Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Optimal Sampled–Data State Feedback Control of Linear Systems ⋆

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Abstract: This paper addresses and solves two optimal state-feedback control design problems for sampled-data systems. First, we reformulate the closed-loop system as a special linear hybrid system. Then, two theorems are developed to evaluate the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ performances of hybrid systems using specific two-point boundary value problems. Both theorems are adapted to provide optimal control conditions, based on linear matrix inequalities (LMIs), for the state-feedback problems under consideration. These results are generalised to cope with non-uniform data-rates in the communication channel between the controller and the plant.

Keywords: Sampled-data control, Sampled-data systems, Hybrid systems, $\mathcal{H}_\infty$ control, Linear systems.

1. INTRODUCTION

In the well-established literature on control of linear systems, there is a vast framework that focuses on sampled-data techniques. Their importance stems from the wide use of sampled-data control systems in real world applications, mainly due to the flexibility induced by the adoption of digital controllers [Chen and Francis, 1995, Ragazzini and Franklin, 1958, Franklin et al., 1997] and communication networks [Hespanha et al., 2007, Wang and Liu, 2008] in the architecture of the closed-loop system. However, it is clear that, even though there are several advantages of using sampled-data controllers, the designer would have to consider the constraints on the information flow that is available for feedback, [Seron et al., 1997]. Thus, the classical control results for linear, time-invariant (LTI) systems have to be adapted to cope with these limitations.

Important early results on sampled-data control systems can be found in [Ragazzini and Franklin, 1958], where the authors study stability and closed-loop performance properties when conventional and digital control techniques are applied. In [Chen and Francis, 1995], the authors revisit the classical results on sampled-data systems with an optimal control point of view, which includes the adoption of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Hardy spaces and their respective induced norms, developing a more general and formal theoretical framework. The $\mathcal{H}_2$ sampled-data control problem is studied in [Khargonekar and Sivashankar, 1991, Bamieh and Pearson, 1992] under the continuous-time lifting background and in [Chen, 1993] with a more basic derivation. Additionally, continuous-time lifting techniques are also applied to the $\mathcal{H}_\infty$ sampled-data problem in [Bamieh and Pearson Jr., 1992].

Hybrid dynamical systems [Goebel et al., 2009] combine continuous-time and discrete-time behaviour in their dynamics. This important class of systems includes, as particular cases, switched systems [Liberzon, 2003] and Markov jump linear systems [Costa et al., 2013], which are very recurring in the literature to date. The application of hybrid systems results to solve sampled-data control problems provides a natural time-domain based framework and circumvents the use of the rather complicated lifting techniques. Stability conditions and $\mathcal{H}_2$ performance results for sampled-data control problems using a hybrid state-space formulation are done in [Hara et al., 1994, Chen. and Francis, 1991]. In [Kabamba. and Hara, 1993, Sun et al., 1993], the $\mathcal{H}_\infty$ sampled-data problem is studied on this background, where the latter reference also considers the sampled-data filtering problem.

In this paper, we present novel techniques for the design of state-feedback sampled-data controllers with a hybrid dynamical systems approach. We first provide stability and $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance conditions based on two-point boundary value problems, whose solution, whenever exists, can be efficiently computed. These results allow us to formulate and solve $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problems when the sampling period is known and constant. Finally, the periodic sampling constraint is relaxed in order to provide robust feedback controllers when the data-rate is bounded but unknown.

The notation used throughout the paper is standard. For square matrices $\mathbf{tr}(\cdot)$ denotes the trace function. For real matrices or vectors $(')$ indicates transpose. For symmetric matrices, the symbol $(\bullet)$ denotes each of its symmetric

⋆ This work was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP/Brazil) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq/Brazil).

blocks. The sets of real, nonnegative real and natural numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$. For any symmetric matrix X, we denote $X > 0$ ($X \geq 0$) to state that X is positive (semi)definite. Specifically, the notation $\xi(t_k^-)$ for $t_k \geq 0$, $k \in \mathbb{N}$, given, indicates the limit of $\xi(t)$ as t goes to t_k from the left.

