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Disturbance decoupling by measurement feedback

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Abstract: The paper addresses the disturbance decoupling problem for MIMO discrete-time nonlinear systems. A sufficient conditions are derived to solve the problem by dynamic measurement feedback, i.e. the feedback that depends on measurable outputs only. The solution to the disturbance decoupling problem, described in this paper, is based on the input-output linearization, which is used to linearize certain functions. Two examples are added to illustrate the results.

1. INTRODUCTION

The disturbance decoupling problem (DDP) is one of the fundamental problems in control theory. There are a lot of papers, that solve the problem by state feedback, see Aranda-Bricaire and Kotta [2001, 2004], Fliegner and Nijmeijer [1994], Grizzle [1985], Monaco and Normand-Cyrot [1984] for nonlinear discrete-time systems and Conte et al. [2007], Isidori [1995], Nijmeijer and van der Schaft [1990] for nonlinear continuous-time systems. For output or measurement feedback, the problem lacks the full solution.

The first paper that applied measurement feedback to solve the DDP was Isidori et al. [1981], where sufficient solvability conditions were given for continuous-time systems, and the feedback that was used was restricted to the so-called pure dynamic measurement feedback. In Kaldmäe et al. [2013], similar results as in Isidori et al. [1981] were given for discrete-time systems (though, more general feedback was used), using algebraic approach (lattice theory), that is able to address also certain type of non-smooth systems. A more general feedback, where the state of the compensator is not a function of the state of the system, but can be chosen independently of it, was used in Xia and Moog [1999] and Kaldmäe and Kotta [2012b], where sufficient conditions for the solvability of the problem by dynamic measurement feedback were given for continuous- and discrete-time SISO systems, respectively. For static measurement feedback solutions see Pothin et al. [2002] and Kaldmäe and Kotta [2012a].

In this paper, we extend the results of Kaldmäe and Kotta [2012b] for MIMO discrete-time systems [1]. However, the extension is not direct since we relax certain integrability conditions. The result of this paper depends heavily on the solution of the input-output linearization problem, see Kaldmäe and Kotta [2014]. We show that a feedback that linearizes certain functions also solves the disturbance decoupling problem. It is our conjecture that our results can be generalized directly for continuous-time systems, though the computations are different because the differential operator and forward-shift operator act differently on the set of functions.

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[1] Note that there are no solutions for MIMO continuous-time systems.

2. PRELIMINARIES

2.1 Algebraic tools

In this paper, x stands for $x(t)$ and for $k \geq 1$, $x^{[k]}$ stands for kth-step forward time shift of x, defined by $x^{[k]} := x(t+k)$. Similar notations are used for the backward shift and the other variables.

Consider a nonlinear system, described by the equations

$$\begin{aligned} x^{[1]} &= f(x, u, w) \\ y &= h_*(x) \\ z &= h(x), \end{aligned} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$ is the state, $u \in U \subset \mathbb{R}^m$ is the controlled input, $w \in W \subset \mathbb{R}^\iota$ is the disturbance input, $y \subset Y \in \mathbb{R}^p$ is the controlled output and $z \subset Z \in \mathbb{R}^q$ is the measured output. It is assumed that the functions f, h_* and h are meromorphic. Also, we assume, that the system (1) is submersive, meaning that generically, i.e. everywhere except on a set of measure zero,

$$\operatorname{rank}\Big[\frac{\partial f}{\partial(x(t), u(t))}\Big] = n. \tag{2}$$

Also, throughout the paper it is assumed that $i = 1, \dots, p$.

