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Functional Series Expansions for Nonlinear Input-Output Systems with Delay

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Abstract: A new type of Chen-Fliess series is first introduced which depends on the input and delayed versions of the input. It is then shown how a class of analytic differential delay systems with a single delay and constant initial conditions has input-output maps representable in terms of this new functional series. As in the classical case, the coefficients can be computed by iterated Lie derivatives, but here the method is applied to an infinite dimensional embedding of the original state space system. Finally, the more technical issue of series convergence is addressed. Sufficient conditions are produced to guarantee convergence in both a local and global sense.

Keywords: Nonlinear systems, delay systems, Chen-Fliess series, algebraic system theory

1. INTRODUCTION

A staple of linear control theory is the ability to describe a given system in terms of either a state space realization or an input-output map, e.g., a transfer function, and to move easily back and forth between the two system representations for the purposes of analysis and design. For a control-affine nonlinear state space realization

$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z)u_i, \quad z(0) = z_0 \tag{1a}$$

$$y = h(z), \tag{1b}$$

where the vector fields g_i and output function h are analytic, the corresponding input-output map $F_c : u \mapsto y$ can be described locally in the time domain by a functional series known as a *Chen-Fliess series*, or equivalently, as a *Fliess operator* (Fliess, 1981, 1983; Isidori, 1995). Such an operator is parameterized by a noncommutative formal power series, c, whose coefficients can be interpreted as a generalization of system Markov parameters. Not every Fliess operator is necessarily realizable by a finite dimensional state space realization of the form (1), so in this way the model class is more general. But when a realization does exist, the generating series c can be computed in terms of iterated Lie derivatives of h with respect to the vector fields g_i. The main advantage of using input-output models, in both linear and nonlinear systems analysis, is that they are independent of any state space coordinate system or state space embedding. Therefore, it is often easier in this setting to distinguish input-output invariants from those system characteristics that are dependent on a particular choice of state space coordinates.

An important class of state space models used in applications incorporates the presence of state delay in the dynamics. Such delay differential systems have a wide literature, especially on fundamental questions such as the existence and uniqueness of solutions (Hale, 1977) and on control theoretic concepts such as stability (Dugard & Verriest, 1998). On the other hand, the problem of characterizing the input-output map of a delay system beyond the linear case (which can be treated in many cases by traditional Laplace transform techniques (Richard, 2003)) appears not to have been addressed. So the purpose of this paper is to take some initial steps in this direction. Specifically, a type of Fliess operator is first introduced which depends on the input and delayed versions of the input. It is then shown how a class of analytic differential delay systems with a single delay and constant initial conditions has input-output maps representable at least formally in terms of such a Fliess operator. As in the classical case, the coefficients can be computed by iterated Lie derivatives, but here the method is applied to an infinite dimensional embedding of the original state space system. Finally, the more technical issue of series convergence is addressed. Sufficient conditions are produced to guarantee convergence in both a local and global sense. The paper is concluded by suggesting how this set up could be extended to handle the case where the initial conditions are not constant.

2. FUNCTIONAL SERIES WITH DELAY

Consider a system with m inputs $\{u_1, u_2, \ldots, u_m\}$ defined on $[t_0, t_1]$ and a corresponding alphabet $X(0) = \{x_0(0), x_1(0), \ldots, x_m(0)\}$, where the letter $x_0(0)$ will always refer to the constant input $u_0 = 1$. This fictitious input is useful for describing a nonhomogeneous system F, that is, a system where $F[0] \neq 0$. For a fixed $T > 0$ define the delay operator $\sigma : u_i(t) \mapsto u_i(t-T)$. For any integer $j \geq 0$ associate $\sigma^j u_i$ with the i-th letter of the alphabet $X(j) = \{x_0(j), x_1(j), \ldots, x_m(j)\}$. Define $X = \cup_{j\geq 0} X(j)$. Any finite sequence of letters from X is called a *word* over X. Let X^* denote the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. All formal power series in X with coefficients in $\mathbb{R}^\ell$ will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. In particular, (c, η) represents the coefficient of c corresponding to the word $\eta \in X^*$. For each $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ the goal is to formally associate an m-input, ℓ-output operator, F_c. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \le i \le m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \le R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. For each $\eta \in X^*$ define recursively the mapping $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by first setting $E_\emptyset[u] = 1$, and then letting

$$E_{x_i(j)\bar{\eta}}[u](t) = \int_{t_0}^{t} (\sigma^j u_i(\tau))\, E_{\bar{\eta}}[u](\tau)\, d\tau,$$

where $x_i(j) \in X(j)$ and $\bar{\eta} \in X^*$. It is assumed throughout that $u_i(t) = u_i(t)\mathbf{1}(t - t_0)$ for $i = 1, 2, \ldots, m$, where $\mathbf{1}(t - t_0)$ denotes the unit step function which switches from zero to unity at $t = t_0$. The main class of input-output systems under consideration in this paper is described below.

Definition 1. Given any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ the corresponding *Fliess operator* is the functional series

$$F_c[u](t) = \sum_{\eta\in X^*} (c, \eta)\, E_\eta[u](t), \tag{2}$$

where $t \in [t_0, t_1]$.