2. PROBLEM STATEMENT AND PRELIMINARIES

In this paper we consider a linear, time-invariant (LTI) dynamical system given by its minimal state-space realisation

$$\mathcal{S} \ : \ \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ew(t), & x(0) = 0 \\ z(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state, $u : \mathbb{R}_+ \to \mathbb{R}^m$ and $w : \mathbb{R}_+ \to \mathbb{R}^p$ are the control and the exogenous input, respectively, and $z : \mathbb{R}_+ \to \mathbb{R}^q$ is the controlled output. The control signal u belongs to a special class of admissible functions of the form

$$u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}) \tag{2}$$

where the sequence of sampling instants $\{t_k\}_{k\in\mathbb{N}}$ is such that $t_0 = 0$, $t_{k+1} - t_k > 0$ and $\lim_{k\to\infty} t_k = \infty$. Our main goal is to provide design conditions for the determination of the gain matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop system $\mathcal{S}$ is globally asymptotically stable and either its $\mathcal{H}_2$ or its $\mathcal{H}_\infty$ norm is minimised; see [Colaneri et al., 1997] for details on Hardy spaces. Thus, considering the system evolves from zero initial condition, our main purpose in this paper is to discuss and solve from the theoretical and numerical viewpoint the following optimal control problems that take into account constraint (2), which characterises a sampled-data state feedback control law. For each problem the optimal solution will be fully characterised by necessary and sufficient conditions expressed by LMIs.

- **(P1) Optimal $\mathcal{H}_2$ Control:** we wish to solve the optimisation problem

$$\inf_K \|\mathcal{S}\|_2^2 = \inf_K \sum_{i=1}^{p} \int_0^\infty z_i(t)' z_i(t) dt \tag{3}$$

  where z_i is the output associated to the exogenous impulsive perturbation $w(t) = e_i\delta(t)$, $t \in \mathbb{R}_+$, considering that $e_i \in \mathbb{R}^p$ is the i-th column of the identity matrix of compatible dimension.
- **(P2) Optimal $\mathcal{H}_\infty$ Control:** the optimisation problem to be solved is

$$\inf_K \|\mathcal{S}\|_\infty^2 = \inf_{K,\gamma} \left\{ \gamma^2 : \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0 \right\} \tag{4}$$

  for any external disturbance $w \in \mathcal{L}_2(\mathbb{R}_+)$, where $\|\cdot\|_2$ is the usual norm in $\mathcal{L}_2(\mathbb{R}_+)$.

It is important to keep in mind that both problems described above, are classical from the viewpoint of optimal control and are also well-defined in the sampled-data framework. Therefore, it is of interest to adequately consider the intersampling continuous-time behaviour of the plant to attempt to keep these properties whenever a sampled-data strategy has to be designed.

In order to solve problems **(P1)** and **(P2)** stated before, we notice the LTI system (1), together with the constraint on u imposed by (2), can be rewritten in the alternative albeit equivalent form

$$\mathcal{H} \ : \ \begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} E \\ 0 \end{bmatrix} w(t), \quad \xi(0^-) = 0 \\ z(t) = \begin{bmatrix} C & D \end{bmatrix} \xi(t) \\ \xi(t_k) = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix} \xi(t_k^-) \end{cases} \tag{5}$$

which is valid for all $t \in [t_k, t_{k+1})$. Note that the equivalence between the *hybrid linear system* $\mathcal{H}$ given in (5) and the original one stems from the particular choice of the augmented state vector $\xi(t) = [x(t)' \ u(t)']'$.

3. HYBRID SYSTEMS ANALYSIS

Motivated by the previous formulation, we will analyse the following hybrid linear system with realisation

$$\mathcal{H} \ : \ \begin{cases} \dot{\xi}(t) = \mathcal{A}\xi(t) + \mathcal{E}w(t), \quad \xi(0^-) = 0 \\ z(t) = \mathcal{C}\xi(t) \\ \xi(t_k) = \mathcal{K}\xi(t_k^-) \end{cases} \tag{6}$$

which is valid for all $t \in [t_k, t_{k+1})$. For the moment, we assume the jump rate is constant, that is, $t_{k+1} - t_k = T > 0$ for all $k \in \mathbb{N}$. This assumption will be relaxed afterwards in order to cope with non uniform data-rates.

