Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Architectural issues in the control of tall multiple-input multiple-output plants

Pedro Albertos * **Mario E. Salgado** **

* Department of System Engineering and Automatica, Polytechnic University of Valencia, Spain, (e-mail: pedro@aii.upv.es)
** Department of Electronic Engineering, Universidad Técnica Federico Santa María, Valparaíso, Chile (e-mail: mario.salgado@usm.cl)

Abstract: This paper deals with the problem of controlling linear discrete-time tall multiple-input multiple-output plants, using a cascade philosophy approach. The main idea is to use a cascade architecture to regulate a subset of the plant outputs while keeping the rest of them bounded and, additionally, aiming at achieving a satisfactory disturbance compensation and a appreciable degree of modelling error robustness. These ideas assume that the two subsets of outputs have been assigned different importance for the control designer.

1. INTRODUCTION

Most, if not all, industrial plants can be modelled as multiple-input multiple-output (MIMO) systems. When the number of manipulable input signals is at least equal to the number of plant variables to be controlled, there are several control design approaches which can be successfully used (Maciejowski [1989], Goodwin et al. [2001], Albertos and Sala [2004], Skogestad and Postlethwaite [1996]). That is not the case for tall plants. Tall MIMO plants are those systems where the number of manipulable inputs is smaller than the number of plant variables of interest. This feature poses a fundamental limitation in the control of these plants: it is not possible to drive all those plant outputs to track arbitrary references and/or to fully compensate arbitrary disturbances. From a different perspective, we could expect this limitation, since it is impossible to build an inverse for the plant model, which is a paradigm in control design Goodwin et al. [2001] Some examples of tall systems include distillation columns Treiber [1984], dams Litrico [2002], magnetic bearing systems Morse et al. [1998], chemical reactors Munro [1990], etc. Tall systems also arise when distributed systems are approximated by finite dimensional ones (see, e.g., Moheimani et al. [2003]).

It is well known that, regarding step references, perfect tracking is only possible if the reference vector lies in the space generated by the columns of the matrix modelling the plant dc gain. Several authors have researched this central aspect in tall MIMO control (see e.g. Chen et al. [2002] and Freudenberg and Middleton [1998]). One possible way to deal with this structural constraint is described in García [2011] where stationary control errors in all channels are accepted. In this paper, our contribution follows a different approach: we consider the control of a plant with p outputs and m inputs ($p > m$), and we focus on a twofold objective, namely

- Regulating m outputs, say $y_1[k] \in \mathbb{R}^m$, such that they can be driven to prescribed set-points with zero steady state errors
- Maintaining the remaining $p - m$ outputs, $y_2[k] \in$ within acceptable values, even when the transfer function matrix from the inputs to these outputs is unstable.

A fundamental observation is that, by prioritizing the regulation of $y_1[k]$, it becomes useless to specify a reference for $y_2[k]$, given that the main objective will force a particular control signal $u[k]$, which in turn will define $y_2[k]$, at least in steady state. On the other hand, these constraints do not preclude the improvement in the transient behaviour of $y_2[k]$, at the expense of the transient performance of $y_1[k]$. This difference in performance specifications makes sense when the control designer assigns different hierarchies to the two plant output subsets.

In this work, we address the above fundamental issues related to the proposed architecture. The controller design itself is only briefly addressed without going into the intricacies of that subject .

2. BASIC SETTING

We consider a discrete-time linear MIMO tall plant with a strictly proper transfer function matrix $\mathbf{G}[\mathbf{z}] \in \mathbb{C}^{p\times m}$, $p > m$, where

$$Y[z] = \begin{bmatrix} Y_1[z] \\ Y_2[z] \end{bmatrix} = \mathbf{G}[\mathbf{z}]U[z]; \tag{1}$$

$$\text{with } \mathbf{G}[\mathbf{z}] = \begin{bmatrix} \mathbf{G_1}[\mathbf{z}] \\ \mathbf{G_2}[\mathbf{z}] \end{bmatrix} = \begin{bmatrix} \mathbf{G_b}[\mathbf{z}] \\ \mathbf{G_c}[\mathbf{z}] \end{bmatrix} \mathbf{G_a}[\mathbf{z}] \tag{2}$$