As will be seen shortly, many of the algebraic structures associated with the finite alphabet version of this series, like the shuffle algebra (Fliess, 1981), naturally extend in this setting. Convergence conditions, on the other hand, require an expanded treatment over what appears in the classical case (Gray & Wang, 2002). Before pursuing that topic, the relationship between this operator type and a class of delay differential systems is developed.

3. ANALYTIC STATE SPACE SYSTEMS WITH DELAY

Consider an n dimensional delay system

$$\dot{z}(t) = \sum_{i=0}^{m} g_i(z(t), \sigma z(t))u_i(t), \quad z(t) = z_0, \quad t \le 0. \tag{3a}$$

$$y(t) = h(z(t), \sigma z(t), \sigma^2 z(t), \ldots), \tag{3b}$$

where each vector field g_i is analytic and the output function h is analytic. Associated with this system is the augmented system

$$\dot{Z}(t) = \sum_{i,j=0}^{m,\infty} g_{i(j)}(Z(t))\sigma^j u_i(t) \tag{4a}$$

$$y(t) = h(Z(t)), \tag{4b}$$

where

$$Z(t) = [z^T(t) \;\; \sigma z^T(t) \;\; \sigma^2 z^T(t) \;\; \cdots]^T,$$

$g_{i(j)} := e_{j+1} \otimes \sigma^j g_i$, e_i is an infinite column vector with a one in the i-th position and zeros elsewhere, and $\otimes$ denotes the Kronecker product. In this setting, the Lie derivative of $\varphi(Z)$ with respect to $g_{i(j)}(Z)$ is

$$L_{g_{i(j)}}\varphi(Z) = \frac{\partial\varphi}{\partial Z}(Z)g_{i(j)}(Z) = \frac{\partial\varphi}{\partial\sigma^j z}(Z)g_i(\sigma^j z, \sigma^{j+1} z)$$

(cf. Oguchi, et al. (2002)). A main result of the paper is given next.

Theorem 2. The state space system (3) has an input-output mapping with a formal representation of the form (2), where

$$(c, \eta) = L_{g_\eta} h(Z_0) = L_{g_{i_1(j_1)}} \cdots L_{g_{i_k(j_k)}} h(Z_0)$$

with $\eta = x_{i_k}(j_k)\cdots x_{i_1}(j_1)$ and $Z_0 = [z_0^T \;\; z_0^T \;\; \cdots]^T$.

The proof of this theorem requires a few preliminaries. First, the $\mathbb{R}$-vector space $\mathbb{R}\langle\langle X\rangle\rangle$ forms a commutative and associative $\mathbb{R}$-algebra under the shuffle product, that is, the $\mathbb{R}$-bilinear mapping $\mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle$ uniquely specified by the shuffle product of two words $\eta = x_{i_1}(j_1)\eta'$ and $\xi = x_{i_2}(j_2)\xi'$ defined iteratively by

$$\eta \sqcup\!\sqcup\, \xi = x_{i_1}(j_1)(\eta' \sqcup\!\sqcup\, \xi) + x_{i_2}(j_2)(\eta \sqcup\!\sqcup\, \xi')$$

and $\eta \sqcup\!\sqcup\, \emptyset = \eta$ (Fliess, 1981). The definition is extended to $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ in a componentwise fashion. A key property in this case is the identity $E_\eta[u]E_\xi[u] = F_{\eta \sqcup\!\sqcup\, \xi}[u]$. Such products appear naturally in the nonlinear setting. For example, if $z_l = E_{x_{i_l(j_l)}}[u]$, $l = 1, 2$ and $y = z_1 z_2$ then $y = F_{x_{i_1}(j_1) \sqcup\!\sqcup\, x_{i_2}(j_2)}[u]$. Of central importance in this context is Fliess's fundamental formula (Fliess, 1981; Isidori, 1995; Wang, 1990). Consider any set of real-valued functions v_l, $l = 1, 2, \ldots, n$ which can be written in the form

$$v_l(t) = \sum_{\eta\in X^*} L_{g_\eta}\lambda_l(z_0)E_\eta[u](t),$$

where $\lambda_l : \mathbb{R}^n \to \mathbb{R}$, and the vector fields $g_{i(j)}$ are all analytic on a neighborhood U of $z_0 \in \mathbb{R}^n$. If $v = [v_1 \;\; v_2 \;\; \cdots \;\; v_n]$ is composed with some $h : U \to \mathbb{R}$, which is also analytic on U, then the fundamental formula says that

$$(h \circ v)(t) = \sum_{\eta\in X^*} L_{g_\eta}(h \circ \lambda)(z_0)E_\eta[u](t),$$

where $\lambda = [\lambda_1 \;\; \lambda_2 \;\; \cdots \;\; \lambda_n]$. The proof of this identity relies in large part on the shuffle algebra associated with $\mathbb{R}\langle\langle X\rangle\rangle$ (see Gray & Thitsa (2012) for a more general discussion on this topic). The fact that X in the present application is not finite, has no significant consequences on these algebraic structures.