Note that the realisation defined in (6) reduces to the hybrid system described in (5) whenever the matrices are properly specified. Thus, it is of interest to analyse systems with this structure since the obtained results can be adapted to the sampled-data control problems to be considered. To this end, we first state the following theorem that provides a way to evaluate the $\mathcal{H}_2$ norm of the hybrid system $\mathcal{H}$.

Theorem 1. If there exists a positive definite matrix $S > 0$ satisfying the two-point boundary value problem defined by the linear differential equation

$$\dot{P}(t) + \mathcal{A}'P(t) + P(t)\mathcal{A} + \mathcal{C}'\mathcal{C} = 0 \tag{7}$$

together with the initial $P(0) = S^{-1}$ and final $P(T) > \mathcal{K}'S^{-1}\mathcal{K}$ conditions, then the hybrid linear system $\mathcal{H}$ with realisation (6) is asymptotically stable and is such that

$$\|\mathcal{H}\|_2^2 < \mathbf{tr}(\mathcal{E}'\mathcal{K}'S^{-1}\mathcal{K}\mathcal{E}) \tag{8}$$

Proof: First we set $\xi(0^-) = \xi_0$ and $w \equiv 0$ in (6) and define the following quadratic function

$$V(\xi(t)) = \xi(t)'P(t)\xi(t) \tag{9}$$

for all $t \in [t_k, t_{k+1}), \forall k \in \mathbb{N}$. Since (7) is time-invariant then its solution in the first time interval $[0, T)$ remains the same in the subsequent ones provided that we set $P(t_k) = P(0)$ and $P(t_{k+1}^-) = P(T)$ for all $k \geq 1$. Since $P(t)$ is not assumed to be positive definite for all $t \in [t_k, t_{k+1}), \forall k \in \mathbb{N}$ then $V(\cdot)$ cannot be considered as a Lyapunov function candidate associated to the hybrid system $\mathcal{H}$. Nevertheless, it is essential to evaluate the norm indicated in (8). Indeed, at any time instant $t \in (t_k, t_{k+1})$, its time derivative satisfies

$$\begin{aligned} \dot{V}(\xi(t)) &= \xi(t)' \left(\dot{P}(t) + \mathcal{A}'P(t) + P(t)\mathcal{A} \right) \xi(t) \\ &= -z(t)'z(t) \end{aligned} \tag{10}$$

where we have used the fact that $P(\cdot)$ is the solution of (7). Moreover, simple integration of both sides of this equality with respect to $t \in [t_k, t_{k+1})$ provides

$$\int_{t_k}^{t_{k+1}} z(t)'z(t)dt = V(\xi(t_k)) - V(\xi(t_{k+1}^-)) \tag{11}$$

Thus, if we consider the initial and final conditions, we can readily verify that $V(\xi(t_k)) = \xi(t_k)'S^{-1}\xi(t_k)$ and taking into account the discontinuity of $\xi(t)$ at the time instants t_{k+1}^- and t_{k+1}, we have

$$\begin{aligned} V(\xi(t_{k+1}^-)) &= \xi(t_{k+1}^-)'P(t_{k+1}^-)\xi(t_{k+1}^-) \\ &> \xi(t_{k+1}^-)'\mathcal{K}'S^{-1}\mathcal{K}\xi(t_{k+1}^-) \\ &> \xi(t_{k+1})'S^{-1}\xi(t_{k+1}) \end{aligned} \tag{12}$$

Hence, considering these relations, we define the positive definite quadratic function $v(\cdot) > 0$, given by $v(\xi(t_k)) = \xi(t_k)'S^{-1}\xi(t_k)$, and obtain

$$v(\xi(t_{k+1})) - v(\xi(t_k)) < -\int_{t_k}^{t_{k+1}} z(t)'z(t)dt < 0 \tag{13}$$

which implies that $v(\cdot)$ is a valid Lyapunov function associated to the discrete-time process $\xi(t_k) \to \xi(t_{k+1}), \forall k \in \mathbb{N}$. Hence, $v(\xi(t_{k+1})) \to 0$ as $k \to \infty$ and it follows that

$$\begin{aligned} \int_0^\infty z(t)'z(t)dt &= \sum_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} z(t)'z(t)dt \\ &< \sum_{k\in\mathbb{N}} (v(\xi(t_k)) - v(\xi(t_{k+1}))) \\ &< v(\xi(0)) = \xi_0'\mathcal{K}'S^{-1}\mathcal{K}\xi_0 \end{aligned} \tag{14}$$