Let $\mathcal{K}$ denote the field of meromorphic functions which depend on finite number of variables from the set $\{x, u^{[k]}, w^{[k]}; k \geq 0\}$. Introduce the forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$, defined by the equations (1); in particular

$$\delta x := f(x, u, w)$$

and for $k \geq 0$, $\delta u^{[k]} := u^{[k+1]}$, $\delta w^{[k]} := w^{[k+1]}$. Moreover,

$$\delta\varphi(x,u,w,\dots,u^{[k]},w^{[s]}) := \varphi(f(x,u,w),u^{[1]},w^{[1]},\dots,u^{[k+1]},w^{[s+1]})$$

for $\varphi \in \mathcal{K}$. Under the submersivity assumption (2), the pair $(\mathcal{K},\delta)$ is a difference field. In general, this difference field is not inversive, i.e. the operator δ is not inversive in $\mathcal{K}$. However, one can always find an overfield $\mathcal{K}^*$ of $\mathcal{K}$, called the inversive closure of $\mathcal{K}$, which is inversive. See Aranda-Bricaire et al. [1996], Aranda-Bricaire and Kotta [2004] for details how to compute $\mathcal{K}^*$. From now on, we assume that difference field $(\mathcal{K},\delta)$ is inversive and denote it by $\mathcal{K}$. Note that then there exists an operator δ^{-1}, which is called backward-shift operator. By δ^k and δ^{-k} we denote the k-fold application of operators δ and δ^{-1}, respectively.

Define the vector space of one-forms as $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$. Also, define $\mathcal{X} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$, $\mathcal{W} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}w^{[k]}, k \geq 0\}$. The operators δ and δ^{-1} are extended to $\mathcal{E}$ by the rules

$$\delta\Big(\sum_j a_j \mathrm{d}\varphi_j\Big) = \sum_j \delta(a_j)\mathrm{d}(\delta\varphi_j)$$
$$\delta^{-1}\Big(\sum_j a_j \mathrm{d}\varphi_j\Big) = \sum_j \delta^{-1}(a_j)\mathrm{d}(\delta^{-1}\varphi_j),$$

where $a_j, \varphi_j \in \mathcal{K}$. A one-form ω is called exact, if it is a differential of some function $\xi \in \mathcal{K}$, i.e $\omega = \mathrm{d}\xi$. Let $y = (y_1,\dots,y_p)$ be the controlled output vector of the system (1). The relative degree r_i of an output y_i with respect to input u is defined by $r_i := \min\{k \in \mathbb{N} \mid \mathrm{d}y_i^{[k]} \notin \mathcal{X} + \mathcal{W}\}$. If there does not exist such integer k, then set $r_i := \infty$.

In general, a one-form ω is a linear combination over $\mathcal{K}$ of finite number of standard basis elements of $\mathcal{E}$, i.e. $\{\mathrm{d}x, \mathrm{d}u^{[k]}, \mathrm{d}w^{[k]}; k \geq 0\}$. However, it is often possible to find a linearly independent set of exact one-forms with less elements than those basis elements of $\mathcal{E}$ in terms of which ω can be expressed.

Definition 1. A number $\gamma \in \mathbb{N}$ is called the rank of a one-form ω, if γ is minimal number of linearly independent exact one-forms necessary to express a one-form ω. The set of these exact one-forms is called the basis of ω.

Next we define two subspaces Ω and Ω_u of $\mathcal{X}$ in the following way:

$$\Omega = \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N} : \delta^k\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]},\dots,\mathrm{d}y_i^{[r_i+k-1]}\}\}. \tag{3}$$

and

$$\Omega_u = \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N} : \delta^k\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{[k-1]}, \mathrm{d}y_i^{[r_i]},\dots,\mathrm{d}y_i^{[r_i+k-1]}\}\}. \tag{4}$$

By definitions, $\Omega \subseteq \Omega_u$. For SISO systems $\Omega = \Omega_u$, since $\mathrm{d}u$ can be written as a linear combination of $\mathrm{d}x$ and $\mathrm{d}y^{[r]}$, where r is the relative degree of output y with respect to input u.

Following lemmas give procedures for computing subspaces Ω and Ω_u.