Proof of Theorem 2. It is first shown for any $l = 1, 2, \ldots, n$ that

$$z_l(t) = \sum_{\eta\in X^*} L_{g_\eta} e_l^T Z_0 E_\eta[u](t) \tag{5}$$

formally satisfies the delay equation (3a). Taking the derivative of (5) yields

$$\begin{aligned}\dot{z}_l(t) &= \sum_{i,j=0}^{m,\infty}\left(\sum_{\eta\in X^*} L_{g_\eta}L_{g_{i(j)}}e_l^T Z_0 E_\eta[u](t)\right)\sigma^j u_i(t)\\ &= \sum_{i,j=0}^{m,\infty}\left(\sum_{\eta\in X^*} L_{g_\eta}e_l^T g_{i(j)}(Z_0)E_\eta[u](t)\right)\sigma^j u_i(t).\end{aligned}$$

Applying Fliess's fundamental formula gives

$$\begin{aligned}\dot{z}_l(t) &= \sum_{i,j=0}^{m,\infty} e_l^T g_{x_i(j)}(Z(t))\sigma^j u_i(t)\\ &= \sum_{i=0}^{m} e_l^T g_i(z(t), \sigma z(t))u_i(t).\end{aligned}$$

Now employ Fliess's fundamental formula a second time to $y = h(z, \sigma z, \sigma^2 z, \ldots)$, and the theorem is proved.

Example 3. Consider the scalar autonomous linear system

$$\dot{z}(t) = az(t - T), \quad z(t) = z_0, \quad t \le 0, \quad y(t) = z(t).$$

When written in the form (4), $g_{0(j)}(Z) = a\sigma^{j+1}(z)e_{j+1}$, $j = 0, 1, \ldots$. Therefore, using Theorem 2 it follows that

Table 1. Generating series coefficients for the system in Example 4

η	(c,η)
$\emptyset$	Cz_0
$x_0(0)$	$C(A_0+A_1)z_0$
$x_1(0)$	Cb
$x_0^2(0)$	$CA_0(A_0+A_1)z_0$
$x_0(0)x_0(1)$	$CA_1(A_0+A_1)z_0$
$x_0(0)x_1(0)$	CA_0b
$x_0(0)x_1(1)$	CA_1b
$x_0^3(0)$	$CA_0^2(A_0+A_1)z_0$
$x_0^2(0)x_0(1)$	$CA_0A_1(A_0+A_1)z_0$
$x_0(0)x_0^2(1)$	$CA_1A_0(A_0+A_1)z_0$
$x_0(0)x_0(1)x_0(2)$	$CA_1^2(A_0+A_1)z_0$
$x_0^2(0)x_1(0)$	CA_0^2b
$x_0^2(0)x_1(1)$	CA_0A_1b
$x_0(0)x_0(1)x_1(1)$	CA_1A_0b
$x_0(0)x_0(1)x_1(2)$	CA_1^2b

$$
\begin{aligned}
y(t) &= \sum_{k=0}^{\infty} a^k z_0 E_{x_0(0)x_0(1)\cdots x_0(k-1)}[u](t) \\
&= \sum_{k=0}^{\infty} a^k z_0 \frac{(t-(k-1)T)^k}{k!}\mathbf{1}(t-(k-1)T) \\
&= \sum_{k=0}^{\lfloor \frac{t}{T}\rfloor+1} a^k z_0 \frac{(t-(k-1)T)^k}{k!},
\end{aligned}
$$

which is the same solution for every $t \geq 0$ as that found by the usual method of steps (Bellman & Cooke, 1963).

Example 4. Consider a linear SIMO delay system of dimension n

$$
\begin{aligned}
\dot{z}(t) &= (A_0 + A_1\sigma)z(t) + bu(t), \quad z(t) = z_0, \quad t \leq 0 \\
y(t) &= Cz(t),
\end{aligned}
$$

With $g_{0(j)}(Z) = e_{j+1} \otimes (A_0\sigma^j z + A_1\sigma^{j+1}z)$ and $g_{1(j)} = e_{j+1} \otimes b$ for $j \geq 0$, it follows from Theorem 2 that the (nonzero) series coefficients are as given in Table 1.

Example 5. Consider the scalar bilinear delay system from Hale (1977)

$$
\dot{z}(t) = z(t) + z(t-T)u(t), \quad z(t) = 1, \quad t \leq 0, \quad y(t) = z(t). \tag{6}
$$

Here

$$
g_{0(j)}(Z) = \sigma^j z e_{j+1}, \quad g_{1(j)}(Z) = \sigma^{j+1} z e_{j+1}, \quad h(Z) = z.
$$

