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Robust stability analysis of discrete-time systems with parametric and switching uncertainties

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Abstract: Robust stability analysis is investigated for discrete-time linear systems with rational dependency with respect to polytopic type uncertainties. Two type of uncertainties are considered: constant parametric uncertainties and time-varying switching uncertainties. Results are in LMI formalism and proofs involve parameter-dependent, quadratic in the state, Lyapunov functions. The new proposed conditions are shown to extend and merge two important existing results. Conservatism reduction is tackled via a model augmentation technique. Numerical complexity is contained by exploiting the structure of the models with respect to the uncertainties.

Keywords: Uncertain discrete-time systems, Robust stability

1. INTRODUCTION

In the early 2000, two results have been produced that we aim at studying and merging in the present paper. Both results consider linear systems with uncertainties of affine polytopic type. The first one, Peaucelle et al. [2000], that extends results of Oliveira et al. [1999a,b], is dedicated to robust analysis assuming uncertainties are constant over time. The second, Daafouz and Bernussou [2001], assumes time-varying uncertainties with possibly unbounded time-variations. To distinguish both cases, the former is designated at the the parametric case, while the latter is the switching uncertainty case. Results of Daafouz and Bernussou [2001] have indeed been mainly used for the study of switching systems, see for example Daafouz et al. [2002]. An intermediate case between the two is when uncertainties are time-varying with bounded rates. This intermediate case is not considered in the present paper.

In both parametric and switching cases, the upper cited papers prove stability using Lyapunov certificates of the same polytopic type. Results are formulated in terms of a finite number of LMI constraints, yet these are quite different. The contribution of the present paper is to explore the links between the two results. A by product is the illustration that all these results can be easily extended to systems rationally-dependent on the polytopic uncertainties. To do so, the contribution is to consider descriptor type models. The models are such that the E matrix is parameter-dependent and left-hand side invertible. These two features for E differ from assumptions in Bara [2011], Barbosa et al. [2012].

The other difference with the last cited papers is that we provide conditions that relax the assumption of parameter-indepedent slack variables. Such assumption happens to be needed only for parametric uncertainties and can be relaxed when considering switching uncertainties. The resulting conservatism reduction is at the expense of increased numerical burden. To limit the effect of this increased burden, we provide methods that exploit the structure of the uncertain models. The methods limit the size of the slack variables. These results are improved versions of that in Peaucelle [2009].

The outline of the paper is as follows. Preliminaries are devoted to exposure of the two central results from Peaucelle et al. [2000] and Daafouz and Bernussou [2001]. The section that follows is devoted to the main results for rationally uncertainty dependent switching systems. The fourth section gives the techniques for reducing the size of LMI problems both in terms of number of variables and of size of the constraints. The fifth section treats the mixed switching and parametric uncertainties case and recalls a simple technique for conservatism reduction. It is followed by an illustrative numerical example. Some conclusions are given in the closing section.

2. PRELIMINARIES

Notation: I stands for the identity matrix. A^T is the transpose of the matrix A. $\{A\}^{\mathcal{S}}$ stands for the symmetric matrix $\{A\}^{\mathcal{S}} = A + A^T$. For a matrix $A \in \mathbb{R}^{n\times m}$ or rank r, $A^\perp \in \mathbb{R}^{(n-r)\times n}$ stands for the matrix of maximal rank such that $A^\perp A = 0$. A^+ stands for the Moore-Penrose of A. $A \prec B$ is the matrix inequality stating that $A - B$ is negative definite. $\Xi_{\bar{v}} = \{\theta_{v=1\ldots\bar{v}} \geq 0,\ \sum_{v=1}^{\bar{v}} \theta_v = 1\}$ is the unit simplex in $\mathbb{R}^{\bar{v}}$. Its vertices are the $\bar{v}$ vectors $\theta^{[v]}$ with all zeros coefficients except one equal to 1.

The considered systems are linear discrete-time:

$$x_{k+1} = A(\theta_k)x_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the vector state at time $k \in \mathbb{N}$. The matrix $A(\theta_k)$ is assumed to depend of a vector of uncertainties $\theta_k \in \Xi_{\bar{v}}$ and is for a start considered to be affine in the uncertainties:

$$A(\theta_k) = \sum_{v=1}^{\bar{v}} \theta_{k,v} A^{[v]}. \tag{2}$$

$A^{[v=1\ldots\bar{v}]}$ are given vertex matrices. The system is said to be affine polytopic. The important key feature of this uncertain model is that $A(\theta)$ lies for all $\theta \in \Xi_{\bar{v}}$ in the convex hull of the finite number of vertices. This feature allows to prove robust stability, that is stability for all the infinitely many possibles realizations of the uncertainties, by LMI feasibility tests that involve only the vertices.