Lemma 1. Kaldmäe and Kotta [2012a] The subspace Ω may be computed as the limit of the following algorithm:

$$\Omega^0 = \mathcal{X} \tag{5}$$
$$\Omega^{k+1} = \{\omega \in \Omega^k \mid \delta\omega \in \Omega^k + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i^{[r_i]}\}\}.$$

Lemma 2. The subspace Ω_u may be computed as the limit of the following algorithm:

$$\Omega^0 = \mathcal{X} \tag{6}$$
$$\Omega^{k+1} = \{\omega \in \Omega^k \mid \delta\omega \in \Omega^k + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y_i^{[r_i]}\}\}.$$

Suppose $\Omega = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\theta_1,\dots,\mathrm{d}\theta_s\}$. Next define the k-time forward-shift of subspace Ω elementwise by $\Omega^{[k]} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\theta_1^{[k]},\dots,\mathrm{d}\theta_s^{[k]}\}$ for $k \geq 1$.

2.2 Problem statement

The DDP by measurement feedback can be stated as follows. Find a dynamic measurement feedback of the form

$$\eta^{[1]} = F(\eta, z, v) \tag{7}$$
$$u = H(\eta, z, v),$$

where $\eta \in \mathbb{R}^\rho$ and $v \in \mathbb{R}^m$, such that controlled outputs y_i of the closed-loop system do not depend on disturbance w at any time instant, i.e.

$$\mathrm{d}y_i^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \quad k < \tilde{r}_i$$
$$\mathrm{d}y_i^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}v, \dots, \mathrm{d}v^{[k-\tilde{r}_i]}\} \quad k \geq \tilde{r}_i,$$

where $\tilde{r}_i$ is the relative degree of output y_i of the closed loop system with respect to u.

Lemma 3. If the relative degrees r_i of outputs y_i with respect to u are finite then system (1) is disturbance decoupled if and only if

$$\mathrm{d}y_i^{[r_i]} \in \Omega_u + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}. \tag{8}$$

Proof: Necessity. Since r_i is the relative degree of output y_i with respect to input u,

$$\mathrm{d}y_i^{[r_i]} = \omega_0 + \sum_{j=1}^m b_{i,j}\mathrm{d}u_j,$$

where $b_{i,j} \in \mathcal{K}$ and $\omega_0 \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$. We show that $\omega_0 \in \Omega_u$. Assume contrary that $\omega_0 \notin \Omega_u$. Then there exists $k \in \mathbb{N}$ such that

$$\delta^k\omega_0 \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{[k-1]}\}.$$

This means that one-form ω_0 is not disturbance decoupled and thus y_i also is not disturbance decoupled. This is a contradiction and thus $\omega_0 \in \Omega_u$.

Sufficiency. If (8) is true, then by Lemma 2 $\Omega_u^{[1]} \subseteq \Omega_u + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}$. Thus, Ω_u is invariant with respect to the system dynamics and since $\mathrm{d}y \in \Omega_u$, the system is disturbance decoupled. ■

3. MAIN RESULTS

3.1 Input-output linearization

Since our solution of the DDP depends on the solution of the input-output (i/o) linearization problem, we start with the statement of the i/o linearization problem. For more information, see Kaldmäe and Kotta [2014]. In this section, let $l = 1, \ldots, q$.

Consider a discrete-time multi-input multi-output (MIMO) nonlinear system, described by the difference equations

$$z_l^{[n_l]} = \Phi_l(z_\tau, \ldots, z_\tau^{[n_{l\tau}]}, u_j, \ldots, u_j^{[n_l-1]}) \tag{9}$$

for $\tau = 1, \ldots, q$, $j = 1, \ldots, m$, where Φ_l are supposed to be meromorphic functions of their arguments and the indices in (9) satisfy the relations

$$\begin{aligned} &n_1 \leq n_2 \leq \cdots \leq n_q, \quad n_{l\tau} < n_\tau \\ &n_{l\tau} < n_l, \quad \tau \leq l \\ &n_{l\tau} \leq n_l, \quad \tau > l. \end{aligned} \tag{10}$$

Also, we assume, that system (9) is submersive, i.e. the map $\Phi = (\Phi_1, \ldots, \Phi_q)^T$ satisfies generically the condition

$$\operatorname{rank}\Big[\frac{\partial \Phi}{\partial(z, u)}\Big] = q,$$

where $z = (z_1, \ldots, z_q)$ and $u = (u_1, \ldots, u_m)$.

