_i|^{n/i} \left(S_{\bar{l}} \, |G(j\omega_k)| \right)^n = |G(j\omega_k)|^n \tag{18}$$

where $\omega_k \in \{\omega_1, \ldots, \omega_m\}$. From (18) one immediately gets:

$$|G(j\omega_k)| = \left(\sum_{i=1}^{n} |\tilde{c}_i(\omega_k)|^{n/i} \right)^{1/n} \quad \text{for all} \quad \omega_k \in \{\omega_1, \ldots, \omega_m\} \tag{19}$$

This uniquely determines the set of frequency gain modulus $|G(j\omega_k)| \; (k = 1 \ldots m)$ in terms of elements of the vector parameters $\tilde{C}(\omega_k) = [\tilde{c}_0(\omega_k) \ldots \tilde{c}_n(\omega_k)]^T$. Substituting the right side of (19) to $|G(j\omega_k)|$ in (12) yields for $i = 1 \ldots n$:

$$\tilde{c}_i(\omega_k) = c_i \left(\sum_{i=1}^{n} |\tilde{c}_i(\omega_k)|^{n/i} \right)^{i/n} \quad \text{for all } \omega_k \in \{\omega_1, \ldots, \omega_m\} \qquad (20)$$

The expressions in (20) show that, for a given $i = 1 \ldots n$, each coefficient c_i comes in linearly in m equations. Therefore, it is judicious to get benefit of all m equations involving a given coefficient to determine that coefficient. To this end, we proceed by adding, side by side, all m equations involving c_i in (20) and solving the resulting expression with respect to c_i. Doing so, one gets:

$$c_i = \frac{\sum_{k=1}^{m} \tilde{c}_i(\omega_k)}{\sum_{k=1}^{m} \left(\sum_{i=1}^{n} |\tilde{c}_i(\omega_k)|^{n/i} \right)^{i/n}} \quad \text{for } i = 1 \ldots n \qquad (21a)$$

Clearly, from (12), only the first coefficients c_0 is uniquely determined, i.e. $c_0 = \tilde{c}_0(\omega_k)$. Then, this coefficient can be determined using the following expression:

$$c_0 = \frac{1}{m} \sum_{k=1}^{m} \tilde{c}_0(\omega_k) \qquad (21b)$$

Then, it follows from A2 that $c_0 = 0$, subsequently the calculation of c_0, for all $\omega_k \in \{\omega_1, \ldots, \omega_m\}$, is not useful. Finally, the equations (19) and (21a-b) suggest the estimators of Table 2 for the frequency gain $|G(j\omega_k)|$ and the coefficients of the output nonlinearity.

Table 2. Estimators of Gain and Output Nonlinearity

Step 1: Minimizing the error (14) and using (10), the estimate $\hat{\tilde{C}}(\omega_k, N) = \left[\hat{\tilde{c}}_0(\omega_k, N) \ldots \hat{\tilde{c}}_n(\omega_k, N)\right]^T$ can be provided.

Step 2: Calculate the gain estimate:

$$\left|\hat{G}(j\omega_k, N)\right| = \left(\sum_{i=1}^{n} \left|\hat{\tilde{c}}_i(\omega_k, N)\right|^{n/i} \right)^{1/n} \quad \text{for all } \omega_k \in \{\omega_1, \ldots, \omega_m\} \qquad (22)$$

Step 3: Determine the coefficient of the output nonlinearity:

$$\hat{c}_i(N) = \frac{\sum_{k=1}^{m} \hat{\tilde{c}}_i(\omega_k, N)}{\sum_{k=1}^{m} \left(\sum_{i=1}^{n} \left|\hat{\tilde{c}}_i(\omega_k, N)\right|^{n/i} \right)^{i/n}} \quad \text{for } i = 1 \ldots n \text{ and } c_0 = 0 \qquad (23)$$

Recall that the identification of the output nonlinearity and frequency gain modulus can be achieved only using the recorded data in Table 1, without needed any other information.

Theorem 2. Consider the problem statement of Propositions1-2. The frequency gain estimator (22) and the output nonlinearity (the coefficient) estimator (23) are consistent in the sense that the following two statements hold w.p.1:

(i) $\lim_{N \to \infty} \left|\hat{G}(j\omega_k, N)\right| = |G(j\omega_k)|$ (for all $\omega_k \in \{\omega_1, \ldots, \omega_m\}$)