The state of the art at the end of the twentieth century for addressing this problem, known as the "quadratic stability" result of Barmish [1985], was to search for a parameter independent quadratic Lyapunov function $V_k = x_k^T P x_k$ and states as follows:

Theorem 1. If there exists a matrix $P = P^T \succ 0 \in \mathbb{R}^{n\times n}$ such that the following LMI conditions hold for all vertices $v = 1 \dots \bar{v}$

$$A^{[v]T} P A^{[v]} - P \prec 0 \tag{3}$$

then the uncertain system defined by (1-2) is robustly stable with respect to any time varying uncertainty $\theta_k \in \Xi_{\bar{v}}$.

Proof The proof is well known. We reproduce it here only for pedagogical purpose to illustrate that the next to come proofs follow similar lines. First note that the constraints $A^T P A - P \prec 0$ are convex with respect to the matrices A. To check this fact apply a Schur complement argument to get the equivalent constraint $\begin{bmatrix} -P & PA \\ A^T P & -P \end{bmatrix} \prec 0$. It is an affine matrix inequality with respect to A, hence, convex in A. Assuming (3) holds, since, as stated upper, it is convex in the $A^{[v]}$s, the inequality also holds for any convex combination of the vertices. That is, for all $\theta_k \in \Xi_{\bar{v}}$ one has $A(\theta_k)^T P A(\theta_k) - P \prec 0$. Pre and post multiply this inequality by x_k^T and its transpose to get along the trajectories of (1) $x_{k+1}^T P x_{k+1} - x_k^T P x_k < 0$ for all non-zero x_k. Stability is hence proved whatever sequence $\{\theta_k\}_{k\geq 0}$ with the decreasing along trajectories, positive definite Lyapunov function $V_k = x_k^T P x_k$. ■

Theorem 1 is known to be conservative. In the case of parametric ($\theta_k = \theta$ constant) conservatism comes from the choice of a parameter-independent Lyapunov matrix P proving stability for all values inside the polytopic convex set. A result that allows to reduce the conservatism is as follows (Peaucelle et al. [2000])

Theorem 2. If there exist $\bar{v}$ matrices $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$ and a matrix $G \in \mathbb{R}^{2n\times n}$ such that the following LMI conditions hold for all vertices $v = 1 \dots \bar{v}$

$$\begin{bmatrix} P^{[v]} & 0 \\ 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G \left[\, I \;\; -A^{[v]} \,\right] \right\}^{\mathcal{S}} \tag{4}$$

then the uncertain system defined by (1-2) is robustly stable with respect to any parametric uncertainty $\theta_k = \theta \in \Xi_{\bar{v}}$. Moreover, if conditions (3) hold, then conditions (4) hold as well.

Proof The proof of robust stability starts as upper by noticing that the LMI constraints are affine, and hence convex, in both the matrices $P^{[v]}$ and $A^{[v]}$. Defining the affine polytopic matrix $P(\theta) = \sum_{v=1}^{\bar{v}} \theta_v P^{[v]}$ one therefore gets for all $\theta \in \Theta$:

$$\begin{bmatrix} P(\theta) & 0 \\ 0 & -P(\theta) \end{bmatrix} \prec \left\{ G \left[\, I \;\; -A(\theta) \,\right] \right\}^{\mathcal{S}}.$$

Pre and post multiply this matrix inequality by $\left(\, x_{k+1}^T \;\; x_k^T \,\right)$ and its transpose respectively to get exactly $x_{k+1}^T P(\theta) x_{k+1} - x_k^T P(\theta) x_k < 0$ along non zero trajectories of (1). Proof of stability is as upper but with the parameter-dependent Lyapunov function $V_k(\theta) = x_k^T P(\theta) x_k$.

Now we prove that conditions (4) are no more conservative than that of (3). Assume the latter hold and apply the Schur complement argument to get $\begin{bmatrix} -P & PA^{[v]} \\ A^{[v]T} P & -P \end{bmatrix} \prec 0$. This inequality happens to be exactly that of (4) when choosing $P^{[v]} = P$ and $G^T = [\, P \;\; 0 \,]$. ■

As seen from the proof, the conservatism reduction of Theorem 2 is thanks to the decoupling of P and A matrices that allows the introduction of the slack-variables G. Thus obtaining convexity. Yet, if looking at (3) the LMIs are already convex in both A and P as soon as one or the other is fixed. This fact is at the core of the following result:

Theorem 3. If there exist $\bar{v}$ matrices $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$ such that the following LMI conditions hold for all pairs of vertices $v = 1 \dots \bar{v}$, $w = 1 \dots \bar{v}$

$$A^{[v]T} P^{[w]} A^{[v]} - P^{[v]} \prec 0 \tag{5}$$

then the uncertain system defined by (1-2) is robustly stable with respect to any time-varying uncertainty $\theta_k \in \Xi_{\bar{v}}$. Moreover, if conditions (3) hold, then conditions (5) hold as well.