In this section, let $\mathcal{K}$ be the field of meromorphic functions in variables z, u and a finite number of their independent forward shifts, i.e. variables from the set $\mathcal{C} = \{z_l, \ldots, z_l^{[n_l-1]}, u_j^{[k]}; k \geq 0\}$. Also, let $\mathcal{E}^k := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z_l, \ldots, \mathrm{d}z_l^{[k-1]}, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{[k-1]}\}$ for any $k \in \mathbb{N}$ and r_l denotes the relative degree of the output z_l with respect to the input u.

Given a discrete-time MIMO nonlinear control system of the form (9), we say that system (9) is i/o linearized by feedback (7), if the differentials of the input-output equations of the closed-loop system satisfy the relations

$$\mathrm{d}z_l^{[n_l]} \in \operatorname{span}_{\mathbb{R}}\{\mathrm{d}z_\tau^{[n_{l\tau}]}, \ldots, \mathrm{d}z_\tau, \mathrm{d}v\} \tag{11}$$

for $\tau = 1, \ldots, q$. In case when

$$\mathrm{d}z_l^{[n_l]} \in \operatorname{span}_{\mathbb{R}}\{\mathrm{d}v\},$$

system (9) is said to be strictly i/o linearized.

We say that functions $\varphi_l(z, \ldots, z^{[s-1]}, u, \ldots, u^{[s-1]})$ are linearizable (strictly linearizable) if the system

$$z_l^{[s]} = \varphi_l(z, \ldots, z^{[s-1]}, u, \ldots, u^{[s-1]})$$

is i/o linearizable (strictly i/o linearizable).

Let

$$\tilde{\omega}_l := \mathrm{d}z_l^{[n_l]} \mod \operatorname{span}_{\mathbb{R}}\{\mathrm{d}z_\tau^{[n_{l\tau}]}, \ldots, \mathrm{d}z_\tau\},$$

where $\tau = 1, \ldots, q$. [2] For solvability of the i/o linearization problem, it is necessary that [3]

$$\tilde{\omega}_l \in \mathcal{E}^{n_l - r_l + 1}, \tag{12}$$

since otherwise nonlinearities appear before the input u starts to affect the output y_i.

First, let ω_{l_*}, $l_* = 1, \ldots, q_*$, be the basis elements of $\operatorname{span}_{\mathbb{R}}\{\tilde{\omega}_l\}$. In the rest of this section assume that $l_*, \tau = 1, \ldots, q_*$ and $j = 1, \ldots, m$.

Let σ_{l_*} be such that

$$\omega_{l_*} \in \mathcal{E}^{\sigma_{l_*}}.$$

Next, define the one-forms

$$\bar{\omega}_{l_*,\lambda} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z^{[\sigma_{l_*}-\lambda]}, \ldots, \mathrm{d}z^{[\sigma_{l_*}-1]}, \mathrm{d}u^{[\sigma_{l_*}-\lambda]}, \ldots, \mathrm{d}u^{[\sigma_{l_*}-1]}\},$$

where $\lambda = 1, \ldots, \sigma_{l_*} - 1$, such that

$$\omega_{l_*} - \bar{\omega}_{l_*,\lambda} \in \mathcal{E}^{\sigma_{l_*}-\lambda} \tag{13}$$

and

$$\bar{\omega}_{l_*,\sigma_{l_*}} := \omega_{l_*}. \tag{14}$$

It means that the one-forms $\bar{\omega}_{l_*,\lambda}$ depend on the $(\sigma_{l_*}-\lambda)$th and higher order terms of the one-forms ω_{l_*}. Let $\gamma_{l_*,\lambda}$ be the rank of a one-form $\bar{\omega}_{l_*,\lambda}$ for $\lambda = 1, \ldots, \sigma_{l_*}$. Then there exist $\gamma_{l_*,\lambda}$ functions $\tilde{\phi}^k_{l_*,\lambda}(z^{[\sigma_{l_*}-\lambda]}, \ldots, z^{[\sigma_{l_*}-1]}, u^{[\sigma_{l_*}-\lambda]}, \ldots, u^{[\sigma_{l_*}-1]})$ such that

