Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

An underwater vehicles dynamics in the presence of noise and Fokker-Planck equations

Shambhu N. Sharma*, Ravish H. Hirpara **

**S.V. National Institute of Technology, Surat, 395007 INDIA (e-mail: snsvolterra@gmail.com)*
*** S.V. National Institute of Technology, Surat, 395007 INDIA (e-mail: ravishhirpara@gmail.com)*

Abstract: In systems and control literature, underwater vehicles are studied extensively. The translational equations of motion coupled with rotational equations, which encompass the evolution of linear velocity vector in combination with angular velocity vector, describe the equation of motion of underwater vehicles. As a result of this, the underwater vehicle dynamics assumes the structure of a six-dimensional ordinary differential equation. Ocean current circulations influence stochastically the motion of underwater vehicles. In this paper, first we account the underwater vehicle-ocean current interaction in the stochastic framework that leads to a multi-dimensional Stochastic Differential Equation (SDE). Secondly, we accomplish the noise analysis of underwater vehicles using the Fokker-Planck technique. This paper will be of interest to mechanists, control theorists, naval scientists looking for research in autonomous systems under noise influence. Further research on stochastically perturbed underwater vehicles will refine existing results available in literature.

Keywords: Kolomogrov-Fokker-Planck equation, Kolomogrov backward equation, underwater Itô stochastic differential equation, underwater vehicle-ocean current random interactions

1. INTRODUCTION

Autonomous Underwater Vehicles (AUVs) have found wide applications in ocean observations, bathymetric surveys, and ocean floor analysis, military, fisheries applications, deep ocean modeling. The potential applications of autonomous underwater vehicles for scientific and naval purposes are well credited in Bovio *et al.* (2006). The underwater vehicle is an appealing example of underactuated autonomous systems. The popular examples of underactuated autonomous systems are robotic manipulators, walking robots, wheeled robots, aircrafts, missiles, helicopters and underwater vehicles. Here, we restrict our discussions to current research activities in underwater vehicles. A control strategy for forcing the underwater state trajectories to track space trajectories is explained nicely in Alonge *et al.* (2001). In their paper, the kinematic and dynamic controllers were the subject of investigation. After accounting the concept of observers in control systems, the concept of dynamic controllers arises. In Aguiar and Pascoal (2007), control of autonomous underwater vehicles was the subject of investigation in which two controllers were designed successfully: (i) non-linear dynamic controller (ii) adaptive non-linear controller design approaches. The Lyapunov method was adopted to design control laws. In their paper, the vehicle-ocean current interaction was accounted deterministically in lieu of stochastically. David Mumford argues striking influence of 'stochastiicity' on the general theory of dynamical systems, see a famous paper of Mumford (2000). He recommends that stochastic methods will change the shape of modern research. Aguiar and Hespanha (2003) published an interesting paper, which is about control of underactuated vehicles. Most notably, they described control of underwater vehicles with noise influence as an open problem that warrants further research, see Aguiar and Hespanha (2003, p. 1993). Despite recommendations on 'noise analysis' of underwater vehicles, literature on 'noise-influenced underwater vehicles' is relatively very scarce. The noise equations of the underwater vehicle involve the multi-dimensional state vector as well as construction of the stochastic differential in lieu of the ordinary differential. This seems to be one of the reasons that why stability, estimation and control of underwater vehicles with noise processes are not available in literature in a greater detail. Much is left to be done.

The intent of this paper is to develop and analyse underwater vehicle dynamics using a formal stochastic interpretation. The ocean current circulations introduce random perturbations into the vehicle dynamics in lieu of the deterministic. For this reason, it is realistic to develop randomly perturbed vehicle dynamics. Subsequently, the Fokker-Planck approach, a celebrated stochastic method, is utilized to accomplish the noise analysis of the stochastic problem considered here. In contrast to Aguiar and Pascoal (2007), this paper accounts heave, pitch and roll motions as well as the stochastic character of the ocean current circulations.

2. AN AUTONOUMOUS UNDERWATER VEHICLE 'SDE'

The dynamical equations of underwater vehicles are derived in the body-fixed frame. The dimension of the state vector, which describes the underwater dynamics, is six. A schematic diagram of autonomous underwater vehicles is given in figure (1).

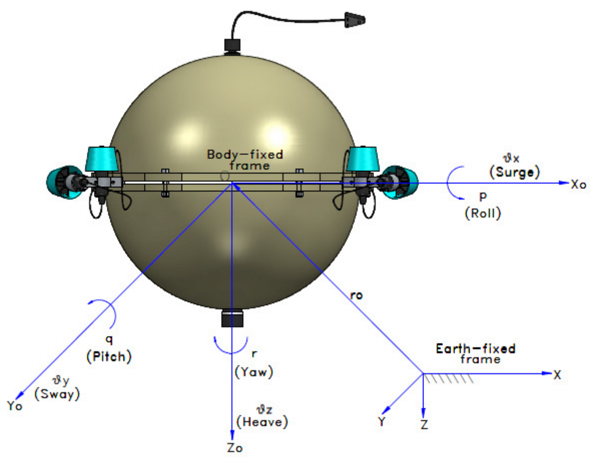

Figure 1: An autonomous underwater vehicle

Making the use of general theory of mechanics and vector calculus, we derive the underwater vehicle dynamics. A greater detail about the noise-free underwater vehicle dynamics is nicely explained in a famous book authored by Fossen (1994). The deterministic structure of the underwater vehicle system is the following:

$$m(\dot{v}_x - v_y r + v_z q) = X,\ \ m(\dot{v}_y - v_z p + v_x r) = Y,$$
$$m(\dot{v}_z - v_x q + v_y p) = Z,\ I_x\dot{p} + (I_z - I_y)qr = K, \qquad (1)$$
$$I_y\dot{q} + (I_x - I_z)rp = M,\ \ I_z\dot{r} + (I_y - I_x)pq = N,$$

where the term m denotes the mass of the body, the vector $(v_x, v_y, v_z)^T$ is the linear velocity vector of the vehicle in the rotating body-fixed frame. The vector $(p,\ q,\ r)^T$ is the angular velocity vector of the co-ordinates of the vehicle. The vector $(X, Y, Z)^T$ denotes the external forces acting on the vehicle. On the other hand, the vector $(K,\ M,\ N)^T$ has interpretation as the vector moment about the co-ordinates of the vehicle. In order to arrive at equation (1), we consider the velocity of underwater vehicle is relatively less as well as the centre of gravity of the vehicle coincides with the origin of the body-fixed frame. One can arrive at the vehicle dynamics stated in equation (1) using the theory of Lagrangian mechanics and dynamical systems as well. Since the procedure the lengthy, the details are omitted. The random underwater perturbation is chiefly attributed to ocean current circulations. The atmospheric wind system, heat exchange at sea surface coupled with salinity changes, the Co-riolis force, tidal components arising from planetary interactions contribute to the stochastic character of the ocean current circulations. Because of these factors, it becomes more realistic to consider the AUV dynamics in the stochastic framework in lieu of the deterministic. The deterministic AUV model is available in celebrated books and seminal papers, e.g. Aguiar and Pascoal (2007) and Fossen (1994). After accounting the stochastic correction terms in the external forces and the moment of external forces in the white noise setting, we arrive at the underwater vehicle SDE. Replace the terms X, Y, Z, K, M, N of equation (1) with the terms

$X(1+\ \gamma_x w_t^x),\ Y(1+\gamma_y w_t^y),\ Z(1+\gamma_z w_t^z),\ K(1+\gamma_k w_t^k),$

$M(1+\gamma_m w_t^m),\ N(1+\gamma_n w_t^n)$. The standard approach in the theory of SDEs is to account stochastic correction terms in the ODEs using the concept of the first-order stochastic perturbation-theoretic approach. The term $(\gamma_x w_t^x,\ \gamma_y w_t^y,\ \gamma_z w_t^z,\ \gamma_k w_t^k,\ \gamma_m w_t^m,\ \gamma_n w_t^n)^T$ denotes the stochastic perturbation parameter vector. Thus, equation (1) can be recast as

$$m(\dot{v}_x - v_y r + v_z q) = X(1+\gamma_x w_t^x),$$
$$m(\dot{v}_y - v_z p + v_x r) = Y(1+\gamma_y w_t^y),$$
$$m(\dot{v}_z - v_x q + v_y p) = Z(1+\gamma_z w_t^z),$$
$$I_x\dot{p} + (I_z - I_y)qr = K(1+\gamma_k w_t^k),$$
$$I_y\dot{q} + (I_x - I_z)rp = M(1+\gamma_m w_t^m),$$
$$I_z\dot{r} + (I_y - I_x)pq = N(1+\gamma_n w_t^n).$$

After recasting the above system of equations as well as considering simpler and brief notations, we have

$$\frac{X}{m}\gamma_x = \gamma_1,\ \frac{Y}{m}\gamma_y = \gamma_2,\ \frac{Z}{m}\gamma_z = \gamma_3,$$
$$\frac{K}{I_x}\gamma_k = \gamma_4, \frac{M}{I_y}\gamma_m = \gamma_5, \frac{N}{I_z}\gamma_n = \gamma_6,$$
$$dv_x = (\frac{X}{m} + v_y r - v_z q)dt + \gamma_1 w_t^x dt,$$
$$dv_y = (\frac{Y}{m} + v_z p - v_x r)dt + \gamma_2 w_t^y dt,$$
$$dv_z = (\frac{Z}{m} + v_x q - v_y p)dt + \gamma_3 w_t^z dt,$$
$$dp = (\frac{K}{I_x} + \frac{(I_y - I_z)}{I_x}qr)dt + \gamma_4 w_t^k dt, \qquad (2)$$
$$dq = (\frac{M}{I_y} + \frac{(I_z - I_x)}{I_y}rp)dt + \gamma_5 w_t^m dt,$$
$$dr = (\frac{N}{I_z} + \frac{(I_x - I_y)}{I_z}pq)dt + \gamma_6 w_t^n dt.$$

Note that a theoretical interpretation of the white noise process is the Itô setting. In the Itô setting, the term $dB_t = w_t dt$ is the subject of investigation (Kuo 2005). The above set of scalar stochastic differential equations can be further recast as the following vector Itô stochastic differential equation. Thus, equation (2) becomes

$$d\xi_t = a(\xi_t, t)dt + b(\xi_t, t)dB_t, \qquad (3)$$

where

$$\xi_t = (v_x, v_y, v_z, p, q, r)^T,\ a(\xi_t, t) = (a_t^T(\xi_t, t), a_r^T(\xi_t, t))^T,$$

$$a_t(\xi_t, t) = \left(\frac{X}{m} + v_y r - v_z q,\ \frac{Y}{m} + v_z p - v_x r,\ \frac{Z}{m} + v_x q - v_y p\right)^T,$$

$$a_r(\xi_t, t) = \left(\frac{K}{I_x} + \frac{(I_y - I_z)}{I_x} qr,\ \frac{M}{I_y} + \frac{(I_z - I_x)}{I_y} rp,\ \frac{N}{I_z} + \frac{(I_x - I_y)}{I_z} pq\right)^T,$$

$$(b_{ii}(\xi_t, t))_{1 \le i \le 6} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6)^T,$$

$$b_{ij}(\xi_t, t)_{i \ne j} = 0, i \ne j,$$

$$dB_t = \left(dB_t^{v_x}, dB_t^{v_y}, dB_t^{v_z}, dB_t^{p}, dB_t^{q}, dB_t^{r}\right)^T.$$