This result is originated from Daafouz and Bernussou [2001]. As a matter of fact, in that paper the result is not formulated that simply. It involves some unnecessary additional variables. As already noticed in Daafouz et al. [2002], and as shown in the following section these slack variables are useless in this case. It should be noted as well that Daafouz and Bernussou [2001] proves that the condition is not only sufficient but also necessary, as long as one restricts the Lyapunov function to the following polytopic in the uncertainties, quadratic in the state form:

$$V_k(\theta) = x_k^T P(\theta_k) x_k \;\; : \;\; P(\theta_k) = \sum_{v=1}^{\bar{v}} \theta_{v,k} P^{[v]}. \tag{6}$$

Proof The proof of robust stability starts as previously in terms of convexity. First, notice the LMIs are convex in both the vertex matrices $P^{[v]}$ and $A^{[v]}$ with indexes v. Hence, defining the affine polytopic matrix $P(\theta_k)$ as in (6), one gets for all $w = 1 \dots \bar{v}$ and all $\theta_k \in \Xi_{\bar{v}}$:

$$A(\theta_k) P^{[w]} A(\theta_k) - P(\theta_k) \prec 0$$

These inequalities being convex in the matrices $P^{[w]}$, one gets for all $\theta_k \in \Xi_{\bar{v}}$ and all $\theta_{k+1} \in \Xi_{\bar{v}}$

$$A(\theta_k) P(\theta_{k+1}) A(\theta_k) - P(\theta_k) \prec 0.$$

Pre and post multiply this matrix inequality by x_k^T and its transpose respectively to get exactly $x_{k+1}^T P(\theta_{k+1}) x_{k+1} - x_k^T P(\theta_k) x_k < 0$ along trajectories of (1). Proof of stability is as previously but with the parameter-dependent time-varying Lyapunov function given in (6).

The last part of the theorem that states that it is no more conservative than Theorem 1 is trivial taking $P^{[v]} = P$ for all vertices. ■

The goal of this paper is to analyze the links and differences of the results of Theorems 2 and 3 which both improve results of Theorem 1 using similar polytopic Lyapunov functions but for different assumptions on time evolutions of the uncertainty θ.

3. MAIN RESULTS

3.1 Descriptor models of systems with rationally dependent switching parameters

Let us consider now that the system (1) depends rationally of the uncertainties θ. In such case, using classical linear-fractional transformation (LFT), it can be equivalently rewritten in the following feedback-loop configuration (see Doyle et al. [1991])

$$\begin{cases} x_{k+1} = Ax_k + Bw_k, \\ z_k = Cx_k + D_k \end{cases} \quad w_k = \Delta(\theta_k) z_k \tag{7}$$

where $\Delta(\theta)$ is affine in the uncertainties and can be written as $\Delta(\theta) = \sum_{v=1}^{\bar{v}} \theta_v \Delta^{[v]}$. By construction, the LFT is said well posed if the matrix $I - D\Delta(\theta)$ is invertible for all $\theta \in \Xi_{\bar{v}}$ and the model (1) is recovered by the following formula:

$$A(\theta) = A + B\Delta(\theta)(I - D\Delta(\theta))^{-1}C$$

An alternative to that modeling, is the following descriptor representation in which uncertainties enter in an affine fashion:

$$\begin{bmatrix} I \\ 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} -B\Delta(\theta_k) \\ I - D\Delta(\theta_k) \end{bmatrix} z_k = \begin{bmatrix} A \\ C \end{bmatrix} x_k.$$

This model happens to be a sub case of more general descriptor models as proposed by Coutinho et al. [2002], Masubuchi et al. [2003] :

$$E_x(\theta_k)x_{k+1} + E_\pi(\theta_k)\pi_k = F(\theta_k)x_k \tag{8}$$

where $\pi \in \mathbb{R}^q$ are fictive signals used for rendering the model affine. All matrices may be considered as affine polytopic

$$[\, E_x(\theta) \;\; E_\pi(\theta) \;\; -F(\theta) \,] = \sum_{v=1}^{\bar{v}} \theta_v \left[\, E_x^{[v]} \;\; E_\pi^{[v]} \;\; -F^{[v]} \,\right]. \tag{9}$$

In the present paper, we consider only systems that are originally in non-descriptor form. In such case, as seen upper when considering models issued from well-posed LFT representations, the matrix $E(\theta) = [\, E_x(\theta) \;\; E_\pi(\theta) \,]$ is square and invertible for all $\theta \in \Xi_{\bar{v}}$. This assumption guarantees that x_{k+1} and π_k are well defined for all k, and that the system is causal without impulsive modes. Extensions to more general descriptor models are possible following the lines of Bara [2011], Barbosa et al. [2012].