(ii) $\lim_{N \to \infty} \hat{c}_i(N) = c_i$ for $i = 1 \ldots n$ □

4.3 Estimation of the input nonlinearity

The knowledge of the estimates of the output nonlinearity and linear subsystem allowed determining the input nonlinearity. From A1 and (2), the working interval $I = [u_m \; u_M]$ can be decomposed into q subintervals, where the input nonlinearity is linear in each subinterval $[d_{l-1} \; d_l]$ and having a slope S_l (that may be zero). This decomposition of I can be performed easily using the search procedure specified in paragraph 4.1. The frequency $\bar{\omega}$ throughout this part will be kept constant, where $\bar{\omega} \in \{\omega_1, \ldots, \omega_m\}$. Let $\hat{d}_{l,N}$ designates the estimate of d_l ($l = 0 \ldots q$), with $\hat{d}_{0,N} = u_m$ and $\hat{d}_{q,N} = u_M$. The system is excited by the inputs (13), for an arbitrarily small value ε_1 and $V_1 = d_0 + \varepsilon_l = u_m + \varepsilon_l$ ($l = 1$), the resulting steady-state output signal $y(t)$ turns out to be constant (up to noise) or variable, then the slope in this latter case is nonzero and zero in the first case. It is obvious that the curve $C^{\bar{\omega}, \varepsilon_1}_{\phi(\bar{\omega})}$ is identical to a line segment ($S_1 = 0$) or static curve ($S_1 \neq 0$) within $[d_0 \; d_1)$, gradually increasing ε_1 until $C^{\bar{\omega}, \varepsilon_1}_{\phi(\bar{\omega})}$ becomes non-static, let $\hat{\varepsilon}^*_{1,N}$ denotes the first value of ε_1 which leads to a non-static curve of $C^{\bar{\omega}, \varepsilon_1}_{\phi(\bar{\omega})}$. Then, it is readily seen that:

$$\hat{d}_{1,N} = u_m + 2\,\hat{\varepsilon}^*_{1,N} \qquad (24a)$$

The same procedure can be applied to identify $\hat{d}_{2,N}$ i.e. by exciting the system with the signal (13), progressively increasing ε_2 and by observing the curve $C^{\bar{\omega}, \varepsilon_2}_{\phi(\bar{\omega})}$ until it becomes non-static, $\hat{\varepsilon}^*_{2,N}$ designates the first value of ε_2 that leads to a non-static curve. Then: $\hat{d}_{2,N} = \hat{d}_{1,N} + 2\,\hat{\varepsilon}^*_{2,N}$. Generally speaking:

$$\hat{d}_{l,N} = \hat{d}_{l-1,N} + 2\,\hat{\varepsilon}^*_{l,N} \quad \text{for all } d_l \in \{1, \ldots, q-1\} \qquad (24b)$$

Let $\hat{X}_{l,N}$, $\hat{P}_{l,N}$, $\hat{S}_{l,N}$, and $\hat{G}_N(0)$ denote the estimates of X_l, P_l, S_l ($l = 1 \ldots q$), and $G(0)$ respectively. It is interesting to note that the determination of the parameters S_l and P_l (relating to segments having a non-zero slope) can be achieved using only data collected during the identification of d_l ($l = 0 \ldots q$) without resorting out to any other information. Determining $X_{\bar{l}}$ (i.e. the intersection of the curve $\bar{f}(z_{\phi(\omega_k)}(t))$ with the x-axis) for two values of $V_{\bar{l}}$ ($V^1_{\bar{l}}$ and $V^2_{\bar{l}}$), where $\bar{l}$ the segment within which the phase estimate scheme was carried out. It readily follows from (8c) and (16) that:

$$X^1_{\bar{l}} = \left(V^1_{\bar{l}} - D_{\bar{l}} + P_{\bar{l}}\right)G(0); \quad X^2_{\bar{l}} = \left(V^2_{\bar{l}} - D_{\bar{l}} + P_{\bar{l}}\right)G(0) \qquad (25)$$

where $\left(D_{\bar{l}}, P_{\bar{l}}\right)$ is any point belonging to the segment $\bar{l}$. Within each subinterval $[d_{l-1} \; d_l)$ ($l = 1 \ldots q$), with $S_l \neq 0$, the elements $G(0)$ and $P_{\bar{l}}$ can be provided easily from (25). Then, for two values of $V_l = d_{l-1} + \varepsilon_l$ with $\varepsilon_l < \hat{\varepsilon}^*_{l,N}$. Let X^1_l and X^2_l be the corresponding values of X_l respectively.

The parameters S_l and P_l can be provided using $G(0)$, X_l^1 and X_l^2. In the case where the search procedure leads to a zero slope $(S_l = 0)$, the curve $C_{\phi(\bar{\omega})}^{\bar{\omega},\varepsilon_l}$ is identical to a line segment, then the input nonlinearity is constant within this segment having the value P_l, subsequently the value of the curve $C_{\phi(\bar{\omega})}^{\bar{\omega},\varepsilon_l}$ is $f\left(P_l G(0)\right)$. Under these conditions, if $f(.)$ is invertible at this point (i.e. there is a single value $P_l G(0)$), then the parameter P_l can be estimated directly. Else, there exists then a set $\left\{P_1^* G(0), \dots, P_r^* G(0)\right\}$ such that:

$$f\left(P_1^* G(0)\right) = \dots = f\left(P_r^* G(0)\right) \text{ with } r \text{ is any integer} \tag{26}$$

For all values $x_i(t) \in \left\{P_1^*, \dots, P_r^*\right\}$ (constant), the steady-state system output converges to a unique value, however these different values have different transitory regimes. Then, to keep only the true value P_l one must perform an audit to remove the other candidates. The verification can be done by exciting the system with a periodic signal, e.g. a square wave with two values, one belongs to the segment $\bar{l}$ already known and the other belongs to the current segment, the determination of P_l can be easily obtained by measuring only one harmonic (e.g. the fundamental frequency) of the decomposition of $\hat{w}(t, N)$, since the linear block and the output nonlinearity are already determined.