The only nonzero coefficients of the corresponding Fliess operator, F_c, are found to be

$$
(c, x_0(0)^{n_0} x_1(0) x_0(1)^{n_1} \cdots x_1(j-1) x_0(j)^{n_j}) = 1
$$

for $j \geq 0$, $n_j \geq 0$. Using the identity $\sum_{n\geq 0} x_i(j)^n = (1-x_i(j))^{-1}$, the generating series can be written in the rational form

$$
c = (1-x_0(0))^{-1} x_1(0)(1-x_0(1))^{-1} x_1(1)\cdots
$$

It is instructive to check the answer by directly constructing (6) from the generating series c. Define the state

$$
z = F_c = \sum_{j=0}^{\infty} \sum_{n_0\cdots n_j=0}^{\infty} E_{x_0(0)^{n_0} x_1(0) x_0(1)^{n_1}\cdots x_1(j-1) x_0(j)^{n_j}},
$$

which clearly satisfies the boundary condition $z(0) = 1$. Using the identities:

$$
\begin{aligned}
\frac{d}{dt} E_{x_i(j)\eta}[u](t) &= u_i(j) E_\eta[u](t) \\
\sigma E_\eta[u](t) &= E_{\sigma\eta}[u](t),
\end{aligned}
$$

where $\sigma(x_{i_1}(j_1)\cdots x_{i_k}(j_k)) := x_{i_1}(j_1+1)\cdots x_{i_k}(j_k+1)$, it follows that

$$
\begin{aligned}
\dot{z} &= u_0(0) \sum_{j=0}^{\infty} \sum_{n_0=0\cdots n_j=0}^{\infty} E_{x_0(0)^{n_0} x_1(0) x_0(1)^{n_1}\cdots x_1(j-1) x_0(j)^{n_j}} \\
&\quad + u_1(0) \sum_{j=0}^{\infty} \sum_{n_0\cdots n_j=0}^{\infty} E_{x_0(1)^{n_0} x_1(1) x_0(2)^{n_1}\cdots x_1(j) x_0(j+1)^{n_j}} \\
&= z + \sigma(z)u.
\end{aligned}
$$

Example 6. Consider a variation of the previous example $\dot{z}(t) = z(t-T) + z(t)u(t)$, $z(t) = 1$, $t \leq 0$, $y(t) = z(t)$. Here, the vector fields of the realization in Z are reversed, i.e.,

$$
g_{0(j)}(Z) = \sigma^{j+1} z e_{j+1}, \quad g_{1(j)}(Z) = \sigma^j z e_{j+1}, \quad h(Z) = z,
$$

and, therefore, the roles of x_0 and x_1 are interchanged giving the rational generating series

$$
c = (1-x_1(0))^{-1} x_0(0)(1-x_1(1))^{-1} x_0(1)\cdots
$$

Of course, this example and the previous example are identical when $T = 0$. It can be verified in this case that $c = (1-x_0(0)-x_1(0))^{-1}$.

Example 7. Consider the scalar autonomous nonlinear delay system

$$
\dot{z}(t) = z^2(t-T), \quad z(t) = 1, \quad t \leq 0, \quad y(t) = z(t).
$$

In this case,

$$
g_{0(j)}(Z) = (\sigma^{j+1} z)^2 e_{j+1}, \quad h(Z) = z.
$$

The support of this sequence is again compute directly from Theorem 2, but it is surprisingly complex. Brute force calculations show that the fastest growing subseries is

$$
(c, x_0(0) x_0^2(1) \cdots x_0^{2^j}(j)) = 1!\,2!\cdots 2^j!, \quad j \geq 0,
$$

which will be shown shortly to determine its convergence characteristics.

4. CONVERGENCE CONDITIONS

4.1 Local Convergence

Convergence issues are considered next. The following theorem describes a sufficient condition under which (2) converges in a local sense. That is, a bound on the size of the input is imposed and the interval of convergence is finite. The method can be viewed as a generalization of the treatment for the non-delay case in Duffaut Espinosa (2009); Duffaut Espinosa, et al. (2009).

Theorem 8. Suppose $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is a series with coefficients that satisfy

$$
|(c,\eta)| \leq K \prod_{i,j=0}^{m,\infty} M_{ij}^{|\eta|_{x_i(j)}} |\eta|!, \quad \forall \eta \in X^* \tag{7}
$$

for some real numbers $K, M_{ij} \geq 0$, where $|\eta|_{x_i(j)}$ denotes the number of times the letter $x_i(j)$ appears in η, and $|(c,\eta)| := \max_i |(c_i,\eta)|$. Then the series (2) converges absolutely and uniformly for any $u \in L_1^m[t_0,t_1]$ if

$$
R := \max\{\|u\|_1, t_1 - t_0\} < \frac{1}{\sum_{i,j=0}^{m,\lfloor\frac{t_1-t_0}{T}\rfloor} M_{ij}}. \tag{8}
$$