For a brevity of presentations, we adopt more familiar notations of SDEs, i.e.

$$\xi_t = (\xi_i(t))_{1 \le i \le 6} = (v_x, v_y, v_z, p, q, r)^T = (x_1, x_2, x_3, x_4, x_5, x_6)^T$$

$$a_t(\xi_t, t) = \left(\frac{X}{m} + x_2 x_6 - x_3 x_5, \frac{Y}{m} + x_3 x_4 - x_1 x_6, \frac{Z}{m} + x_1 x_5 - x_2 x_4\right)^T,$$

$$a_r(\xi_t, t) = \left(\frac{K}{I_x} + \frac{(I_y - I_z)}{I_x} x_5 x_6, \frac{M}{I_y} + \frac{(I_z - I_x)}{I_y} x_4 x_6, \frac{N}{I_z} + \frac{(I_x - I_y)}{I_z} x_4 x_5\right)^T.$$

More importantly, the vector $(dB_t^{v_x}, dB_t^{v_y}, dB_t^{v_z})^T$ denotes the random contribution of ocean current circulations to the translation motion of co-ordinates of underwater vehicles. Furthermore, the vector $(dB_t^{p}, dB_t^{q}, dB_t^{r})^T$ denotes the random contribution of ocean current circulations to the rotational motion of the co-ordinates of underwater vehicles. More importantly, equation (3) describes a vehicle non-linear Itô SDE, the cornerstone formalism of this paper, that can be treated as an advanced and refined system of dynamical equations of the vehicle in contrast to the vehicle dynamics available in literature. Since equation (3) accounts non-linearity and stochasticity, this suggests ability of equation (3) to account a greater qualitative characteristics of underwater vehicle dynamics. On the other hand, vehicle dynamical equations available in literature ignore stochastic considerations. The estimation theory, stability and control of vehicle by exploiting the vehicle Itô SDE stated in equation (3) would be more stringent in lieu of the vehicle deterministic dynamics. A schematic diagram of the vehicle stochastic dynamics can be constructed from the system-theoretic viewpoint. Figure (2) is a consequence of the underwater vehicle SDE.

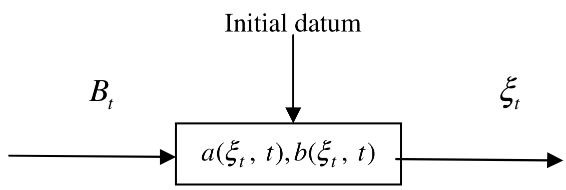

Figure 2: autonomous underwater vehicle SDE diagram

3. NOISE ANALYSIS OF AUVs USING FOKKER PLANCK EQUATION

The Fokker-Planck equation is about the evolution of conditional probability density for given states for Markov processes. The Fokker-Planck equation has found striking applications for the noise analysis of stochastic differential systems. The usefulness of the Fokker-Planck equation in systems and control literature can be found in the seminal papers, e.g. Bierbaum *et al.* (2002), Risken (1984), and Sharma (2008). In this regard, Sharma and Parthasarathy (2007), Chow and Zhou (2007) will be also useful. Making the use of the Kolmogorov-Fokker-Planck equation for Markov processes (Karatzas and Shreve 1988), the equation for the randomly perturbed autonomous system of the paper is