where $Y_1[z] \in \mathbb{C}^m$, $Y_2[z] \in \mathbb{C}^{p-m}$, $U[z] \in \mathbb{C}^m$, $\mathbf{G_1}[\mathbf{z}] \in \mathbb{C}^{m\times m}$, $\mathbf{G_2}[\mathbf{z}] \in \mathbb{C}^{(p-m)\times m}$, $\mathbf{G_a}[\mathbf{z}] \in \mathbb{C}^{m\times m}$, $\mathbf{G_b}[\mathbf{z}] \in \mathbb{C}^{m\times m}$, $\mathbf{G_c}[\mathbf{z}] \in \mathbb{C}^{(p-m)\times m}$

To attain the goals defined above we propose to use the cascade architecture shown in Figure 1. In that structure,

* This work was supported by CONICYT grants 1130700 and ACT53, and by MECESUP FSM1204

we assume that we want to drive the vector output $y_1[k]$ to track step references, as specified by $r_1[k]$; also, we aim to keep the vector output $y_2[k]$ bounded. It is straightforward to see that these two goals are achieved if the whole loop is internally stable with $\mathbf{C_1}[\mathbf{z}]$ having integral action in all channels. Although no disturbance has been explicitly considered, it is known that the presence of integration in $\mathbf{C_1}[\mathbf{z}]$ suffices to completely compensate the impact of disturbances on $y_1[k]$, when those disturbances tend to be constant as $k \to \infty$.

Bearing in mind that Youla parametrization of all stabilizing controllers will be used in the examples to carry out the control synthesis, we define the following quantities and relationships

$$\mathbf{Q_2}[\mathbf{z}] = (\mathbf{I} + \mathbf{C_2}[\mathbf{z}]\mathbf{G_2}[\mathbf{z}])^{-1}\mathbf{C_2}[\mathbf{z}] \tag{3}$$
$$= \mathbf{C_2}[\mathbf{z}](\mathbf{I} + \mathbf{G_2}[\mathbf{z}]\mathbf{C_2}[\mathbf{z}])^{-1} \tag{4}$$
$$\mathbf{C_2}[\mathbf{z}] = \mathbf{Q_2}[\mathbf{z}](\mathbf{I} - \mathbf{G_2}[\mathbf{z}]\mathbf{C_2}[\mathbf{z}])^{-1} \tag{5}$$
$$\mathbf{G_e}[\mathbf{z}] = \mathbf{G_1}[\mathbf{z}](\mathbf{I} - \mathbf{Q_2}[\mathbf{z}]\mathbf{G_2}[\mathbf{z}]) \tag{6}$$
$$\mathbf{Q_1}[\mathbf{z}] = (\mathbf{I} + \mathbf{C_1}[\mathbf{z}]\mathbf{G_e}[\mathbf{z}])^{-1}\mathbf{C_1}[\mathbf{z}] \tag{7}$$
$$\mathbf{C_1}[\mathbf{z}] = \mathbf{Q_1}[\mathbf{z}](\mathbf{I} - \mathbf{G_e}[\mathbf{z}]\mathbf{Q_1}[\mathbf{z}])^{-1} \tag{8}$$