3.2 Slack variables result

Theorem 4. If there exist $2\bar{v}$ matrices $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$, $G^{[v]} \in \mathbb{R}^{(2n+q)\times(n+q)}$ such that the following LMI conditions hold for all pairs of vertices $v = 1\ldots\bar{v}$, $w = 1\ldots\bar{v}$

$$\begin{bmatrix} P^{[w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G^{[w]} \left[\, E_x^{[v]} \;\; E_\pi^{[v]} \;\; -F^{[v]} \,\right] \right\}^{\mathcal{S}} \tag{10}$$

then the uncertain system defined by (8-9) is robustly stable with respect to any time-varying uncertainty $\theta_k \in \Xi_{\bar{v}}$.

Proof The LMIs are convex in both the vertex matrices $P^{[v]}$, $E_x^{[v]}$, $E_\pi^{[v]}$ and $F^{[v]}$ with indexes v. Hence, defining the affine polytopic matrix $P(\theta_k)$ as in (6) one gets for all $w = 1\ldots\bar{v}$ and all $\theta_k \in \Xi_{\bar{v}}$:

$$\begin{bmatrix} P^{[w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P(\theta_k) \end{bmatrix} \prec \left\{ G^{[w]} \left[\, E_x(\theta_k) \;\; E_\pi(\theta_k) \;\; -F(\theta_k) \,\right] \right\}^{\mathcal{S}}.$$

These inequalities being convex in the matrices $P^{[w]}$ and $G^{[w]}$, defining $G(\theta_{k+1}) = \sum_{w=1}^{\bar{w}} \theta_{w,k+1} G^{[w]}$ one gets for all $\theta_k \in \Xi_{\bar{v}}$ and all $\theta_{k+1} \in \Xi_{\bar{v}}$

$$\begin{bmatrix} P(\theta_{k+1}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P(\theta_k) \end{bmatrix} \prec \{ G(\theta_{k+1}) [\, E_x(\theta_k) \;\; E_\pi(\theta_k) \;\; -F(\theta_k) \,] \}^{\mathcal{S}}.$$

Pre and post multiply this matrix inequality by $\left(x_{k+1}^T \;\; \pi_k^T \;\; x_k^T \right)$ and its transpose respectively to get exactly $x_{k+1}^T P(\theta_{k+1}) x_{k+1} - x_k^T P(\theta_k) x_k < 0$ along trajectories of (8). ■

As illustrated by the proof, Theorem 4 is a direct extension of Theorem 3 for rationally-dependent uncertain systems. The extension is made possible thanks to the affine descriptor modeling of the systems and by introducing as in Theorem 2 some slack variables. A question that arises naturally is whether the additional variables are necessary or do they artificially complexify the numerical problem to solve. The following section aims at giving some answers to this question.

Before that, let us study the conservatism of Theorem 4. One source of conservatism is the choice of a quadratic in the state Lyapunov function $V_k(\theta) = x_k^T P(\theta_k) x_k$. Assuming this choice is done, let us look at the other possible sources of conservatism. Lyapunov stability implies that for all $\theta_k \in \Xi_{\bar{v}}$ and $\theta_{k+1} \in \Xi_{\bar{v}}$ the following quadratic form is negative for all vectors satisfying the linear constraint:

$$\eta_k^T \begin{bmatrix} P(\theta_{k+1}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P(\theta_k) \end{bmatrix} \eta_k < 0$$
$$\forall \eta_k = \begin{pmatrix} x_{k+1} \\ \pi_k \\ x_k \end{pmatrix} \neq 0 \;:\; [\, E_x(\theta_k) \;\; E_\pi(\theta_k) \;\; -F(\theta_k) \,] \eta_k = 0.$$

Equivalently, by Finsler lemma (see Skelton et al. [1998]), it writes as the existence of a parameter-dependent $G(\theta_k, \theta_{k+1})$ matrix such that

$$\begin{bmatrix} P(\theta_{k+1}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P(\theta_k) \end{bmatrix} \prec \{ G(\theta_k, \theta_{k+1}) [\, E_x(\theta_k) \;\; E_\pi(\theta_k) \;\; -F(\theta_k) \,] \}^{\mathcal{S}}.$$

The second source of conservatism appears at this point. It amounts to looking for θ_k independent slack matrices $G(\theta_k, \theta_{k+1}) = G(\theta_{k+1})$. Assume this conservative choice is made. Let $P^{[v]} = P(\theta^{[v]})$ and $G^{[w]} = G(\theta^{[w]})$ be the parameter dependent matrices evaluated at the vertices of the simplex. Since the upper defined conditions holds for all $\theta_k \in \Xi_{\bar{v}}$ and $\theta_{k+1} \in \Xi_{\bar{v}}$ it also holds for all pairs of vertices $v = 1\ldots\bar{v}$, $w = 1\ldots\bar{v}$ which is exactly the condition of Theorem 4.