$$\bar{\omega}_{l_*,\lambda} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\tilde{\phi}^1_{l_*,\lambda}, \ldots, \mathrm{d}\tilde{\phi}^{\gamma_{l_*,\lambda}}_{l_*,\lambda}\}.$$

Finally, define the function $\phi^k_{l_*,\lambda}$ as a $(\sigma_{l_*} - \lambda)$ step backward shift of the function $\tilde{\phi}^k_{l_*,\lambda}$, i.e.

$$\phi^k_{l_*,\lambda} := (\delta^{-1})^{\sigma_{l_*}-\lambda}\tilde{\phi}^k_{l_*,\lambda} = \delta^{\lambda-\sigma_{l_*}}\tilde{\phi}^k_{l_*,\lambda}$$

for $\lambda = 1, \ldots, \sigma_{l_*}$ and $k = 1, \ldots, \gamma_{l_*,\lambda}$.

Theorem 1. Kaldmäe and Kotta [2014] Under the assumption (12) the system (9) is input-output linearizable by dynamic output feedback of the form (7) if and only if

$$\dim(\operatorname{span}_{\mathcal{K}}\{\mathrm{d}\phi^k_{l_*,\lambda}\}) = \operatorname{rank}_{\mathcal{K}}\frac{\partial \phi^k_{l_*,\lambda}}{\partial(u, \delta\phi^k_{l_*,\lambda^*})}, \tag{15}$$

for $\lambda = 1, \ldots, \sigma_{l_*}$, $\lambda^* = 1, \ldots, \sigma_{l_*} - 1$, $k = 1, \ldots, \gamma_{l_*,\lambda}$ and functions $\phi^1_{l_*,\sigma_{l_*}}$ are independent from all the other functions.

3.2 Sufficient conditions for solvability of the DDP

The theorem below gives sufficient solvability conditions of the DDP by dynamic measurement feedback.

Theorem 2. Under the assumption that all the relative degrees r_i of outputs y_i with respect to u are finite, the DDP by dynamic measurement feedback is solvable for system (1), if

(i) there exist one-forms $\omega_i \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z, \ldots, \mathrm{d}z^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ with rank $\omega_i =: \gamma_i$ such that
$$\mathrm{d}y_i^{[r_i+s-1]} - \omega_i \in \Omega + \cdots + \Omega^{[s-1]}$$
for some $s \geq 1$;

(ii) for $\omega_i = \sum_{j=1}^{\gamma_i} \beta_{i,j}\mathrm{d}\alpha_{i,j}(z, \ldots, z^{[s-1]}, u, \ldots, u^{[s-1]})$ from (i), the functions $\alpha_{i,j}$ are strictly linearizable by dynamic measurement feedback.

Proof: We show that the feedback that linearizes strictly the functions $\alpha_{i,j}$ in (ii), solves the disturbance decoupling problem.

Note that the relative degree of y_i with respect to input v is $\bar{r}_i = r_i + s - 1$. Since for the closed-loop system $\omega_i \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}v\}$, one gets from (i) that

$$\mathrm{d}y_i^{[\bar{r}_i]} \in \Omega + \cdots + \Omega^{[s-1]} + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}v\}.$$

Next, we show that $\bar{\Omega} = \Omega + \cdots + \Omega^{[s-1]}$, where $\bar{\Omega}$ is the subspace Ω for the closed-loop system. From the definition of the subspace Ω,

$$\Omega + \cdots + \Omega^{[s-1]} \subseteq \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]}, \ldots, \mathrm{d}y_i^{[r_i+s-2]}\}.$$

[2] In the case of strict linearizability, one has to take $\tilde{\omega}_l := \mathrm{d}z_l^{[n_l]}$.