leading to

$$\tilde{U}[z] = \mathbf{Q_1}[\mathbf{z}]R_1[z] \tag{9}$$
$$U[z] = (\mathbf{I} - \mathbf{Q_2}[\mathbf{z}]\mathbf{G_2}[\mathbf{z}])\tilde{U}[z] \tag{10}$$
$$= (\mathbf{I} - \mathbf{Q_2}[\mathbf{z}]\mathbf{G_2}[\mathbf{z}])\mathbf{Q_1}[\mathbf{z}]R_1[z] \tag{11}$$
$$Y_2[z] = (\mathbf{I} - \mathbf{G_2}[\mathbf{z}]\mathbf{Q_2}[\mathbf{z}])\mathbf{G_2}[\mathbf{z}]\tilde{U}[z] \tag{12}$$
$$Y_1[z] = \mathbf{G_e}[\mathbf{z}]\mathbf{Q_1}[\mathbf{z}]R_1[z] \tag{13}$$
$$= \mathbf{G_1}[\mathbf{z}](\mathbf{I} - \mathbf{Q_2}[\mathbf{z}]\mathbf{G_2}[\mathbf{z}])\mathbf{Q_1}[\mathbf{z}]R_1[z] \tag{14}$$

The above relations are specially useful for a stable MIMO plant. In this case, to achieve internal stability it is necessary and sufficient to choose the Youla parameters $\mathbf{Q_1}[\mathbf{z}]$ and $\mathbf{Q_2}[\mathbf{z}]$ stable. The unstable plant case will be dealt with in a later section.

In the expressions above the secondary or inner loop has the main function of keeping the output $y_2[k]$ bounded. Thus, a first logical design step is to choose $\mathbf{Q_2}[\mathbf{z}]$, which allows to compute the equivalent plant $\mathbf{G_e}[\mathbf{z}]$, to choose $\mathbf{Q_1}[\mathbf{z}]$. Since we are aiming to achieve zero steady state errors for the regulation of $y_1[k]$ when the references are step signals we parametrize $\mathbf{Q_1}[\mathbf{z}]$ as

$$\mathbf{Q_1}[\mathbf{z}] = (1 - z^{-1})\tilde{\mathbf{Q}}_\mathbf{1}[\mathbf{z}] + (\mathbf{G_e}[\mathbf{1}])^{-1} \tag{15}$$

This construction ensures perfect inversion at frequency zero.

These ideas are next illustrated with an example.

Example 1. (A 3×2 plant). Assume a 3×2 ($p = 3$, $m = 2$ plant with

$$G[z] = \frac{\begin{bmatrix} z(z-0.8) & (z-0.5)(z-0.8) \\ 0.5(z-0.5)(z-0.8) & 0.2z(z-0.5) \\ 0.5z(z-0.8) & 0.1z(z-0.5) \end{bmatrix}}{z(z-0.5)(z-0.8)} \tag{16}$$

It is also assumed that the outputs to be controlled are the first two, that is $y[k] = [y_1[k] \quad y_2[k]]^T$, where $y_1[k] \in \mathbb{R}^2$ and $y_2[k] \in \mathbb{R}$, with

$$y_1[k] = [y_{11}[k] \quad y_{12}[k]]^T \tag{17}$$

With the above choice we have that

$$G_1[z] = \frac{\begin{bmatrix} z(z-0.8) & (z-0.5)(z-0.8) \\ 0.5(z-0.5)(z-0.8) & 0.2z(z-0.5) \end{bmatrix}}{z(z-0.5)(z-0.8)}; \tag{18}$$

$$G_2[z] = \frac{[0.5z(z-0.8) \quad 0.1z(z-0.5)]}{z(z-0.5)(z-0.8)} \tag{19}$$

Say we choose

$$\mathbf{Q_2}[\mathbf{z}] = \begin{bmatrix} \dfrac{z-0.5}{2(z-0.1)} & \dfrac{2.5(z-0.8)}{z-0.1} \end{bmatrix}^T \tag{20}$$

With this choice,

$$\mathbf{I} - \mathbf{G_2}[\mathbf{z}]\mathbf{Q_2}[\mathbf{z}] = \frac{z-0.6}{z-0.1} \tag{21}$$

and the equivalent plant is

$$\mathbf{G_e}[\mathbf{z}] = \begin{bmatrix} \dfrac{z-0.6}{(z-0.5)(z-0.1)} & \dfrac{z-0.6}{z(z-0.1)} \\ \dfrac{0.5(z-0.6)}{z(z-0.1)} & \dfrac{0.2(z-0.6)}{(z-0.8)(z-0.1)} \end{bmatrix} \tag{22}$$