$$\begin{aligned}
dp &= \Big(-\sum_i \frac{\partial a_i(\xi, t) p}{\partial \xi_i} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 (bb^T)_{ij}(\xi, t) p}{\partial \xi_i \partial \xi_j}\Big) dt \\
&= (-((\frac{X}{m} + x_2 x_6 - x_3 x_5)\frac{\partial p}{\partial x_1} \\
&+ (\frac{Y}{m} + x_3 x_4 - x_1 x_6)\frac{\partial p}{\partial x_2} \\
&+ (\frac{Z}{m} + x_1 x_5 - x_2 x_4)\frac{\partial p}{\partial x_3} + (\frac{K}{I_x} + \frac{(I_y - I_z)}{I_x} x_5 x_6)\frac{\partial p}{\partial x_4} \\
&+ (\frac{M}{I_y} + \frac{(I_z - I_x)}{I_y} x_4 x_6)\frac{\partial p}{\partial x_5} \\
&+ (\frac{N}{I_z} + \frac{(I_x - I_y)}{I_z} x_4 x_5)\frac{\partial p}{\partial x_6}) \\
&+ \frac{1}{2}(\gamma_1^2 \frac{\partial^2 p}{\partial x_1^2} + \gamma_2^2 \frac{\partial^2 p}{\partial x_2^2} + \gamma_3^2 \frac{\partial^2 p}{\partial x_3^2} + \gamma_4^2 \frac{\partial^2 p}{\partial x_4^2} \\
&+ \gamma_5^2 \frac{\partial^2 p}{\partial x_5^2} + \gamma_6^2 \frac{\partial^2 p}{\partial x_6^2}) + \gamma_1\gamma_2 \frac{\partial^2 p}{\partial x_1 \partial x_2} + \gamma_1\gamma_3 \frac{\partial^2 p}{\partial x_1 \partial x_3} \\
&+ \gamma_1\gamma_4 \frac{\partial^2 p}{\partial x_1 \partial x_4} + \gamma_1\gamma_5 \frac{\partial^2 p}{\partial x_1 \partial x_5} \\
&+ \gamma_1\gamma_6 \frac{\partial^2 p}{\partial x_1 \partial x_6} + \gamma_2\gamma_3 \frac{\partial^2 p}{\partial x_2 \partial x_3} \\
&+ \gamma_2\gamma_4 \frac{\partial^2 p}{\partial x_2 \partial x_4} + \gamma_2\gamma_5 \frac{\partial^2 p}{\partial x_2 \partial x_5} \\
&+ \gamma_2\gamma_6 \frac{\partial^2 p}{\partial x_2 \partial x_6} + \gamma_3\gamma_4 \frac{\partial^2 p}{\partial x_3 \partial x_4} \\
&+ \gamma_3\gamma_5 \frac{\partial^2 p}{\partial x_3 \partial x_5} + \gamma_3\gamma_6 \frac{\partial^2 p}{\partial x_3 \partial x_6} \\
&+ \gamma_4\gamma_5 \frac{\partial^2 p}{\partial x_4 \partial x_5} + \gamma_4\gamma_6 \frac{\partial^2 p}{\partial x_4 \partial x_6} + \gamma_5\gamma_6 \frac{\partial^2 p}{\partial x_5 \partial x_6}) dt, \qquad (4)
\end{aligned}$$

where the term p denotes the conditional probability density $p = p(\xi, t | \xi_{t_0}, t_0)$. A comment on the Kolmogorov-Fokker-Planck operator and the Kolmogorov backward operator for the Itô SDE and the vehicle SDE of this paper is briefly explained in the '*appendix*' of the paper.

Making the use of the Kolmogorov-Fokker-Planck equation, Kolmogorov backward equation, and the definition of the evolution of the conditional expectation of the scalar function of the state ξ_t, we arrive at exact evolutions of conditional mean and variance. A proof of the exact estimation equations for the Itô SDE can be found in Jazwinski (1970, p. 137).

The stochastic autonomous system of this paper accounts linear terms, square non-linearity terms and higher-order non-linearity terms vanish. Suppose the vector system non-linearity $a(\xi_t, t)$, and the diffusion coefficient matrix $bb^T(\xi_t, t)$ are bounded continuous as well as they have bounded continuous double derivatives, the following conditional mean and variance evolution equations will be useful for the noise analysis of the autonomous system of this paper:

$$d\langle \xi_i(t)\rangle = (a_i(\langle \xi_t\rangle, t) + \frac{1}{2}\sum_{p,q} P_{pq} \frac{\partial^2 a_i(\langle \xi_t\rangle, t)}{\partial\langle \xi_p\rangle \partial\langle \xi_q\rangle})dt, \quad (5a)$$

$$dP_{ij} = (\sum_p P_{ip} \frac{\partial a_j(\langle \xi_t\rangle, t)}{\partial\langle \xi_p\rangle} + \sum_p P_{jp} \frac{\partial a_i(\langle \xi_t\rangle, t)}{\partial\langle \xi_p\rangle} + (bb^T)_{ij}(\langle \xi_t\rangle, t) + \frac{1}{2}\sum_{p,q} P_{pq} \frac{\partial^2 (bb^T)_{ij}(\langle \xi_t\rangle, t)}{\partial\langle \xi_p\rangle \partial\langle \xi_q\rangle})dt, \quad (5b)$$

where the state $\xi_t = (\xi_i(t))$ and the vector system non-linearity $a(\xi_t, t) = (a_i(\xi_t, t))$ and $\langle \xi_i\rangle = E(\xi_i(t)|\xi_{t_0}, t_0)$, $P_{ij} = \langle (\xi_i - \langle \xi_i\rangle)(\xi_j - \langle \xi_j\rangle)\rangle$. Note that equation (5) denotes equations (5a)-(5b). More precisely, the notation '$\langle\ \rangle$' denotes the action of the conditional expectation operator E on random variables. For deterministic initial states, the operator E becomes the expectation in lieu of the conditional expectation. The system of equations stated in equation (5) is the approximate for the estimation of the exact Itô SDE. However, the system of equations has ability to account linear terms and square non-linearity terms completely. Thus, the above system of equations becomes the exact estimation equations for vehicle SDEs that encompass linear and square non-linearity terms only. Notably, the above system of equations has found applications as well as demonstrated its effectiveness for the noise analysis of appealing cases arising from mechanics and dynamical systems, eg. Sharma and Parthasarathy (2007).