Finally, the results obtained suggest the following estimator for the input nonlinearity:

Table 3. The input nonlinearity Estimator (INE)

Step 1: Let $l = \bar{l}$, for two values of $V_{\bar{l}}$ ($V_{\bar{l}}^1$ and $V_{\bar{l}}^2$), determine $X_{\bar{l}}$ ($\hat{X}_{\bar{l},N}^1$ and $\hat{X}_{\bar{l},N}^2$). Then:

$$\hat{G}_N(0) = \frac{\hat{X}_{\bar{l},N}^1 - \hat{X}_{\bar{l},N}^2}{\left(V_{\bar{l}}^1 - V_{\bar{l}}^2\right)} \tag{27a}$$

$$\hat{P}_{\bar{l},N} = \frac{\hat{X}_{\bar{l},N}^1\left(V_{\bar{l}}^2 - D_{\bar{l}}\right) - \hat{X}_{\bar{l},N}^2\left(V_{\bar{l}}^1 - D_{\bar{l}}\right)}{\hat{X}_{\bar{l},N}^2 - \hat{X}_{\bar{l},N}^1}\,; \quad S_{\bar{l}} = 1 \tag{27b}$$

Step 2: Select $\bar{\omega} \in \{\omega_1, \dots, \omega_m\}$, let $\hat{d}_{0,N} = u_m$ and $l = 1$.

Step 3: Apply the sine input $u(t) = \hat{d}_{l-1,N} + \varepsilon_l\left(1 + \sin(\bar{\omega} t)\right)$, where ε_l is initialized to a small value. If the resulting curve $C_{\phi(\bar{\omega})}^{\bar{\omega},\varepsilon_l}$ is a horizontal segment, then go to step 4. Otherwise (i.e. $C_{\phi(\bar{\omega})}^{\bar{\omega},\varepsilon_l}$ is static and $S_l \neq 0$), for two values of V_l (V_l^1 and V_l^2), estimate X_l ($\hat{X}_{l,N}^1$ and $\hat{X}_{l,N}^2$). So, determine S_l and P_l:

$$\hat{S}_{l,N} = \frac{\left(\hat{X}_{l,N}^1 - \hat{X}_{l,N}^2\right)}{\left(V_l^1 - V_l^2\right)\hat{G}_N(0)} \tag{28a}$$

$$\hat{P}_{l,N} = \frac{\hat{X}_{l,N}^1\left(V_l^2 - D_l\right) - \hat{X}_{l,N}^2\left(V_l^1 - D_l\right)}{\left(V_l^2 - V_l^1\right)\hat{G}_N(0)} \tag{28b}$$

If needed increase ε_l. Then, go to step 5.

Step 4: If $f(.)$ is invertible at $P_l G(0)$ determine P_l directly. Else, conduct an audit in order to choose the correct estimate of P_l.

Step 5: Increasing progressively ε_l until the curve $C_{\phi(\bar{\omega})}^{\bar{\omega},\varepsilon_l}$ becomes non-static or $u(t) = u_M$. Let $\hat{\varepsilon}_{l,N}^*$ the corresponding value of ε_l. Then:

$$\hat{d}_{l,N} = \hat{d}_{l-1,N} + 2\,\hat{\varepsilon}_{l,N}^* \tag{29}$$

If $u(t) < u_M$ then $l = l + 1$ and go to step 3. Else $q = l$ (end).

Theorem 3. Consider the problem statement of Propositions1-2. The system is subject to assumptions A1-2. Then, the estimator described in Table 3, is consistent in the sense that the following statements hold w.p.1:

(i) $\lim_{N\to\infty} \hat{d}_{l,N} = d_l$ (for all $l \in \{0, \dots, q\}$)

(ii) $\lim_{N\to\infty} \hat{P}_{l,N} = P_l$ and $\lim_{N\to\infty} \hat{S}_{l,N} = S_l$ (for all $l \in \{1, \dots, q\}$) □

6. CONCLUSION

In this paper, a frequency-domain solution has been developed to deal with Hammerstein-wiener system identification in presence of hard input nonlinearity or piecewise affine with a limited number of segments. The identification method is designed using analytic geometry tools. Interestingly, the linear part of the system can be of unknown structure. The output nonlinearity is not necessarily invertible, except in the origin. The method only necessitates simple experiments involving sine input excitation. All estimators are shown to be consistent.

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