Therefore, the Youla parameter which guarantees integral action is

$$\mathbf{Q_1}[\mathbf{z}] = (1 - z^{-1})\tilde{\mathbf{Q}}_\mathbf{1}[\mathbf{z}] + \mathbf{G_e}[\mathbf{1}]^{-1} \tag{23}$$

For simplicity, we choose $\tilde{\mathbf{Q}}_\mathbf{1}[\mathbf{z}] = I$, and a reference given by

$$r[k] = [\mu[k-1] \quad -2\mu[k-25]]^T \tag{24}$$

where $\mu[k - k_o]$ is a unit step at time $k = k_o$ The results for that reference are shown in Figure 2.

We appreciate that the tracking error for $y_1[k]$ tends to zero, leading to a steady state value for $y_2[k]$ which depends, not on $r_2[k]$, but on the reference $r_1[k]$ and the d.c. gain of G_2.

3. IMPACT OF DISTURBANCES

Given that $G_2[z]$ is a stable transfer matrix, one might think that it would be acceptable to just focus on the control design to regulate $y_1[k]$, leaving $G_2[z]$ in open loop; however, there are several reasons why the proposed architecture exhibits potentially better performance in a more realistic setting. In this section we consider the presence of an output disturbance $d_2[k] \in \mathbb{R}^{p-m}$, as shown in Figure 3.

We can quantify the impact of this disturbance on $y_2[k]$, after some analytic work we obtain

$$Y_2[z]|_{d_2} = (\mathbf{I} - \mathbf{G_2}[\mathbf{z}](\mathbf{I} - \mathbf{G_e}[\mathbf{z}]\mathbf{Q_1}[\mathbf{z}])\mathbf{Q_2}[\mathbf{z}])\, D_2[z] \tag{25}$$

from where it can be appreciated that, through a sensible choice of the design parameters $\mathbf{Q_1}[\mathbf{z}]$ and $\mathbf{Q_2}[\mathbf{z}]$, the impact of the disturbance on $y_2[k]$ can be damped.

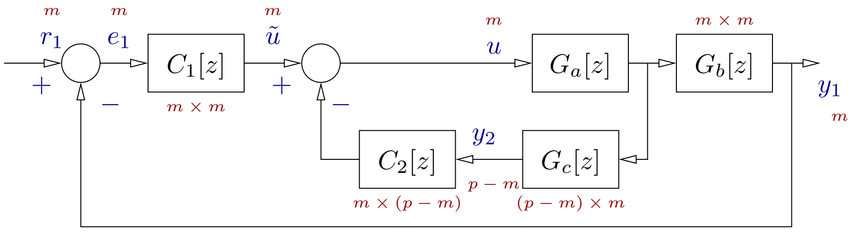

Fig. 1. Undisturbed cascade architecture, with indication of the dimensions of every transfer function.

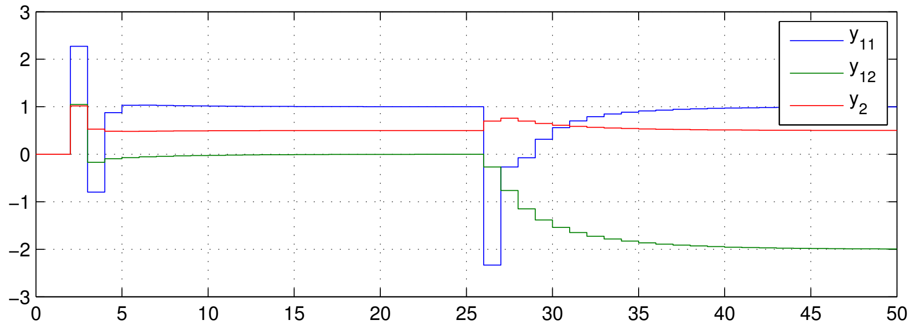

Fig. 2. Plant outputs under control (Example 1)

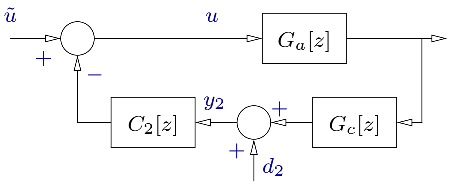

Fig. 3. Section of the loop, showing a disturbance at the output of $G_2[z]$.