After combining equations (3) and (5), we get the following set of coupled conditional moment evolution equations for the stochastic problem of concern here:

$$d\langle \xi_t\rangle = A(\langle \xi_t\rangle, P_t)dt, \quad (6a)$$

$$dP_t = B(\langle \xi_t\rangle, P_t)dt. \quad (6b)$$

For the stochastic autonomous system of this paper,

$$A(\langle \xi_t\rangle, P_t) = (A_i(\langle \xi_t\rangle, P_t))_{1\le i\le 6},$$

$$B(\langle \xi_t\rangle, P_t) = (B_{ij}(\langle \xi_t\rangle, P_t))_{\substack{1\le i\le 6,\\ 1\le j\le 6}},$$

or

$$A_1(\langle \xi_t\rangle, P_t) = \frac{X}{m} + \langle x_2\rangle\langle x_6\rangle - \langle x_3\rangle\langle x_5\rangle + P_{26} - P_{35},$$

$$A_2(\langle \xi_t\rangle, P_t) = \frac{Y}{m} + \langle x_3\rangle\langle x_4\rangle - \langle x_1\rangle\langle x_6\rangle + P_{34} - P_{16},$$

$$A_3(\langle \xi_t\rangle, P_t) = \frac{Z}{m} + \langle x_1\rangle\langle x_5\rangle - \langle x_2\rangle\langle x_4\rangle + P_{15} - P_{24},$$

$$A_4(\langle \xi_t\rangle, P_t) = \frac{K}{I_x} + \frac{(I_y - I_z)}{I_x}\langle x_5\rangle\langle x_6\rangle + \frac{(I_y - I_z)}{I_x}P_{56}, \quad (7)$$

$$A_5(\langle \xi_t\rangle, P_t) = \frac{M}{I_y} + \frac{(I_z - I_x)}{I_y}\langle x_4\rangle\langle x_6\rangle + \frac{(I_z - I_x)}{I_y}P_{46},$$

$$A_6(\langle \xi_t\rangle, P_t) = \frac{N}{I_x} + \frac{(I_x - I_y)}{I_z}\langle x_4\rangle\langle x_5\rangle + \frac{(I_x - I_y)}{I_z}P_{45}.$$

The diagonal elements of the matrix $B(\langle \xi_t\rangle, P_t)$ of equation (6b) are the following:

$$B_{11}(\langle \xi_t\rangle, P_t) = 2(P_{12}\langle x_6\rangle - P_{13}\langle x_5\rangle - P_{15}\langle x_3\rangle + P_{16}\langle x_2\rangle) + \gamma_1^2,$$

$$B_{22}(\langle \xi_t\rangle, P_t) = 2(-P_{12}\langle x_6\rangle + P_{23}\langle x_4\rangle + P_{24}\langle x_3\rangle - P_{26}\langle x_1\rangle) + \gamma_2^2,$$

$$B_{33}(\langle \xi_t\rangle, P_t) = 2(P_{13}\langle x_5\rangle - P_{23}\langle x_4\rangle - P_{34}\langle x_2\rangle + P_{35}\langle x_1\rangle) + \gamma_3^2,$$

$$B_{44}(\langle \xi_t\rangle, P_t) = 2\frac{(I_y - I_z)}{I_x}(P_{45}\langle x_6\rangle + P_{46}\langle x_5\rangle) + \gamma_4^2, \quad (8)$$

$$B_{55}(\langle \xi_t\rangle, P_t) = 2\frac{(I_z - I_x)}{I_y}(P_{45}\langle x_6\rangle + P_{56}\langle x_4\rangle) + \gamma_5^2,$$

$$B_{66}(\langle \xi_t\rangle, P_t) = 2\frac{(I_x - I_y)}{I_z}(P_{46}\langle x_5\rangle + P_{56}\langle x_4\rangle) + \gamma_6^2.$$

Equations (6a) and (6b) in combination with equations (7)-(8) are the estimation *results* of the paper, which are not available in literature, that can be utilized for the underwater vehicle state trajectory estimation.

4. NUMERICAL SIMULATIONS

Here, we simulate the underwater vehicle dynamics accounting the underwater vehicle-ocean current interaction in the stochastic framework. Equations (1) and (3) of this paper are the subject of numerical experimentations. Two different sets of initial conditions, stochastic autonomous system parameters as well as noise perturbations are exploited to accomplish numerical experimentations. Since the paper formalizes the underwater vehicle dynamics in stochastic differential equation framework, the numerical simulations of the propagated state estimations, equation (6), are accomplished as well. Under Gaussian assumptions, the mean and variance propagations will suffice to accomplish the noise analysis of the dynamical system considered here. After considering the Gaussianity, the even and odd powers of moments of the system state can be expressed in terms of first-and second-order moments. For the brevity of presentations, we demonstrate the variance trajectories in the tabular form, tables (1)-(3), in lieu of graphical interpretations. In this section, first we demonstrate the random system state trajectories and subsequently, the propagated estimated state trajectories. The first set of data is the following (Singh *et al.* 2009, Smith 2008):

$$m = 123.8kg, I_x = 5.46kg/m^2, I_y = 5.29kg/m^2, I_z = 5.72kg/m^2$$

$$x_1 = 0.03m/\sec, x_2 = 0.04m/\sec, x_3 = 0.045m/\sec,$$

$x_4 = 0.252 rad/\sec$, $x_5 = -0.189 rad/\sec$, $x_6 = 0.147 rad/\sec$,

The first set of the noise intensity parameter vector, diffusion parameter vector, is

$$(\gamma_1, \gamma_2,\ \gamma_3, \gamma_4, \gamma_5, \gamma_6)^T = (4\times10^{-4},\ 2\times10^{-4}, 4\times10^{-4},\ 10^{-4},\ 10^{-4}, 8\times10^{-5})^T. \qquad (9)$$

The second set of the diffusion parameter vector is

$$(\gamma_1, \gamma_2,\ \gamma_3, \gamma_4, \gamma_5, \gamma_6)^T = (4\times10^{-3},\ 2\times10^{-3}, 4\times10^{-3},\ 10^{-3},\ 10^{-3}, 8\times10^{-4})^T. \qquad (10)$$

The first set of diffusion parameters, equation (9), denotes 'lower' intensity of noise influence that corresponds to the dash-dash line (--) trajectories of the states. On the other hand, the second set of diffusion parameters, equation (10), denotes the relatively 'larger' intensity of noise influence. That corresponds to the dotted line (…) trajectories of the vehicle states, see figures (3)-(8). Note that the *solid line (-)* denotes the vehicle state trajectories resulting from the *noise-free* vehicle dynamics. The differences between the noise-free and noise-influenced trajectories are attributed to the vector stochastic term $b(\xi_t, t)dB_t$ associated with vehicle dynamics, see equation (3) of the paper.