since $\xi(0) = \mathcal{K}\xi_0$. The $\mathcal{H}_2$ norm is determined considering the system (6) with $\xi(0^-) = \xi(0) = 0$ and $w(t) = \delta(t^-)e_i$. The effect caused by the impulsive disturbances on the system and by the discontinuity at $t = 0^-$ is the same as the one induced by the initial conditions $\xi(0) = \mathcal{K}\mathcal{E}e_i$, for $i = 1, \cdots, p$, which, together with the inequality (14), yields (8) and that completes the proof. □

We notice that the solution of the linear differential matrix equation (7) follows immediately without difficulty and linearity with respect to the matrix variable $S^{-1} > 0$ is observed. In the following theorem, we state conditions that allow us to determine the $\mathcal{H}_\infty$ norm of $\mathcal{H}$.

Theorem 2. If there exists a positive definite matrix $S > 0$ satisfying the two-point boundary value problem defined by the nonlinear differential equation

$$\dot{P}(t) + \mathcal{A}'P(t) + P(t)\mathcal{A} + \gamma^{-2}P(t)\mathcal{E}\mathcal{E}'P(t) + \mathcal{C}'\mathcal{C} = 0 \tag{15}$$

together with the initial $P(0) = S^{-1}$ and final $P(T) > \mathcal{K}'S^{-1}\mathcal{K}$ conditions, then the hybrid linear system $\mathcal{H}$ with realisation (6) is asymptotically stable and is such that

$$\|\mathcal{H}\|_\infty^2 < \gamma^2 \tag{16}$$

Proof: The proof is similar to that of Theorem 1. □

Once again the matrix differential equation (15) is time invariant which simplifies the determination of a solution valid in all time interval of the form $[kT, (k+1)T), \forall k \in \mathbb{N}$, even though it is nonlinear. For the moment we want to stress that both theorems stated in this section provide important results on hybrid linear systems analysis.

4. $\mathcal{H}_2$ SAMPLED-DATA CONTROL

In this section, our main purpose is to solve the $\mathcal{H}_2$ optimal control problem **(P1)** and, for the moment, we still assume that the intersampling time interval is constant, that is, $t_{k+1} - t_k = T > 0$. To this end, we consider the conditions stated in Theorem 1 and observe that

$$P(t) = e^{-\mathcal{A}'t}P(0)e^{-\mathcal{A}t} - \int_0^t e^{\mathcal{A}'(\tau-t)}\mathcal{C}'\mathcal{C}e^{\mathcal{A}(\tau-t)}d\tau \tag{17}$$

valid for $t \in [0, T]$, solves the differential equation (7); see [Abou-Kandil et al., 2003] for details. Thus, (17) together with the initial and final consitions, yield

$$e^{\mathcal{A}'T}\mathcal{K}'S^{-1}\mathcal{K}e^{\mathcal{A}T} < S^{-1} - R_T \tag{18}$$

where [1]

$$R_T = \int_0^T e^{\mathcal{A}'\tau}\mathcal{C}'\mathcal{C}e^{\mathcal{A}\tau}d\tau > 0 \tag{19}$$

Therefore, remembering the realisation $(\mathcal{A}, \mathcal{E}, \mathcal{C}, \mathcal{K})$ is block structured as depicted in (5), our main goal is to solve the optimisation problem

$$\inf_{K,S>0} \left\{ \mathbf{tr}(\mathcal{E}'\mathcal{K}'S^{-1}\mathcal{K}\mathcal{E}) \ : \ e^{\mathcal{A}'T}\mathcal{K}'S^{-1}\mathcal{K}e^{\mathcal{A}T} < S^{-1} - R_T \right\} \tag{20}$$

Surprisingly, (20) can be reformulated as a convex optimisation problem. The key observation to attain this goal is to introduce the new matrix variable $W > 0$ and split the constraint in (20) as $[I\ K']S^{-1}[I\ K']' < W^{-1}$ and $e^{\mathcal{A}'T}[I\ 0]'W^{-1}[I\ 0]e^{\mathcal{A}T} < S^{-1} - R_T$. Moreover, from (5), simple algebraic manipulations point out that

$$\mathcal{K}e^{\mathcal{A}T} = \begin{bmatrix} I \\ K \end{bmatrix} [\, A_T\ B_T \,] \tag{21}$$

where we readily identify the matrices $A_T = e^{AT}$ and $B_T = \int_0^T e^{A\tau}Bd\tau$, which are the classical ZOH discrete-time equivalent matrices [Chen and Francis, 1995, Anderson and Moore, 2007]. All these relations allow us to state the following theorem, that shows the constraints of the optimal control problem (20) can be expressed in terms of linear matrix inequalities.