To appreciate this feature, we consider a simple single-input two-ouput tall plant. Say that

$$\mathbf{G[z]} = \begin{bmatrix} G_1[z] \\ G_2[z] \end{bmatrix} \quad G_1[z] = \frac{0.3}{z-0.8}; \quad G_2[z] = \frac{0.5}{z-0.5} \tag{26}$$

To isolate the performance regarding disturbance compensation, we assume that $r_1[k] = 0 \ \forall k \geq 0$ and that the disturbance $d_2[k]$ is a sequence of two steps, namely

$$d_2[k] = \mu[k-1] - 2\mu[k-20] \tag{27}$$

If G_2 were in open loop, we can anticipate that $y_2[k] = d_2[k]$ for all $k \geq 0$. On the other hand, if G_2 is in closed loop as in Figure 3, we know that $y_2[k]$ will tend to track the disturbance steps (since the control input $u[k]$ will tend to zero, given that C_1 has integration). However, a sensible choice of the controllers may yield an improved transient behaviour in $y_2[k]$, as illustrated in this example.

To synthesize the controllers, we again use the Youla parametrization. We first choose

$$Q_2[z] = 1.6\frac{z-0.5}{z} \Longrightarrow G_e[z] = \frac{0.3}{z} \tag{28}$$

From where we choose a $Q_1[z]$ satisfying $Q_1[1] = G_e[1]^{-1} = 10/3$, to force integration in C_1. For instance

$$Q_1[z] = \frac{4z}{3(z-0.6)} \tag{29}$$

We next simulate the control loop, from where we obtain the results shown in Figure 4.

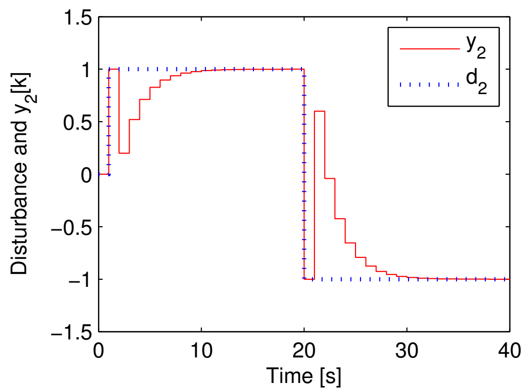

Fig. 4. Compensation of an output disturbance in $y_2[k]$

In this figure we firstly note two unavoidable features of the disturbance response:

- The plant output $y_2[k]$ reproduces the initial steps in the disturbance (at $k = 1$ and at $k = 20$). This behaviour is due to the fact that we are dealing with an output disturbance.
- The ouput $y_2[k]$ tends to the steps stationary values; this behaviour can be observed in the intervals $k \in (10, 20)$ and $k > 30$. This is due to the fact that the reference for the output $y_1[k]$ is zero, driving the stationary value of control signal $u[k]$ to zero.

In spite of these two features, we see that the control loop provides an appreciable damping of the disturbance.

4. ROBUTSNESS ISSUES

As in the previous section one might wonder whether it would be simpler to leave (a stable) $G_2[z]$ in open loop, even if $\mathbf{G_2}$ includes a significant modelling error. To provide an answer, we show next a way to quantify the impact of that error in the configuration shown in Figure 1. Assume first that a calibration model for the second part of the plant is $\mathbf{G_{2T}}[\mathbf{z}]$, such that

$$\mathbf{G_{2T}}[\mathbf{z}] = (\mathbf{I} + \mathbf{G_{\Delta \ell}}[\mathbf{z}])\mathbf{G_2}[\mathbf{z}] \tag{30}$$