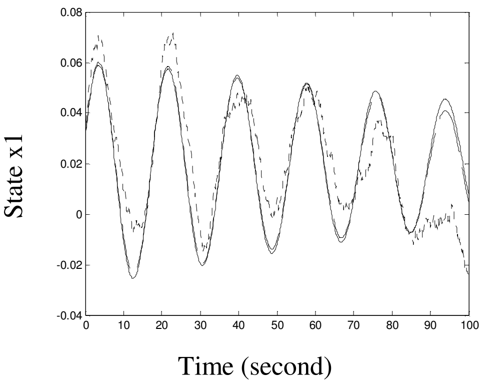

Figure 3: a comparison between three trajectories

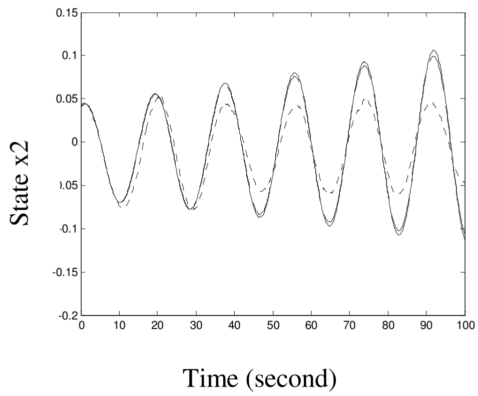

Figure 4: a comparison between three trajectories

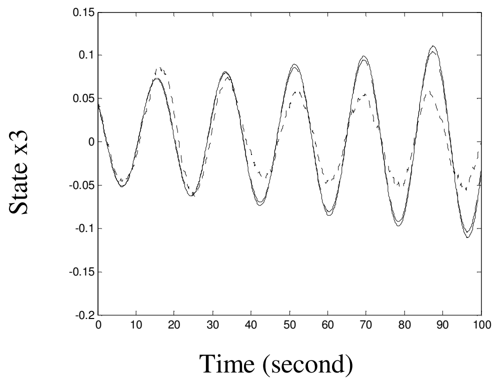

Figure 5: a comparison between three trajectories

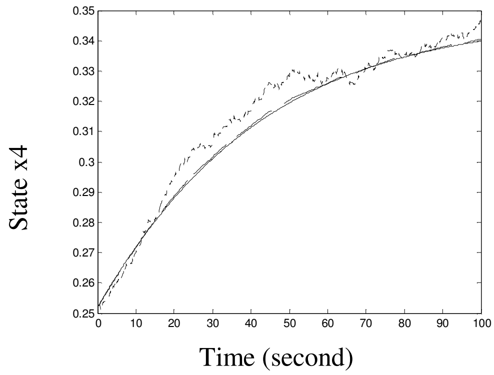

Figure 6: a comparison between three trajectories

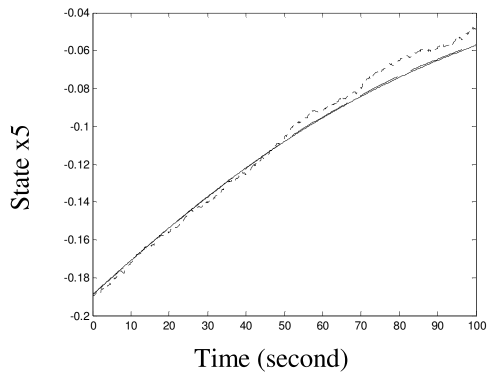

Figure 7: a comparison between trajectories

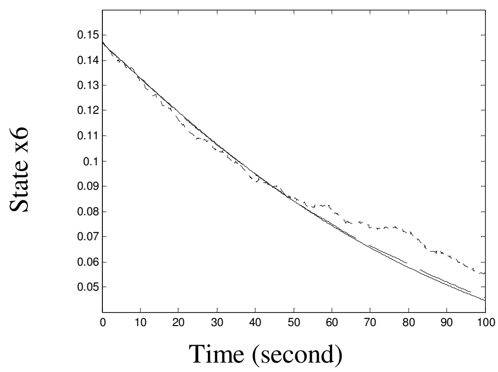

Figure 8: a comparison between trajectories

Importantly, figures (1)-(8) suggest that SDE descriptions are more accurate to describe the vehicle dynamics under noise influence that is attributed to the stochastic character of ocean current circulations. Note that the *dotted line (…)* denotes the numerical simulations of the underwater vehicle SDE under *greater* noise influence. As a result of this, this paper develops the estimation theory of noise-influenced underwater vehicle dynamics, see equations (6a)-(6b) of the paper. The variances of the linear velocity vector $(P_{11}, P_{22}, P_{33})^T$ are illustrated in tables (1)-(3). Table (1) denotes the variances of the state x_1 under two different sets of diffusion parameter vectors, see equations (9)-(10). Tables (2) and (3) denote the variances of the second and third states respectively. The variance tables denote the state variances, which take non-negative, bounded, finite values of the order less than 10^{-3}. That demonstrate the efficacy of the vehicle estimation equations of the paper.