As for the classical slack variable results of Theorem 2 the upper discussion indicates that the only source of conservatism comes from imposing the parameter-dependent slack variable G not to depend of the uncertainties. Here the restriction is only on part of this dependency: dependency to the current uncertainty value θ_k. The fact that the Lyapunov matrix is of affine polytopic-type is a consequence of this choice. That constatation is classical to slack variable results (see for example Oliveira and Geromel [2005]) and is extended here to the switching uncertainty case.

4. REDUCING DIMENSIONS OF LMI

First, let us consider the formula (10) applied to the case of affine polytopic uncertain systems (1-2) it reads as follows:

$$\begin{bmatrix} P^{[w]} & 0 \\ 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G^{[w]} \begin{bmatrix} I & -A^{[v]} \end{bmatrix} \right\}^{\mathcal{S}}. \tag{11}$$

When compared to (5) the LMIs contain many more decision variables and are of doubled size. The upper formulated question is whether this increased numerical complexity is useful in terms of conservatism reduction or not. The answer is clearly no. Indeed, pre and post multiply (11) by $\begin{bmatrix} A^{[v]T} & I \end{bmatrix}$ and its transpose respectively. The result is exactly (5). Conversely, assume (5) holds, then, by a Schur complement argument it reads equivalently as

$$\begin{bmatrix} -P^{[w]} & P^{[w]}A^{[v]} \\ A^{[v]T}P^{[w]} & -P^{[v]} \end{bmatrix} \prec 0$$

which is exactly (11) for the choice $G^{[w]T} = \begin{bmatrix} P^{[w]} & 0 \end{bmatrix}$. Conditions (5) and (11) are equivalent and the former is preferable numerically since it is of reduced dimensions and contains much less decision variables. This same discussion also applies to the results given in Daafouz and Bernussou [2001]. The additional variables it contains are useless, at least for the stability analysis issue.

4.1 Parameter-independent columns

The upper discussion is now generalized allowing to reduce systematically the size of the LMIs (10). Unfortunately, except for the upper case of affine systems (1-2), we were not able to prove that the procedure is fully lossless. The overall procedure contains two steps. The first one concerns parameter independent columns of the $E(\theta)$ matrix. It may be conservative in some cases. The second one, that is exposed later on, is lossless and concerns parameter-independent rows.

Lemma 5. Assume there exists an invertible matrix T such that for all $v = 1 \dots \bar{v}$

$$E^{[v]}T = \begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} \end{bmatrix} T = \begin{bmatrix} E_1 & E_2^{[v]} \end{bmatrix} \ , \ E_1 \in \mathbb{R}^{(n+q)\times p}.$$

Based on this factorization define

$$N_1^{[v]} = E_1^{+} \begin{bmatrix} E_2^{[v]} & -F^{[v]} \end{bmatrix}, \ N_2^{[v]} = E_1^{\perp} \begin{bmatrix} E_2^{[v]} & -F^{[v]} \end{bmatrix}.$$

Moreover, let the following notations

$$\begin{bmatrix} M_{11}(P^{[w]}) & M_{12}(P^{[w]}) \\ M_{12}^T(P^{[w]}) & M_{22}(P^{[w]}) \end{bmatrix} = T^T \begin{bmatrix} P^{[w]} & 0 \\ 0 & 0 \end{bmatrix} T \ , \ M_{11} \in \mathbb{R}^{p\times p}$$

$$\hat{M}(P^{[w]}, P^{[v]}) = \left[\begin{array}{c|cc} M_{11}(P^{[w]}) & M_{12}(P^{[w]}) & 0 \\ \hline M_{12}^T(P^{[w]}) & M_{22}(P^{[w]}) & 0 \\ 0 & 0 & -P^{[v]} \end{array}\right]$$

If the following LMIs in the decisions variables $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$, $\hat{G}^{[v]} \in \mathbb{R}^{(2n+q-p)\times(n+q-p)}$ hold for all all pairs of vertices $v = 1 \dots \bar{v}$, $w = 1 \dots \bar{v}$

$$\begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix}^T \hat{M}(P^{[w]}, P^{[v]}) \begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix} \prec \left\{ \hat{G}^{[w]} N_2^{[v]} \right\}^{\mathcal{S}} \tag{12}$$

then conditions of Theorem 4 are feasible, and hence the uncertain system defined by (8-9) is robustly stable with respect to any time-varying uncertainty $\theta_k \in \Xi_{\bar{v}}$.