[3] Note that if $r_l = 1$, then the condition (12) is always satisfied.

Since $\bar{r}_i = r_i + s - 1$, then in the closed-loop system

$$\Omega + \cdots + \Omega^{[s-1]} \subseteq \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}.$$

Thus,

$$\begin{aligned}\Omega + \cdots + \Omega^{[s-1]} &= \{\bar{\omega} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \mid \forall k \in \mathbb{N} : \\ &\quad \bar{\omega}^{[k]} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}y_i^{[r_i+s-1]}, \ldots, \mathrm{d}y_i^{[r_i+s-k-2]}\}\} \\ &= \bar{\Omega}.\end{aligned}$$

The last equality comes from the definition (3) of the subspace $\bar{\Omega}$.

Since $\bar{\Omega} \subseteq \bar{\Omega}_u$, then by Lemma 3, system (1) is disturbance decoupled. ■

Corollary 1. For SISO systems, the conditions of Theorem 2 are necessary and sufficient.

Proof: It remains to prove the necessity. By Lemma 3, since the closed-loop system is disturbance decoupled,

$$\mathrm{d}y^{[\bar{r}]} \in \bar{\Omega}_u + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}v\}, \tag{16}$$

where $\bar{r}$ is the relative degree of y in the closed-loop system with respect to the new input v and $\bar{\Omega}_u$ is the subspace Ω_u for the closed-loop system. We choose $s \geq 1$ such that $\bar{r} = r + s - 1$.

Since for single input systems $\Omega = \Omega_u$, one can show, as in the proof of Theorem 2, that $\bar{\Omega}_u = \Omega + \cdots + \Omega^{[s-1]}$. Now, one can find the one-form $\omega \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}v\}$, with rank 1, such that we get from (16)

$$\mathrm{d}y^{[r+s-1]} - \omega \in \Omega + \cdots + \Omega^{[s-1]}.$$

Assume that $\omega = \beta \mathrm{d}\alpha$ for some functions $\beta, \alpha \in \mathcal{K}$. Clearly, the feedback that solves the disturbance decoupling problem, also linearizes strictly function α, since for the closed-loop system $\omega \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}v\}$. Thus conditions (i) and (ii) of Theorem 2 are satisfied. ■

Note that if we take $s = 1$ in Theorem 2, we get solvability conditions for DDP by static measurement feedback. In this case the strict linearizability of functions $\alpha_{i,j}$ means that system of equations $\alpha_{i,j}(z, u) = v_\mu$, $\mu = 1, \ldots, m$, is solvable in u.

4. EXAMPLES

Example 1. Consider the system

$$\begin{aligned} x_1^{[1]} &= u_1 \\ x_2^{[1]} &= x_3 u_3 + x_2 x_4 u_2 - x_1 \\ x_3^{[1]} &= u_2 \\ x_4^{[1]} &= x_1 w \\ x_5^{[1]} &= u_1 u_2 x_4 + x_2 \\ y_1 &= x_2 \\ y_2 &= x_5 \\ z &= x_4. \end{aligned} \tag{17}$$

First, note that the relative degrees r_1 and r_2 of outputs y_1 and y_2 with respect to u are both 1. One can also computes subspaces $\Omega = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_2, \mathrm{d}x_5\}$ and $\Omega_u = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_5\}$. Clearly, $\mathrm{d}y_i \notin \Omega_u + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u\}$ for $i = 1, 2$. Therefore, system (17) is not disturbance decoupled.