Proof. It is convenient to define a function $\bar{u} \in L_1^m[t_0,t_1]$ with components $\bar{u}_i = |u_i|$ so that

$$
\left|E_{x_i(j)}[u](t)\right| \leq \max_i E_{x_i(0)}[\bar{u}](t_1) \leq R
$$

for all $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots$; and $t \in [t_0, t_1]$. In which case, it follows from (7) and the multinomial theorem that for any fixed $t \in [t_0, t_1]$

$$\sum_{\eta \in X^*} |(c,\eta) E_\eta[u](t)|$$
$$\leq \sum_{k=0}^{\infty} \sum_{\eta \in X^k} |(c,\eta)|\, |E_\eta[u](t)|$$
$$\leq K \sum_{k=0}^{\infty} k! \sum_{\substack{r_{ij}\geq 0 \\ |r|=k}} \prod_{i,j} \frac{\left(M_{ij} E_{x_i(j)}[\bar{u}](t_1)\right)^{r_{ij}}}{r_{ij}!}$$
$$\leq K \sum_{k=0}^{\infty} R^k k! \sum_{\substack{r_{ij}\geq 0 \\ |r|=k}} \prod_{i,j} \frac{M_{ij}^{r_{ij}}}{r_{ij}!}$$
$$\leq K \sum_{k=0}^{\infty} \left(R \sum_{i,j=0}^{m, \left\lfloor \frac{t_1 - t_0}{T} \right\rfloor} M_{ij} \right)^k,$$

where $|r| := \sum_{i,j} r_{ij}$, and product is taken over all $0 \leq i, j \leq m, \left\lfloor \frac{t_1 - t_0}{T} \right\rfloor$ corresponding to each letter $x_i(j)$ that appears in a given word $\eta \in X^k$. The convergence condition (8) follows directly from the last line.

It is natural to use the smallest possible set of growth constants M_{ij} satisfying (7) on the right-hand side of (8). In which case, consistent with the non-delay theory described in Thitsa & Gray (2012), it is tempting to call the right-side the *radius of convergence* for the class of systems having the same minimal growth constants. But the appearance of t_1 on both sides of this inequality makes the significance of this definition not so obvious. For example, for a fixed value of $\|u\|_1$ (which implicitly depends on t_1), it is not immediate that there even exists a t_1 satisfying (8). For a constant input $u(t) = a \geq 1$ it is clear that (8) reduces to

$$t_1 - t_0 < \frac{1}{\sum_{i,j=0}^{m, \left\lfloor \frac{t_1 - t_0}{T} \right\rfloor} a M_{ij}}. \tag{9}$$

In which case, the smallest possible t_1 satisfying (9) would define the *interval of convergence* for this class of inputs. The following corollary describes another interesting special case.

Corollary 9. Suppose $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is a series with coefficients that satisfy (7), where $M_{ij} = M_i$ for all j. Then the series (2) converges absolutely and uniformly for any $u \in L_1[t_0, t_1]$ satisfying

$$\max\{\|u\|_1, t_1 - t_0\} < \frac{1}{\left(\left\lfloor \frac{t_1 - t_0}{T} \right\rfloor + 1\right) \sum_{i=0}^m M_i}.$$

Example 10. It is useful for comparison to begin with the non-delay case since the convergence condition (8) constitutes a refinement of the known results described first in Gray & Wang (2002) and then improved upon in Duffaut Espinosa (2009); Duffaut Espinosa, et al. (2009). Setting $M_{ij} = 0$ for $j > 0$ and defining $M_i = M_{i0}$, it is immediate that a sufficient condition for convergence is

$$R := \max\{\|u\|_1, t_1 - t_0\} < \frac{1}{\sum_{i=0}^m M_i} \tag{10}$$

and (7) reduces to

$$|(c,\eta)| \leq K M_0^{|\eta|_{x_0}} \cdots M_m^{|\eta|_{x_m}} |\eta|!, \quad \forall \eta \in X^*$$

with $X = \{x_0, x_1, \ldots, x_m\}$. If $M = M_i$ for all $i = 0, 1, \ldots, m$ then this theory further reduces to exactly the classical case, where the radius of convergence is defined as $1/(m+1)M$. It is shown next that the bound in (10) is determined by a specific series in the equivalence class of series having the minimal growth constants M_i, $i = 0, 1, \ldots, m$. So no larger bound than $1/\sum_{i=0}^m M_i$ is possible. First consider the realization

$$\dot{z}_i = u_i, \quad z_i(0) = 0, \quad i = 0, 1, \ldots, m$$
$$h(z) = \frac{K}{1 - \sum_{i=0}^m M_i z_i}.$$

It has the corresponding generating series

$$\bar{c} = \sum_{\eta \in X^*} K M_0^{|\eta|_{x_0}} \cdots M_m^{|\eta|_{x_m}} |\eta|!\, \eta.$$

By definition, $u_0 = 1$. Setting $u_i = 1$, $i = 1, 2, \ldots, m$ gives an output defined on exactly the interval $[0, t^*)$, where $R = t^* = 1/\sum_{i=0}^m M_i$. The exact same conclusion can be drawn from the *minimal* realization for $F_{\bar{c}} : u \mapsto y$

$$\dot{z} = z^2 \sum_{i=0}^m M_i u_i, \quad z(0) = K, \quad h(z) = z.$$

Example 11. Now a delay version of the previous example is considered. Define the series $\bar{c}$ with coefficients

$$\bar{c} = \sum_{\eta \in X^*} K \prod_{i,j=0}^{m,\infty} M_{ij}^{|\eta|_{x_i(j)}} |\eta|!\, \eta$$
$$= \sum_{k=0}^{\infty} K \left(\sum_{i,j=0}^{m,\infty} M_{ij} x_i(j) \right)^{⧢ k}.$$