where $\mathbf{G_{\Delta \ell}}[\mathbf{z}]$ is the (left) multiplicative modelling error (Goodwin et al. [2001]). Then the achievable transfer function from $\tilde{U}[z]$ to $U[z]$, originally given by (10), is now given by

$$U[z] = (\mathbf{I} - \mathbf{Q_2}[\mathbf{z}]\mathbf{G_2}[\mathbf{z}])(\mathbf{I} + \mathbf{G_{\Delta \ell}}[\mathbf{z}]\mathbf{G_2}[\mathbf{z}]\mathbf{Q_2}[\mathbf{z}])^{-1}\tilde{U}[z] \tag{31}$$

It then becomes evident that the impact of the modelling error is negligible if $\mathbf{Q_2}[\mathbf{z}]$ is chosen in such a way that the product $\mathbf{G_{\Delta \ell}}[\mathbf{z}]\mathbf{G_2}[\mathbf{z}]\mathbf{Q_2}[\mathbf{z}]$ is small for all ω. More specifically, it will suffice to impose the constraint.

$$\overline{\sigma}\left\{\mathbf{G_{\Delta \ell}}[\mathbf{e^{j\omega}}]\mathbf{G_2}[\mathbf{e^{j\omega}}]\mathbf{Q_2}[\mathbf{e^{j\omega}}]\right\} << 1 \qquad \forall \omega \in [-\pi, \pi] \tag{32}$$

where $\overline{\sigma}\{\circ\}$ denotes the maximum singular value of the matrix. Since usually the multiplicative modelling error is small at high frequencies, that requirement can be satisfied by imposing an upper bound for the secondary loop bandwidth.

The satisfaction of constraint (32) will also make $\mathbf{G_e}$, given in (6) , fairly insensitive to the considered modelling error.

5. UNSTABLE $\mathbf{G_2}$

A second fundamental reason to have $\mathbf{G_2}[\mathbf{z}]$ in closed loop appears when $\mathbf{G_2}[\mathbf{z}]$ is unstable. Then, the synthesis of the controller $\mathbf{C_2}[\mathbf{z}]$, using the Youla parametrization follows a sequence of steps which can be summarized as follows (Goodwin et al. [2001]).

(1) Express $\mathbf{G_2}[\mathbf{z}]$ using a right matrix fraction description (RMFD), i.e.

$$\mathbf{G_2}[\mathbf{z}] = \mathbf{G_{2N}}[\mathbf{z}]\mathbf{G_{2D}}[\mathbf{z}]^{-1} \tag{33}$$

(2) Find any stabilizing controller $\mathbf{C_o}[\mathbf{z}]$ and express it in RMFD as

$$\mathbf{C_o}[\mathbf{z}] = \mathbf{C_{oN}}[\mathbf{z}]\mathbf{C_{oD}}[\mathbf{z}]^{-1} \tag{34}$$

(3) Then, all stabilizing controllers $\mathbf{C_2}[\mathbf{z}]$, can be expressed in RMFD as

$$\mathbf{C_2}[\mathbf{z}] = (\mathbf{C_{oN}}[\mathbf{z}] + \mathbf{G_{2D}}[\mathbf{z}]\Omega[z])(\mathbf{C_{oD}}[\mathbf{z}] - \mathbf{G_{2N}}[\mathbf{z}]\Omega[z])^{-1} \tag{35}$$

In fact, the stability of the matrix $\Omega[z]$ is a necessary and sufficient condition for $\mathbf{C_2}[\mathbf{z}]$, given by (35), to stabilize $\mathbf{G_2}[\mathbf{z}]$.

With this controller, the equivalent plant $\mathbf{G_e}[\mathbf{z}]$ is stable, demanding then, the synthesis of an stable Youla parameter $\mathbf{Q_1}[\mathbf{z}]$. Naturally, a similar approach should be followed if $\mathbf{G_1}[\mathbf{z}]$ is also unstable (leading to an unstable $\mathbf{G_e}[\mathbf{z}]$.