Theorem 3. Problem (20) is feasible if, and only if, there exist positive definite symmetric matrices S, W and a matrix M of compatible dimensions such that the following LMIs

$$\begin{bmatrix} W & [\, W\ M' \,] \\ \bullet & S \end{bmatrix} > 0 \tag{22}$$

$$\begin{bmatrix} W - [\, A_T\ B_T \,]\, S \begin{bmatrix} A_T' \\ B_T' \end{bmatrix} & [\, A_T\ B_T \,]\, S \\ \bullet & R_T^{-1} - S \end{bmatrix} > 0 \tag{23}$$

hold.

Proof: First, let us prove the sufficiency. Assume that (22)-(23) hold. Setting $K = MW^{-1}$, we multiply both sides of (22) by $\mathbf{diag}(W^{-1}, I)$ and then calculate the Schur Complement of the last row and column, yielding

$$W^{-1} > [\, I\ K' \,]\, S^{-1} \begin{bmatrix} I \\ K \end{bmatrix} \tag{24}$$

[1] Matrix R_T is positive definite for $T > 0$ whenever the pair $(\mathcal{A}, \mathcal{C})$ is observable. It can be verified that this is always true provided that the pair (A, C) is observable, $C'D = 0$ and $D'D > 0$.

Moreover, (23) is equivalent to $R_T^{-1} > S > 0$ and

$$W > [\, A_T \;\; B_T \,] \left(S^{-1} - R_T\right)^{-1} \begin{bmatrix} A_T' \\ B_T' \end{bmatrix} \tag{25}$$

where we have used the Matrix Inversion Lemma (see [Meyer, 2000]) to get this inequality, which, together with (21) and (24), allows us to verify that

$$\begin{aligned} S^{-1} - R_T &> \begin{bmatrix} A_T' \\ B_T' \end{bmatrix} W^{-1} [\, A_T \;\; B_T \,] \\ &> e^{\mathcal{A}'T} \mathcal{K}' S^{-1} \mathcal{K} e^{\mathcal{A}T} \end{aligned} \tag{26}$$

and, thus, the constraints of the optimisation problem (20) are feasible. Conversely, let us assume that (18) is feasible for a given gain matrix $K \in \mathbb{R}^{m\times n}$ and a given symmetric matrix $S^{-1} > R_T > 0$. These assumptions allow us to write it as

$$S^{-1} - R_T > \begin{bmatrix} A_T' \\ B_T' \end{bmatrix} \Phi [\, A_T \;\; B_T \,] \tag{27}$$

where $\Phi = [I \;\; K']S^{-1}[I \;\; K']' > 0$. Therefore, simple algebraic manipulations allow us to rewrite it equivalently as

$$\Phi^{-1} > [\, A_T \;\; B_T \,] \left(S^{-1} - R_T\right)^{-1} \begin{bmatrix} A_T' \\ B_T' \end{bmatrix} \tag{28}$$

and we conclude that this inequality remains valid if we replace Φ^{-1} by $W = \Phi^{-1} - \epsilon I > 0$, with $\epsilon > 0$ sufficiently small, which gives (23). However, this also implies that $W^{-1} > \Phi$ and consequently

$$W > W\Phi W = [\, W \;\; M' \,]\, S^{-1} \begin{bmatrix} W \\ M \end{bmatrix}$$

where $M = KW$, reproduces the linear matrix inequality (22) and, thus, the proof is complete. □