To find the one-forms ω_i, defined in (i) of Theorem 2, we calculate $\mathrm{d}y_i^{[r_i+s_i-1]}$ for $s_i = 1, 2, \ldots$, until

$$\begin{aligned}\mathrm{d}y_i^{[r_i+s_i-1]} &\in \Omega + \cdots + \Omega^{[s_i-1]} \\ &\quad + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z, \ldots, \mathrm{d}z^{[s_i-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s_i-1]}\}.\end{aligned}$$

For system (17), we calculate

$$\begin{aligned}\mathrm{d}y_1^{[1]} &= u_3 \mathrm{d}x_3 - \mathrm{d}x_1 + z u_2 \mathrm{d}x_2 + x_3 \mathrm{d}u_3 + x_2 \mathrm{d}(z u_2) \\ &\notin \Omega + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}z\} \\ \mathrm{d}y_2^{[1]} &= \mathrm{d}x_2 + \mathrm{d}(u_1 u_2 z) \\ &\in \Omega + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}z\}.\end{aligned}$$

Thus, $s_2 = 1$. Compute $\Omega + \Omega^{[1]} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_2, \mathrm{d}x_5, \mathrm{d}x_2^{[1]}, \mathrm{d}x_5^{[1]}\}$. Now,

$$\begin{aligned}\mathrm{d}y_1^{[2]} &= \mathrm{d}(u_3^{[1]} u_2 - u_1) + z^{[1]} u_2^{[1]} \mathrm{d}x_2^{[1]} \\ &\quad + x_2^{[1]} \mathrm{d}(z^{[1]} u_2^{[1]}) \\ &\in \Omega + \Omega^{[1]} + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}u^{[1]}, \mathrm{d}z, \mathrm{d}z^{[1]}\},\end{aligned}$$

meaning that $s_1 = 2$. Next, we can choose the one-forms ω_i as

$$\begin{aligned}\omega_1 &= \mathrm{d}(u_3^{[1]} u_2 - u_1) + x_2^{[1]} \mathrm{d}(z^{[1]} u_2^{[1]}) \\ \omega_2 &= \mathrm{d}(u_1 u_2 z).\end{aligned}$$

Obviously, $\operatorname{rank} \omega_1 = 2$ and $\operatorname{rank} \omega_2 = 1$. It remains to check whether the functions $\alpha_{1,1} = u_3^{[1]} u_2 - u_1$, $\alpha_{1,2} = z^{[1]} u_2^{[1]}$ and $\alpha_{2,1} = u_1 u_2 z$ are linearizable. One can find, that the dynamic feedback

$$\begin{aligned}\eta_1^{[1]} &= \frac{z(\eta_2 v_1 + v_3)}{\eta_2^2} \\ \eta_2^{[1]} &= v_2 \\ u_1 &= \frac{v_3}{\eta_2} \\ u_2 &= \frac{\eta_2}{z} \\ u_3 &= \eta_1,\end{aligned} \tag{18}$$

linearizes functions $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{2,1}$ and also decouples disturbances from the controlled outputs y_1 and y_2. Really, in the closed-loop system

$$\begin{aligned}y_1^{[2]} &= v_1 + x_2^{[1]} v_2 \\ y_2^{[1]} &= v_3 + x_2\end{aligned}$$

and since $\bar{\Omega}_u = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_5, \mathrm{d}x_2^{[1]}, \mathrm{d}\eta_2\}$, the conditions of Lemma 3 are satisfied. This means that the closed-loop system is disturbance decoupled.

Example 2. The next example is taken from Kaldmäe et al. [2013]. The system in Figure 1 is a typical subsystem in many applications and consists of linear subsystems $W_1 = k_1/(1 + T_1 \frac{\mathrm{d}}{\mathrm{d}t})$, $W_2 = k_2/(1 + T_2 \frac{\mathrm{d}}{\mathrm{d}t})$, $W_3 = k_3 T_3 \frac{\mathrm{d}}{\mathrm{d}t}/(1 + T_3 \frac{\mathrm{d}}{\mathrm{d}t})$, $W_4 = k_4/\frac{\mathrm{d}}{\mathrm{d}t}$ and saturation operation,

$$\sigma(x) = \begin{cases} x, & \text{if } |x| \leq x_0 \\ x_0 \operatorname{sign} x, & \text{if } |x| > x_0 \end{cases}$$