With states $Z_{i+jm+1} := E_{x_i(j)}$ for $i = 0, 1, \ldots m$, $j = 0, 1, \ldots$, $F_{\bar{c}}$ is realized in the form of (4), where $g_i(j) = e_{i+jm+1}$, $h(Z) = K/(1 - \sum_{i,j\geq 0}^{m,\infty} M_{ij} Z_{i+jm+1})$, and $Z(0) = 0$, or in the form of (3) with dimension $m+1$ and $g_i = e_{i+1}$, $i = 0, 1, \ldots, m+1$; $h(z, \sigma z, \sigma^2 z, \ldots) = K/(1 - \sum_{j\geq 0}^{\infty} [M_{0j} \cdots M_{mj}] \sigma^j z(t))$; and $z_0 = 0$. In either case, the response to the unit step inputs $u_i = \mathbf{1}$, $i = 1, 2, \ldots, m$ is

$$y(t) = \frac{K}{1 - \sum_{i,j=0}^{m, \left\lfloor \frac{t}{T} \right\rfloor} M_{ij}(t - jT)},$$

which has a singularity at $t^* > 0$ if

$$\sum_{i,j=0}^{m, \left\lfloor \frac{t^*}{T} \right\rfloor} M_{ij}(t^* - jT) = 1. \tag{11}$$

In the special case described in Corollary 9, (11) reduces to

$$\left(\left\lfloor \frac{t^*}{T} \right\rfloor + 1 \right) \left(t^* - \frac{T}{2} \left\lfloor \frac{t^*}{T} \right\rfloor \right) = \frac{1}{\sum_{i=0}^m M_i} =: L. \tag{12}$$

A plot of t^* versus T when $M_0 = M_1 = 1$ is shown in Fig. 1. When (12) is written in the form

$$t^* = \frac{1}{2} \left\lfloor \frac{t^*}{T} \right\rfloor T + \frac{L}{\left(\left\lfloor \frac{t^*}{T} \right\rfloor + 1 \right)},$$

it is easy to see that t^* as a function of T will be piecewise linear with slopes that are integer multiples of $1/2$ and that $t^* = L = 0.5$ whenever $T > t^*$.

The following theorem describes a stricter local growth condition on a generating series that is available in some applications.

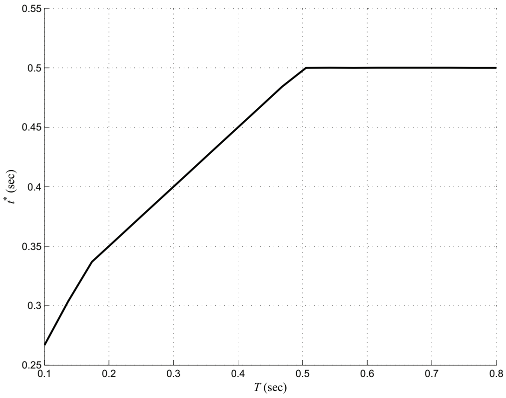

Fig. 1. Plot of t^* versus T in Example 11

Theorem 12. Suppose $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is a series with coefficients that satisfy

$$|(c,\eta)| \leq K \prod_{i,j=0}^{m,\infty} M_{ij}^{|\eta|_{x_i(j)}} \, |\eta|_{x_i(j)}!, \quad \forall \eta \in X^* \tag{13}$$

for some real numbers $K, M_{ij} \geq 0$. Then the series (2) converges absolutely and uniformly for any $u \in L_1^m[t_0,t_1]$ such that

$$\left|E_{x_i(j)}[u](t)\right| \leq \frac{1}{M_{ij}} \tag{14}$$

with $i = 0,1,\ldots m$, $0 \leq j \leq \left\lfloor \frac{t}{T} \right\rfloor$, and $t \in [t_0,t_1]$.

Proof. For any fixed $t \in [t_0,t_1]$ it follows from (13) that

$$\begin{aligned}
&\sum_{\eta \in X^*} |(c,\eta) E_\eta[u](t)| \\
&\qquad \leq \sum_{k=0}^{\infty} \sum_{\eta \in X^k} |(c,\eta)| \, |E_\eta[u](t)| \\
&\qquad \leq K \sum_{k=0}^{\infty} \sum_{\substack{r_{ij} \geq 0 \\ |r|=k}} \prod_{i,j} \left(M_{ij} \left|E_{x_i(j)}[u](t)\right|\right)^{r_{ij}} \\
&\qquad = K \prod_{i,j=0}^{m,\left\lfloor \frac{t}{T} \right\rfloor} \sum_{r_{ij}=0}^{\infty} \left(M_{ij} \left|E_{x_i(j)}[u](t)\right|\right)^{r_{ij}},
\end{aligned}$$

which yields (14).