Time(second)	P_{11} with the first set of diffusion parameters of equation (9)	P_{11} with the second set of diffusion parameters of equation (10)
0	0	0
25	0.393×10^{-5}	0.4×10^{-3}
50	0.761×10^{-5}	0.8×10^{-3}
75	1.039×10^{-5}	1.0×10^{-3}
100	1.03×10^{-5}	10^{-3}

Table 1: the variance P_{11} of the state x_1

Time(second)	P_{22} with the first set of diffusion parameters of equation (9)	P_{22} with the second set of diffusion parameters of equation (10)
0	0	0
25	0.225×10^{-5}	0.2×10^{-3}
50	0.487×10^{-5}	0.5×10^{-3}
75	0.790×10^{-5}	0.8×10^{-3}
100	0.8×10^{-5}	0.7×10^{-3}

Table2: the variance P_{22} of the state x_2

Time(second)	P_{33} with the first set of diffusion parameters of equation (9)	P_{33} with the first set of diffusion parameters of equation (10)
0	0	0
25	0.219×10^{-5}	0.2×10^{-3}
50	0.530×10^{-5}	0.5×10^{-3}
75	1.572×10^{-5}	1.5×10^{-3}
100	1.5×10^{-5}	2×10^{-3}

Table3: the variance P_{33} of the state x_3

5. A STOCHASTIC METHODS FOR UNDERWATER VEHICLE DYNAMICS

In this paper, we develop the vehicle dynamics under stochastic influence using a formal stochastic setting, i.e. the Itô setting. The Itô setting is a standard and celebrated stochastic method in the theory of non-linear stochastic control. The noise analysis of this paper, formalized and stated in equations (6a)-(6b), is useful for valueless observations. The notion of valueless observations arises when observations are completely masked by noise. After accounting observations in the noise analysis of dynamical systems, estimation equations become filtering equations. This paper attempts the first part, noise analysis of dynamical systems with value-less observations. Under the value-less observations situation, the noise analysis is formalized as the prediction problem.

The second part, noise analysis of vehicle dynamics accounting observations, warrants further research. That can be accomplished by considering the sensor noise in the observation equation.

6. CONCLUSION

The main contribution of this paper is to develop the stochastically influenced underwater vehicle dynamics using the Itô SDE, see equation (3) of the paper. Notably, the vehicle Itô SDE accounts stochastic correction terms attributed to the ocean current circulations. Another contribution of this paper is to accomplish the noise analysis of the stochastically influenced underwater vehicle. Equations (6a)-(6b) will be remarkably useful for estimation theory of underwater vehicles for the case in which 'observations are valueless'.

This paper has demonstrated first time the application of the Itô theory to underwater vehicle dynamics. This paper will have influence on further research, since the Itô stochastic description of the underwater vehicle dynamics was the subject of investigation.

This paper will be of interest to naval dynamists aspiring for systems and control methods in stochastic framework for underwater vehicles. In this paper, the Authors have attempted to list and cite recent, seminal papers and celebrated books on the related topics, i.e. underwater vehicle dynamics, stochastic processes and the Fokker-Planck equation. However, we do not claim the completeness of the references. This paper can be treated as a research paper in lieu of bibliographic compilations.

REFERENCES

Aguiar, A. P., and Pascoal, A. M. (2007). Dynamics positioning and way-point tracking of underactuated AUVs in the presence of ocean currents, *International Journal of Control*, vol. 80, pp. 1092-1108.

Aguiar, A. P., and Hespanha, J. P. (2003). Position tracking of underactuated vehicles, *Proceedings of American Control Conference*, vol. 3, pp. 1988-1993.

Alonge, F., D'lppolito, F., and Ramimondi, F.M. (2001). Trajectory tracking of underactuated underwater vehicles, *Proceedings of the 40th IEEE Conference on Decision and Control*.

Bierbaum, M. M., Joseph, R. I., Fry, R. L., and Nelson, J. B. (2002). A Fokker-Planck model for a two-body problem, *Proceedings of AIP Conference,* vol. 617, pp. 340-371.

Bovio, E., Cecchi, D., and Baralli., F. (2006). Autonomous underwater vehicles for scientific and naval operations, *Annual Reviews in Control*, vol. 30, pp. 117-130.

Chow, S. N., and Zhou, H. M. (2007). An analysis of phase noise and Fokker-Planck equations, *Journal of Differential Equations*, vol. 234, pp. 391–411.

Fossen, T. I. (1994). *Guidance and Control of Ocean Vehicles*, John Wiley and Sons, England.

Jazwinski, A. H. (1970). *Stochastic Processes and Filtering Theory,* Academic Press, New York and London.

Karatzas, I., and Shreve, S. E. (1988). *Brownian Motion and Stochastic Calculus,* Springer, New York.

Kuo, H. H. (2005). White noise stochastic integration, stochastic analysis: classical and quantum, *E Proceedings, World Scientific*, pp. 57-71.

Mumford, D. (2000). The dawning of the age of stochasticity in *Mathematics: Frontiers and Perspectives*, V. Arnold, M. Atiyah, P. Lax and B. Mazur, Ed. *American Mathematical Society*, pp. 197-218.

Risken, H. (1984). *The Fokker-Planck Equation: Methods of Solution and Applications,* Springer-Verlag, Berlin.