This result is somewhat surprising because it shows that the feasible set of problem (20) is convex and so can be expressed by LMIs. Furthermore, the objective function that defines a valid upper bound to the $\mathcal{H}_2$ norm of the hybrid system $\mathcal{H}$ under consideration can also be written as a function of the new set of matrix variables introduced in Theorem 3. Indeed, from (5)-(6), (8) and (24) we obtain

$$\begin{aligned} \|\mathcal{S}\|_2^2 &< \mathbf{tr}(\mathcal{E}'\mathcal{K}'S^{-1}\mathcal{K}\mathcal{E}) \\ &< \mathbf{tr}\left(E'W^{-1}E\right) \end{aligned} \tag{29}$$

meaning that problem (20) reduces to the following convex programming problem

$$\inf_{S,W,M} \left\{ \mathbf{tr}(E'W^{-1}E) \; : \; (22) - (23) \right\} \tag{30}$$

Hence, all the results developed so far point out that the boundary value problem in Theorem 1 can be efficiently solved using the numerical methods available in the literature to date. This aspect is of particular importance since for any $T > 0$ we are able to solve problem **(P1)** exactly, that is, with no kind of approximation. Of course, constraint (2) is always satisfied and, in general, is responsible for the performance deterioration as $T > 0$ increases. It can be shown that problem (30) recovers the classical $\mathcal{H}_2$ optimal control problem in continuous-time whenever $T > 0$ becomes arbitrarily small.

Now we consider a more general problem, in which the control signal u, given in (2) is unevenly sampled. Indeed, the only available information is that the intersampling time intervals $T_k = t_{k+1} - t_k > 0, k \in \mathbb{N}$ are time-varying and bounded, that is, $T_k \in [T_\star, T^\star]$, where $T^\star > T_\star > 0$ are given by the designer. This problem has important practical implications since shared networked environments usually do not present uniform data rate [Hespanha et al., 2007, Wang and Liu, 2008]. To this end, we seek a symmetric matrix $S > 0$ and a robust state feedback gain K that solve the problem

$$\inf_{S,W,M} \left\{ \mathbf{tr}(E'W^{-1}E) \; : \; (22) - (23), \; \forall T \in [T_\star, T^\star] \right\} \tag{31}$$

It is clear that the nonlinear dependence of the constraints of problem (31) on the intersampling interval $T \in [T_\star, T^\star]$ makes this problem difficult to solve. However, continuity of the constraints with respect to T allow us to circumvent this difficulty by selecting a large enough number N of evenly spaced points T_i, $i = 1, \ldots, N$, in the interval $[T_\star, T^\star]$ and imposing the constraints of (31) to each $T = T_i$ simultaneously. This problem can be solved efficiently even for N very large, see [Boyd et al., 1994]. The following example validates the theoretical results developed so far in both contexts, namely, constant and time-varying intersampling time intervals.

Example 1. We consider the open-loop unstable sampled-data system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

already discussed in [Souza et al., 2013]. For $T = 0.5$ s, we apply the optimal $\mathcal{H}_2$ design conditions (30) and obtain the same state feedback gain $K = [2.3758 \;\; -1.3907]$, which yields the closed-loop performance $\|\mathcal{S}\|_2^2 = 17.5661$. We also observe that the optimal solution provided by these conditions coincides with the one provided in [Souza et al., 2013] for all $T > 0$ and, thus, the same behaviour of the optimal cost with respect to the sampling period values remains valid; that is, for $T \to 0$, the optimal continuous-time solution is generated and there are pathological sampling frequencies for which the system is uncontrollable [Chen and Francis, 1995, Seron et al., 1997]. Finally, for the robust controller design, we consider the interval $[T_\star, T^\star] = [200, 800]$ ms and apply the robust design conditions stated in (31), with $N = 200$. The robust state feedback gain provided is $K = [4.0766 \;\; -1.2187]$, which ensures the bound $\|\mathcal{S}\|_2^2 < 38.9648$ for the closed-loop system. This controller is then validated through time simulation. Indeed, we compute $2,000$ time simulations with uniformly distributed time-varying intersampling intervals $T_k \in [T_\star, T^\star]$, which yield a mean closed-loop $\mathcal{H}_2$ performance of 28.8998 and a worst case performance of 36.9778.