Proof Starting from conditions (12) by a small perturbation argument, one gets that there exists $\epsilon > 0$ such that:

$$\begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix}^T \left[\begin{array}{c|cc} M_{11}(P^{[w]}) + \epsilon I & M_{12}(P^{[w]}) & 0 \\ \hline M_{12}^T(P^{[w]}) & M_{22}(P^{[w]}) & 0 \\ 0 & 0 & -P^{[v]} \end{array}\right] \begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix} \prec \left\{ \hat{G}^{[w]} N_2^{[v]} \right\}^{\mathcal{S}}.$$

Applying a Schur complement argument, it reads also as

$$\begin{bmatrix} -(M_{11}(P^{[w]}) + \epsilon I) & -(M_{11}(P^{[w]}) + \epsilon I)N_1^{[v]} \\ -N_1^{[v]T}(M_{11}(P^{[w]}) + \epsilon I) & (*) \end{bmatrix} \prec 0$$

where the bottom-right block is $(*) =$

$$\begin{bmatrix} M_{22}(P^{[w]}) & 0 \\ 0 & -P^{[v]} \end{bmatrix} - \left\{ \hat{G}^{[w]} N_2^{[v]} + \begin{bmatrix} M_{12}^T(P^{[w]}) \\ 0 \end{bmatrix} N_1^{[v]} \right\}^{\mathcal{S}}.$$

After some manipulations the inequalities also write as

$$\hat{M}(P^{[w]}, P^{[v]}) \prec \left\{ \breve{G}^{[w]} \begin{bmatrix} E_1 & E_2^{[v]} & -F^{[v]} \end{bmatrix} \right\} - \begin{bmatrix} \epsilon I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\breve{G}^{[w]} = \begin{bmatrix} (M_{11}(P^{[w]}) + \epsilon I)E_1^{+} \\ \hat{G}^{[w]}E_1^{\perp} + \begin{bmatrix} M_{12}^T(P^{[w]}) \\ 0 \end{bmatrix} E_1^{+} \end{bmatrix}$. The final step of the proof needs to pre and post multiply the inequality by $\breve{T}^T = \begin{bmatrix} T^{-T} & 0 \\ 0 & I \end{bmatrix}$ and $\breve{T}$ respectively. It implies exactly (10) with $G^{[w]} = \breve{T}^T \breve{G}^{[w]}$. ■

The reduction of numerical complexity from Theorem 4 to Lemma 5 can be measured in terms of number of decisions variables and number of rows of the LMIs. The difference of the number of variables is

$$\bar{v}((3n + 2q)p - p^2)$$

which is positive since $p \leq n + q$. The reduction of the number of rows in the LMIs is $\bar{v}^2 p$. These values are non negligible, especially when the number of vertices $\bar{v}$ is large. It is often the case, for example when the model is composed of N independent parameters in intervals. In that case $\bar{v} = 2^N$.

4.2 Parameter-independent rows

The upper defined method for reducing the dimensions of the LMI problem is based on a factorization of the $E(\theta)$ matrix taking advantage of parameter independent columns. That procedure can be combined with the next lemma that takes advantage of possible knowledge about parameter independent rows.

Lemma 6. Assume there exists an invertible matrix S such that for all $v = 1 \dots \bar{v}$

$$S \begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} & -F^{[v]} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2^{[v]} \end{bmatrix} \ , \ \begin{array}{l} \mathrm{rank}(F_1) = r \\ F_1 \in \mathbb{R}^{s\times(2n+q)} \end{array}$$

then the conditions of Theorem 4 are feasible, if and only if, there exists $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$, $\tilde{G}^{[v]} \in \mathbb{R}^{(2n+q-r)\times(n+q-s)}$ such that the following conditions hold for all pairs of vertices $v = 1 \dots \bar{v}$, $w = 1 \dots \bar{v}$

$$X^T \begin{bmatrix} P^{[w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[v]} \end{bmatrix} X \prec \left\{ \tilde{G}^{[w]} F_2^{[v]} X \right\}^{\mathcal{S}} \tag{13}$$

where $X = F_1^{T\perp T} \in \mathbb{R}^{(2n+q)\times(2n+q-r)}$ is such that $F_1 X = 0$.

Lemma 7. Assume there exists an invertible matrix $\tilde{S}$ such that for all $v = 1 \dots \bar{v}$

$$S N_2^{[v]} = \begin{bmatrix} F_1 \\ F_2^{[v]} \end{bmatrix} \ , \ \begin{array}{l} \mathrm{rank}(F_1) = r \\ F_1 \in \mathbb{R}^{s\times(2n+q-p)} \end{array}$$

then the conditions of Lemma 5 are feasible, if and only if, there exists $P^{[v]} = P^{[v]T} \succ 0 \in \mathbb{R}^{n\times n}$, $\tilde{G}^{[v]} \in \mathbb{R}^{(2n+q-p-r)\times(n+q-p-s)}$ such that the following conditions hold for all pairs of vertices $v = 1\ldots\bar{v}$, $w = 1\ldots\bar{v}$

$$X^T \begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix}^T \hat{M}(P^{[w]}, P^{[v]}) \begin{bmatrix} N_1^{[v]} \\ -I \end{bmatrix} X \prec \left\{ \tilde{G}^{[w]} F_2^{[v]} X \right\}^{\mathcal{S}} \tag{14}$$

where $X = F_1^{T\perp T} \in \mathbb{R}^{(2n+q-p)\times(2n+q-p-r)}$ is such that $F_1 X = 0$.