Sharma, S. N., and Parthasarathy, H. (2007). Dynamics of a stochastically perturbed two-body problem, *Pro. R. Soc. A, London, The Royal Society*, vol. 463, pp. 979–1003.

Sharma, S. N. (2008). A Kolmogorov-Fokker-Planck approach for a stochastic Duffing-van der Pol system, *Differential Equations and Dynamical Systems (An International Journal for Theory, Application, Computer Simulations)*, vol. 16, no. 4, pp. 351–377.

Singh, S., Sanyal, A., Smith, R., and Nordkvist, N. (2009). Robust tracking control of autonomous underwater vehicals in the presence of disturbance inputs, *Proceedings of the ASME 28th International Conference on Ocean, Offshore and Arctic Engineering*.

Smith, R.N. (2008). Geometric control theory and its application to underwater vehicles, *Ph.D. Thesis, Univ. of Hawaii.*

APPENDIX

Here, we explain briefly the Kolmogorov-Fokker-Planck equation and the Kolmogorov backward equation for Markov processes. Subsequently, we derive the Kolmogorov backward operator for the stochastic autonomous system of this paper. The Fokker-Planck operator $L(.)$ satisfies the following condition:

$$\langle \phi, Lp \rangle = \langle L'\phi, p \rangle, \tag{A.1}$$

where the notation $\langle \, , \rangle$ denotes the inner product, the term ϕ is a scalar non-linear and twice continuously differentiable function. The term p denotes the conditional probability density. Importantly, the operator L' denotes the Kolmogorov backward operator. The structure of the Kolmogorov backward operator is

$$L'(\cdot) = \sum_i a_i(\xi, t)\frac{\partial(\cdot)}{\partial \xi_i} + \frac{1}{2}\sum_{i,j}(bb^T)_{ij}(\xi, t)\frac{\partial^2(\cdot)}{\partial \xi_i \partial \xi_j}. \tag{A.2}$$

The Kolmogorov backward operator has found its applications in 'estimation theory' and 'stability' of Itô SDEs. Making the use of equation (4) in combination with equation (A.1), we arrive at the Kolmogorov backward operator for the stochastic underwater vehicle, i.e

$$\begin{aligned}
&(\frac{X}{m} + x_2 x_6 - x_3 x_5)\frac{\partial(.)}{\partial x_1} + (\frac{Y}{m} + x_3 x_4 - x_1 x_6)\frac{\partial(.)}{\partial x_2} \\
&+ (\frac{Z}{m} + x_1 x_5 - x_2 x_4)\frac{\partial(.)}{\partial x_3} + (\frac{K}{I_x} + \frac{(I_y - I_z)}{I_x} x_5 x_6)\frac{\partial(.)}{\partial x_4} \\
&+ (\frac{M}{I_y} + \frac{(I_z - I_x)}{I_y} x_4 x_6)\frac{\partial(.)}{\partial x_5} + (\frac{N}{I_z} + \frac{(I_x - I_y)}{I_z} x_4 x_5)\frac{\partial(.)}{\partial x_6} \\
&+ \frac{1}{2}(\gamma_1^2 \frac{\partial^2(.)}{\partial x_1^2} + \gamma_2^2 \frac{\partial^2(.)}{\partial x_2^2} + \gamma_3^2 \frac{\partial^2(.)}{\partial x_3^2} + \gamma_4^2 \frac{\partial^2(.)}{\partial x_4^2} \\
&+ \gamma_5^2 \frac{\partial^2(.)}{\partial x_5^2} + \gamma_6^2 \frac{\partial^2(.)}{\partial x_6^2}) + \gamma_1\gamma_2 \frac{\partial^2(.)}{\partial x_1 \partial x_2} + \gamma_1\gamma_3 \frac{\partial^2(.)}{\partial x_1 \partial x_3} \\
&+ \gamma_1\gamma_4 \frac{\partial^2(.)}{\partial x_1 \partial x_4} + \gamma_1\gamma_5 \frac{\partial^2(.)}{\partial x_1 \partial x_5} + \gamma_1\gamma_6 \frac{\partial^2(.)}{\partial x_1 \partial x_6} + \gamma_2\gamma_3 \frac{\partial^2(.)}{\partial x_2 \partial x_3} \\
&+ \gamma_2\gamma_4 \frac{\partial^2(.)}{\partial x_2 \partial x_4} + \gamma_2\gamma_5 \frac{\partial^2(.)}{\partial x_2 \partial x_5} \\
&+ \gamma_2\gamma_6 \frac{\partial^2(.)}{\partial x_2 \partial x_6} + \gamma_3\gamma_4 \frac{\partial^2(.)}{\partial x_3 \partial x_4} \\
&+ \gamma_3\gamma_5 \frac{\partial^2(.)}{\partial x_3 \partial x_5} + \gamma_3\gamma_6 \frac{\partial^2(.)}{\partial x_3 \partial x_6} + \gamma_4\gamma_5 \frac{\partial^2(.)}{\partial x_4 \partial x_5} + \gamma_4\gamma_6 \frac{\partial^2(.)}{\partial x_4 \partial x_6} \\
&+ \gamma_5\gamma_6 \frac{\partial^2(.)}{\partial x_5 \partial x_6}.
\end{aligned} \tag{A.3}$$

Equation (A.3) is a specific case of equation (A.2) of the paper. Equation (A.3) for the underwater vehicle dynamics will be useful to derive the estimation equations alternatively, equations (6a)-(6b). Equation (A.3) will be useful to derive stability conditions of the underwater Itô SDE as well.