Example 13. Reconsider Example 7 where it is evident that the coefficients of c satisfy the growth condition in Theorem 12 with $M_{0j} = 1$ for all $j \geq 0$. Observe for every $T \geq 0$ that $E_{x_0(j)}(t) = t - jT < 1$ on $[0,1)$. A MATLAB simulation of the dynamical system gives outputs when $T = 0$ and $T = 0.1$ as shown in Fig. 2. This confirms that a solution exists on at least $[0,1)$ for every $T \geq 0$.

Example 14. This example is a generalization of Example 7. Consider an autonomous system with generating series

$$c = \sum_{k=0}^{\infty} \sum_{\substack{r_j \geq 0 \\ |r|=k}} \prod_j (M_{0j} x_0(j))^{r_j} r_j!.$$

It has exactly the growth rate described by Theorem 12, so it is possible to compute the interval of convergence exactly. Observe

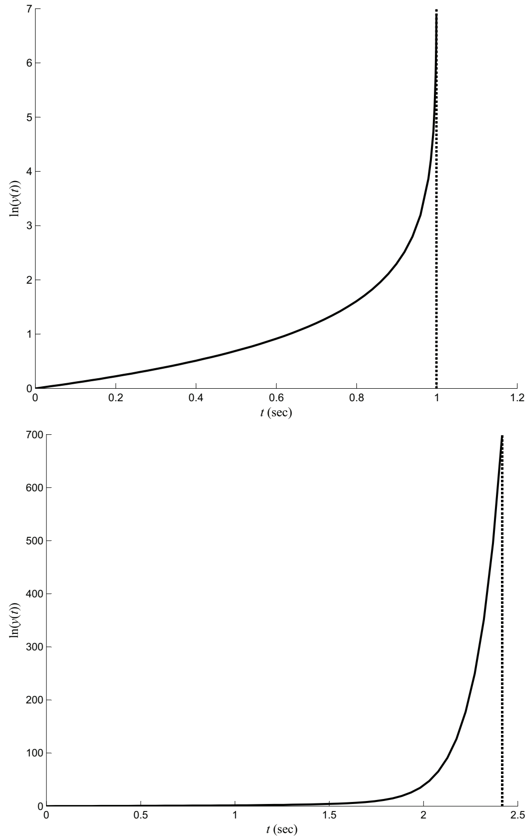

Fig. 2. Outputs for the system in Examples 7 and 13 with $T = 0$ (top) and $T = 0.1$ (bottom)

$$\begin{aligned}
F_c[u](t) &= \prod_{j=0}^{\left\lfloor \frac{t}{T} \right\rfloor} \sum_{r_j=0}^{\infty} \left(M_{0j} E_{x_0(j)}[u](t)\right)^{r_j} \\
&= \prod_{j=0}^{\left\lfloor \frac{t}{T} \right\rfloor} \frac{1}{1 - M_{0j} E_{x_0(j)}(t)},
\end{aligned}$$

which has the realization

$$g_{0(j)}(Z) = 1, \quad \sigma^j z(0) = 0, \quad j \geq 0$$

$$h(Z(t)) = \prod_{j=0}^{\left\lfloor \frac{t}{T} \right\rfloor} \frac{1}{1 - M_{0j} \sigma^j z(t)}.$$

In light of (14), convergence at a given t is assured if for every $0 \leq j \leq \left\lfloor \frac{t}{T} \right\rfloor$

$$\left|E_{x_0(j)}[u](t)\right| = t - jT \leq \frac{1}{M_{0j}}$$

or, equivalently,

$$t \leq \min_{0 \leq j \leq \left\lfloor \frac{t}{T} \right\rfloor} \left\{ \frac{1}{M_{0j}} + jT \right\}.$$

Take as a specific example $M_{0j} = 2^j$ and $T = 0.1$ as shown in Table 2. Then the interval of convergence is at least $[0, 0.425)$. But it is not difficult to see from the state space realization that this is exactly the interval of convergence. The MATLAB simulation shown in Fig. 3 also confirms this conclusion.

4.2 Global Convergence

The final convergence result considered assumes the most stringent growth condition for the generating series. But

Table 2. Data to determine the interval of convergence in Example 14

j	$\frac{1}{M_{0j}}+jT$
0	1
1	0.6
2	0.45
3	0.425
4	0.4625
5	0.53125

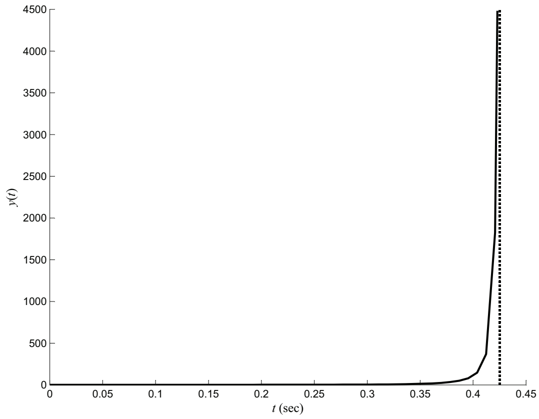

Fig. 3. Output response of system in Example 14

as a result, the corresponding Fliess operator is *globally convergent* in that no upper bound is imposed on either the norm of the input or the duration T.