Proof Only the proof of Lemma 7 is detailed. The proof of Lemma 6 follows exactly the same lines.

Fist we prove that (12) implies (14). By definition of S the right hand side of (12) is $\left\{ \hat{G}^{[w]} S^{-1} \begin{bmatrix} F_1 \\ F_2^{[v]} \end{bmatrix} \right\}^{\mathcal{S}}$. Therefore, pre and post multiplying (12) by X^T and X respectively, one gets exactly (14) with $\tilde{G}^{[w]} = X^T \hat{G}^{[w]} S^{-1} \begin{bmatrix} 0 \\ I_{n+q-p-s} \end{bmatrix}$.

Conversely, since X is full column rank one has $X^T X^{+T} = I$, and (14) can rewritten as:

$$X^T \left(\Psi^{[v,w]} - \left\{ X^{+T} \tilde{G}^{[w]} F_2^{[v]} \right\}^{\mathcal{S}} \right) X \prec 0$$

where $\Psi^{[v,w]}$ is the left-hand side term of (12). Finsler lemma Skelton et al. [1998] implies the existence of positive scalars $\epsilon^{[v,w]}$ such that

$$\Psi^{[v,w]} - \left\{ X^{+T} \tilde{G}^{[w]} F_2^{[v]} \right\}^{\mathcal{S}} \prec \epsilon^{[v,w]} F_1^T F_1 \prec \epsilon^{[w]} F_1^T F_1.$$

The right-hand side inequality is obtained taking $\epsilon^{[w]} > \epsilon^{[v,w]}$ for all $v = 1\ldots\bar{v}$. This last inequality is exactly (12) for the choice $\hat{G}^{[w]} = \begin{bmatrix} \frac{\epsilon^{[w]}}{2} F_1^T & X^{+T} \tilde{G}^{[w]} \end{bmatrix} S$. ■

Only the reduction of numerical complexity from Theorem 4 to Lemma 6 is stated since that to Lemma 7 is the combination of two lemmas but is not the sum of the two. The difference of the number of variables is

$$\bar{v}((2n+q)s + r(n+q) - rs)$$

and the reduction of the number of rows in the LMIs is $\bar{v}^2 r$. Again, these values are non negligible.

5. ROBUSTNESS

In the previous section it is assumed that all uncertainties are time varying. This is a very general case that includes the case of constant, parametric, uncertainties. But, as seen in the preliminaries, there exist as well some slack variable results specific for the situation where all uncertain parameters are constant. The goal of this subsection is to consider the combined case when some uncertainties $\theta_k \in \Xi_{\bar{v}}$ are time varying, while other, $\phi \in \Xi_{\bar{\mu}}$ are constant.

The uncertain models are again assumed in descriptor form

$$E_x(\theta_k, \phi) x_{k+1} + E_\pi(\theta_k, \phi) \pi_k = F(\theta_k, \phi) x_k \tag{15}$$

with left invertible $E(\theta_k, \phi) = [\, E_x(\theta_k, \phi) \;\; E_x(\theta_k, \phi) \,]$ for all uncertainties. The model is considered as affine polytopic in both the uncertainties

$$\begin{aligned} &[\, E_x(\theta,\phi) \;\; E_\pi(\theta,\phi) \;\; -F(\theta,\phi) \,] \\ &= \sum_{\mu=1}^{\bar{\mu}} \sum_{v=1}^{\bar{v}} \phi_\mu \theta_v \left[\, E_x^{[\mu,v]} \;\; E_\pi^{[\mu,v]} \;\; -F^{[\mu,v]} \,\right]. \end{aligned} \tag{16}$$

Without any difficulty, for these models one gets the following general slack variables result:

Theorem 8. If there exist $\bar{v}\bar{\mu}$ matrices $P^{[\mu,v]} = P^{[\mu,v]T} \succ 0 \in \mathbb{R}^{n\times n}$ and $\bar{v}$ matrices $G^{[v]} \in \mathbb{R}^{(2n+q)\times(n+q)}$ such that the following LMI conditions hold for all triples of vertices $\mu = 1\ldots\bar{\mu}$, $v = 1\ldots\bar{v}$, $w = 1\ldots\bar{v}$

$$\begin{bmatrix} P^{[\mu,w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[\mu,v]} \end{bmatrix} \prec \left\{ G^{[w]} \left[\, E_x^{[\mu,v]} \;\; E_\pi^{[\mu,v]} \;\; -F^{[\mu,v]} \,\right] \right\}^{\mathcal{S}} \tag{17}$$

then the uncertain system defined by (15-16) is robustly stable with respect to any time-varying uncertainty $\theta_k \in \Xi_{\bar{v}}$ and any parametric uncertainty $\phi \in \Xi_{\bar{\mu}}$.