Example 2. Consider the same plant as in Example 1, except for the fact that now, $\mathbf{G_2}[\mathbf{z}]$ is unstable, and given by

$$\mathbf{G_2}[\mathbf{z}] = \begin{bmatrix} \dfrac{0.5}{z-1.5} & \dfrac{0.1}{z-0.8} \end{bmatrix} \tag{36}$$

with, for example, the RMFD given by

$$\mathbf{G_{2N}}[\mathbf{z}] = \begin{bmatrix} \dfrac{0.5(z-0.8)}{z^2} & \dfrac{0.1(z-0.5)}{z^2} \end{bmatrix} \tag{37}$$

$$\mathbf{G_{2D}}[\mathbf{z}] = \frac{(z-0.5)(z-0.8)}{z^2} \tag{38}$$

We can verify that $\mathbf{C_o}[z] = [3\ \ 2]^T$ stabilizes $\mathbf{G_2}[\mathbf{z}]$ with a RMFD given by

$$\mathbf{C_{oN}}[z] = [3\ \ 2]^T; \qquad \mathbf{C_{oD}}[z] = 1 \tag{39}$$

Then, using (35), all stabilizing controllers $\mathbf{C_2}[\mathbf{z}]$ can be expressed as a bilinear function of a stable and proper $\Omega[z] \in C^2$. This parameter, as well as $\mathbf{Q_1}[\mathbf{z}]$, should be chosen to satisfy the particular specifications of a given problem.

6. CONCLUSION

A cascade based architecture has been proposed to deal with the control of tall MIMO plants. A key assumption is that the plant outputs can be organized in two subsets, one of them to be regulated, and the second one to be kept within reasonable boundaries. In essence, a sequential design is called for. A fundamental fact is that to achieve a good transient behaviour in the second subset, one must sacrifice the performance in the control of the first subset. As a compensation, we have achieved zero steady state errors in the regulation of that first subsets of plant outputs. Future work should include the formal proposal of a design strategy, as well as the usage of an interaction measure to simplify the procedures arising from that strategy.

REFERENCES

P. Albertos and A. Sala. *Multivariable Control Systems: An Engineering Approach.* Springer, 2004.

G. Chen, J. Chen, and R. Middleton. Optimal tracking performance for SIMO systems. *IEEE Transactions on Automatic Control*, 47(10):1770–1775, 2002.

J. Freudenberg and R. Middleton. Properties of single input, two output feedback systems. *Proceedings of the 1998 American Control Conference*, 4:2055–2060, June 1998. Philadelphia, PA.

M. García. Límites de desempeño y diseño de controladores para sistemas multivariables estables con ms salidas que entradas. Master's thesis, Departamento de Electrónica, Universidad Técnica Federico Santa María, 2011.

G. Goodwin, S. Graebe, and M. Salgado. *Control System Design.* Prentice Hall, 2001.

X. Litrico. Robust IMC flow control of SIMO dam-river open-channel systems. *IEEE Transactions on Control Systems Technology*, 10(3):432–437, 2002.

J. M. Maciejowski. *Multivariable feedback design.* Addison Wesley, 1989.

S.O.R. Moheimani, D. Halim, and A.J. Fleming. *Spatial control of vibration: theory and experiments.* World Scientific Pub Co Inc, 2003.

N. Morse, R. Smith, B. Paden, and J. Antaki. Position sensed and self-sensing magnetic bearing configurations and associated robustness limitations. *Proceedings of the 37th IEEE Conference on Decision and Control*, December 1998. Tampa, FL, USA.

N. Munro. Multivariable control applications: turbine and chemical plant examples. *IEE Colloquium on Successful Industrial Applications of Multivariable Analysis*, pages 1–4, February 1990. London, UK.

S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design.* Wiley-Interscience, 1st edition, 1996. ISBN 0470011688.

S. Treiber. Multivariable control of non-square systems. *Industrial & Engineering Chemistry Process Design and Development*, 23(4):854–857, 1984.