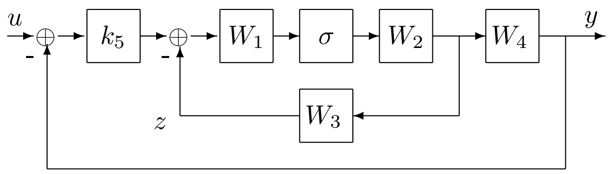

Fig. 1. System with saturation operation.

that corresponds to the amplifier. Here $k_1, \ldots, k_5$, are real coefficients, T_1, T_2 are certain time constants and T_3 may be considered as unknown function of disturbance w because of the unexpected changes in the feedback loop.

After the Euler discretization, one gets a system described by the equations:

$$\begin{aligned} x_1^{[1]} &= k_4 x_2 + x_1 \\ x_2^{[1]} &= \frac{k_2}{T_2}\sigma(x_3) + x_2(1 - \frac{1}{T_2}) \\ x_3^{[1]} &= \frac{1}{T_1}(k_1 k_5(u - x_1) - k_1 k_3(x_2 - x_4)) + x_3(1 - \frac{1}{T_1}) \\ x_4^{[1]} &= \frac{1}{T_3(w)}x_2 + x_4(1 - \frac{1}{T_3(w)}) \\ y &= x_1 \\ z &= k_3(x_2 - x_4). \end{aligned} \tag{19}$$

In Kaldmäe et al. [2013], a dynamic measurement feedback is found that solves the DDP for system (19). However, note that the problem statement of Kaldmäe et al. [2013] is somewhat different from that in this paper. Namely, in Kaldmäe et al. [2013] the state η of a compensator is assumed to be a function of state x, i.e. $\eta = \phi(x)$.

Below we solve the DDP for system (19) using the method described in this paper. Since our method assumes all functions to be meromorphic, we take $\sigma(x_3) = x_3$ in (19), i.e. $|x_3| \leq x_{3,0}$ for some $x_{3,0} \in \mathbb{R}$. Note that if $|x_3| > x_{3,0}$, one can show by Lemma 3 that the system (19) is already disturbance decoupled.

The relative degree of output y with respect to input u is $r = 3$. Next, we have to find, by Lemma 1, the subspace Ω. Compute $\Omega = \Omega^1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$. Since

$$\begin{aligned} y^{[3]} = &\big(1 - \frac{k_1 k_2 k_4 k_5}{T_1 T_2}\big)x_1 + \big(3k_4 - \frac{3k_4}{T_2} + \frac{k_4}{T_2^2}\big)x_2 \\ &+\big(\frac{3k_2 k_4}{T_2} - \frac{k_2 k_4}{T_2^2} - \frac{k_2 k_4}{T_1 T_2}\big)x_3 + \frac{k_1 k_2 k_4}{T_1 T_2}\big(k_5 u - z\big), \end{aligned}$$

one can choose $\omega = k_5 \mathrm{d}u - \mathrm{d}z$. Then condition (i) of Theorem 2 is satisfied for $s = 1$. The rank of the one-form ω is obviously 1 and $\alpha = k_5 u - z$. By taking $v = k_5 u - z$, one gets $u = \frac{1}{k_5}(v + z)$. This static measurement feedback solves the DDP for system (19).

The reason, why we get static solution in this paper, but dynamic solution in Kaldmäe et al. [2013], is that the selection of one-form ω, in Theorem 2, is more restricted, than the selection of certain function, based on which the solution is computed, in Kaldmäe et al. [2013]. In the latter case the choice of a function that leads to static solution is not obvious.

5. CONCLUSION

This paper addressed the DDP by dynamic measurement feedback. Using algebraic methods, sufficient solvability conditions were given. For SISO systems, the conditions are also necessary. The key point of the solution is linearization of certain functions by measurement feedback. It is shown that this feedback also solves the disturbance decoupling problem. The future work will include finding necessary and sufficient solvability conditions for MIMO systems. Two examples were given to illustrate the theory.

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