The proof follows exactly the lines of the previous ones and is therefore not reproduced here. Moreover, similar results as Lemmas 5, 6, 7 are applicable. They are not included in the manuscript for evident reasons of lack of space.

Theorem 8 is the result that merges the two type of results defined in the preliminaries. They happen to be complementary and thanks to the slack variables approach to have a simple mathematical formulation (at least before applying the lemmas for numerical complexity reduction).

6. NUMERICAL EXAMPLE

Let the system described by

$$a_k y_{k+2} + b_k^2 y_{k+1} + a_k b_k y_k = 0. \tag{18}$$

It admits the following usual state-space representation, rational in the uncertainties $a_k \neq 0$ and b_k

$$x_{k+1} = \begin{pmatrix} y_{k+2} \\ y_{k+1} \end{pmatrix} = \begin{bmatrix} -b_k^2/a_k & -b_k \\ 1 & 0 \end{bmatrix} x_k$$

and the following affine descriptor representation

$$\begin{bmatrix} a_k & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} b_k \\ 0 \\ 1 \end{bmatrix} \pi_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ b_k & a_k \end{bmatrix} x_k.$$

The uncertainties are assumed to belong to intervals

$$a_k \in [\,1\,,\,2\,]\,,\; b_k \in [\,-0.5\,,\,\beta].$$

The LMI conditions are tested for different values of β in order to measure their conservatism.

Conditions of Theorem 4 are feasible up to $\beta_1 = 0.81090$, the number of decision variables is 72 and the number of lines of the LMIs (10) is 80. For values of β larger than 0.81096 the LMIs are found unfeasible. In between the two given values, the SDPT3 solver of Toh et al. [1999] does not conclude due to numerical problems (we used default settings).

Conditions of Lemma 6 are tested. From a theoretical point of view these are equivalent to that of Theorem 4. Yet, they are of smaller dimensions (44 variables, 64 rows). This has the effect of reducing the possible numerical problems. Indeed, the LMIs are feasible up to $\beta_2 = 0.81094$ and unfeasible for $\beta = 0.81095$. For results further on, no numerical problems were found and we therefore give only the upper bounds for which the LMIs are feasible. They are unfeasible as soon as the last digit is increased.

For the considered example conditions of Lemma 5 are exactly the same as those of 6 (at the expense of a permutation of some rows and columns). Results are hence trivially identical.

β (nb vars/nb rows)	syst. (18)	syst. (19)
a_k, b_k	0.81094 (44/64)	0.84677 (480/1536)
a, b_k	0.89027 (28/32)	0.90293 (144/192)
a_k, b	0.82658 (28/32)	0.85375 (144/192)
a, b	0.98059 (20/16)	0.99519 (48/24)

Table 1. Summary of the numerical results

Table 1 summarizes results when applying Theorem 8 combined to size reduction technique of Lemma 6. Four cases are tested for all possible combinations of either switching or constant parametric uncertainties. The table provides the largest values of β such that the LMIs are feasible. As expected, comparing a switching situation to a parametric one, the range of admissible values of uncertainties is increased. This is not guaranteed by the theoretical results but does hold on the example.

To reduce conservatism a strategy exposed in Ebihara et al. [2005], Peaucelle et al. [2007] is to apply the LMI tests to an augmented model. The one considered here is with one added sample of time ahead:

$$\begin{array}{c} a_k y_{k+2} + b_k^2 y_{k+1} + a_k b_k y_k = 0 \\ a_{k+1} y_{k+3} + b_{k+1}^2 y_{k+2} + a_{k+1} b_{k+1} y_{k+1} = 0. \end{array} \tag{19}$$

Results for the augmented model are given in the second column of Table 1. The conservatism reduction is non negligible. It is at the expense of highly increased numerical burden, in particular in the al switching uncertainties case.

A simple analysis of the system indicates that the actual robust bound for constant parameters is $\beta^\star = 1$. It is almost attained with test on the augmented model. Actual upper bound for the case of switching parameters are unknown.

7. CONCLUSIONS

The contribution of the paper is to provide a flexible general methodology for the analysis of discrete-time systems with uncertainties. Results illustrate the high flexibility of the slack-variable approach to analyze systems rational in the uncertainties that could be switching (no bounds on their variations), parametric (constant) or a combination of the two. Results are shown to be conservative in general and a technique is shown to provide less conservative results by simply applying the formulas to models augmented with repeated dynamics further ahead in time.

The drawback of the slack variable approach is to introduce many decision variables. As seen on the examples this increase of the numerical burden comes mainly from the switching uncertainties. Some lemmas are provided to contain this numerical complexity augmentation. These prove to be efficient both in reducing the computation time and the robustness to numerical errors.

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