Theorem 15. Suppose $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is a series with coefficients that satisfy

$$|(c,\eta)| \le K \prod_{i,j=0}^{m,\infty} M_{ij}^{|\eta|_{x_i(j)}}, \quad \forall \eta \in X^*$$

for some real numbers $K, M_{ij} \ge 0$. Then the series (2) converges absolutely and uniformly on $[t_0, t_0+T]$ for any $u \in L_1^m[t_0, t_0+T]$ and $T > 0$.

Proof. Similar to the proofs of the other convergence theorems, it follows directly for any fixed $t \ge t_0 = 0$ that

$$\begin{aligned} |F_c[t](t)| &\le K \prod_{i,j=0}^{m,\lfloor \frac{t}{T}\rfloor} \sum_{r_{ij}=0}^{\infty} \frac{\left(M_{ij}\left|E_{x_i(j)}[u](t)\right|\right)^{r_{ij}}}{r_{ij}!} \\ &= K \prod_{i,j=0}^{m,\lfloor \frac{t}{T}\rfloor} \exp(M_{ij}\left|E_{x_i(j)}[u](t)\right|). \end{aligned}$$

Clearly, the right-hand side of the above inequality is finite for every t, which completes the proof.

Is it easily confirmed that the generating series in Examples 3-6 are all globally convergent, so Theorem 15 applies in each case.

5. CONCLUSIONS AND FUTURE WORK

A new type of Chen-Fliess functional series was introduced which depends on the input and delayed versions of the input. It was then shown that a class of analytic differential delay systems with a single delay and constant initial conditions has input-output maps that can be written in terms of such series. Sufficient conditions were given to ensure convergence in both a local and global sense. The following example demonstrates that the method will *not* work for arbitrary time-varying initial conditions but suggests what modifications might be needed in future work to address such a generalization.

Example 16. Reconsider the system in Example 3 except with initial condition $z(t) = \phi(t)$ for $t \le 0$. Using Theorem 2 with $Z_0 = [\phi(0) \;\; \phi(-T) \;\; \phi(-2T) \;\cdots]$ gives

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} a^k \phi(-kT) E_{x_0(0)x_0(1)\cdots x_0(k-1)}[u](t) \\ &= \sum_{k=0}^{\lfloor \frac{t}{T}\rfloor+1} a^k \phi(-kT) \frac{(t-(k-1)T)^k}{k!}, \quad t \ge 0. \end{aligned}$$

For the initial condition $\phi(0) = 1$ and $\phi(t) = 0$, $t < 0$, which defines Hale's fundamental solution (Hale, 1977, Section 1.5), this becomes $y(t) = 1$ and agrees with the actual globally defined solution

$$y(t) = \sum_{k=0}^{\lfloor \frac{t}{T}\rfloor} a^k \frac{(t-kT)^k}{k!}.$$

only over the initial interval $[0,T]$. To recover this global solution in the present context, it is necessary to treat ϕ more like an input in that iterated integrals of ϕ need to be computed. This necessitates the introduction of n addition letters in X if the state space has dimension n.

REFERENCES

Bellman, R. & K. L. Cooke, *Differential-difference Equations*, Academic Press, New York-London, 1963.

Duffaut Espinosa, L. A., Interconnections of Nonlinear Systems Driven by L_2-Itô Stochastic Processes, Doctoral Dissertation, Old Dominion University, 2009.

Duffaut Espinosa, L. A., W. S. Gray & O. R. González, On Fliess operators driven by L_2-Itô random processes, *Proc. 48th IEEE Conf. on Decision and Control*, Shanghai, China, 2009, pp. 7478–7484.

Dugard, L. & E. I. Verriest, Eds., *Stability and Control of Time-delay Systems*, Springer, London, 1998.

Fliess, M., Fonctionnelles causales non linéaires et indéterminées non commutatives, *Bull. Soc. Math. France*, 109 (1981) 3–40.

Fliess, M., Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives, *Invent. Math.*, 71 (1983) 521–537.

Gray, W. S. & M. Thitsa, A unified approach to generating series for mixed cascades of analytic nonlinear input-output systems, *Int. J. Control*, 85 (2012) 1737–1754.

Gray, W. S. & Y. Wang, Fliess operators on L_p spaces: convergence and continuity, *Systems Control Lett.*, 46 (2002) 67–74.

Hale, J., *Theory of Functional Differental Equations*, Springer-Verlag, New York, 1977.

Isidori, A., Nonlinear Control Systems, 3rd Ed., Springer–Verlag, London, 1995.

Oguchi, T., A. Watanabe & T. Nakamizo, Input-output linearization of retarded non-linear systems by using an extension of Lie derivative, *Int. J. Control*, 75 (2002) 582–590.

Richard, J.-P., Time-delay systems: an overview of some recent advances and open problems, *Automatica*, 39 (2003) 1667–1694.

Thitsa, M. & W. S. Gray, On the radius of convergence of interconnected analytic nonlinear input–output systems, *SIAM J. Control Optim.*, 50 (2012) 2786–2813.

Wang, Y., Algebraic Differential Equations and Nonlinear Control Systems, Doctoral Dissertation, Rutgers University, 1990.