5. $\mathcal{H}_\infty$ SAMPLED-DATA CONTROL

Now we focus on the solution of the optimal control problem **(P2)**. To this end, we assume from now on the existence of a symmetric (not necessarily positive definite or stabilising) solution to the algebraic Riccati equation (ARE)

$$\mathcal{A}\bar{\mathcal{Q}} + \bar{\mathcal{Q}}\mathcal{A}' + \gamma^{-2}\mathcal{E}\mathcal{E}' + \bar{\mathcal{Q}}\mathcal{C}'\mathcal{C}\bar{\mathcal{Q}} = 0 \tag{32}$$

which is of the form $\bar{\mathcal{Q}} = \mathrm{diag}\{\bar{Q}, 0\}$ where $\bar{Q}$ is a symmetric solution to the ARE

$$A\bar{Q} + \bar{Q}A' + \gamma^{-2}EE' + \bar{Q}C'C\bar{Q} = 0 \tag{33}$$

and produces the closed-loop matrix

$$\bar{\mathcal{A}} = \mathcal{A} + \bar{\mathcal{Q}}\mathcal{C}'\mathcal{C} = \begin{bmatrix} \bar{A} & \bar{B} \\ 0 & 0 \end{bmatrix} \tag{34}$$

where $\bar{A} = A + \bar{Q}C'C$ and $\bar{B} = B + \bar{Q}C'D$. Finally, considering the LTI differential Lyapunov equation

$$\dot{\mathcal{Z}} + \bar{\mathcal{A}}'\mathcal{Z} + \mathcal{Z}\bar{\mathcal{A}} + \mathcal{C}'\mathcal{C} = 0 \tag{35}$$

which presents the following solution

$$\mathcal{Z}(t) = e^{-\bar{\mathcal{A}}'t}\mathcal{Z}(0)e^{-\bar{\mathcal{A}}t} - \int_0^t e^{-\bar{\mathcal{A}}'(t-\tau)}\mathcal{C}'\mathcal{C}e^{-\bar{\mathcal{A}}(t-\tau)}d\tau \tag{36}$$

for all $t \in [0, T]$, it follows that $P(t) = \left(\mathcal{Z}(t)^{-1} + \bar{\mathcal{Q}}\right)^{-1}$, $\forall t \in [0, T]$. Consequently, expressing this solution in terms of the initial condition $P(0) = S^{-1} > 0$, we obtain

$$P(T)^{-1} - \bar{\mathcal{Q}} = e^{\bar{\mathcal{A}}T}\left(\left(S - \bar{\mathcal{Q}}\right)^{-1} - \bar{R}_T\right)^{-1} e^{\bar{\mathcal{A}}'T} \tag{37}$$

where

$$\bar{R}_T = \int_0^T e^{\bar{\mathcal{A}}'\tau}\mathcal{C}'\mathcal{C}e^{\bar{\mathcal{A}}\tau}d\tau > 0 \tag{38}$$

At this point, we have to impose the final condition provided in Theorem 2. This goal is accomplished if we adopt the same strategy used to handle the $\mathcal{H}_2$ case by introducing an additional matrix variable $W > 0$ satisfying the constraint $[I\ K']S^{-1}[I\ K']' < W^{-1}$. Denoting

$$e^{\bar{\mathcal{A}}T} = \begin{bmatrix} \bar{A}_T & \bar{B}_T \\ 0 & I \end{bmatrix} \tag{39}$$

these algebraic manipulations are used to prove the next theorem. It states that the $\mathcal{H}_\infty$ sampled-data state-feedback control design problem

$$\inf_{K, \mathcal{S}>0, \gamma} \left\{\gamma^2 \ : \ P(T) > \mathcal{K}'\mathcal{S}^{-1}\mathcal{K}\right\} \tag{40}$$

can be converted to a convex programming problem expressed through LMIs. This is somewhat surprising due to the intricate dependence of the constraint with respect to the matrix decision variables.

Theorem 4. Problem (40) is feasible if, and only if, there exist positive definite symmetric matrices S, W and a matrix M of compatible dimensions such that the LMIs (22) and

$$\begin{bmatrix} W - \bar{Q} - \left[\ \bar{A}_T \ \ \bar{B}_T\ \right](S - \bar{\mathcal{Q}})\begin{bmatrix} \bar{A}_T' \\ \bar{B}_T' \end{bmatrix} & \left[\ \bar{A}_T \ \ \bar{B}_T\ \right](S - \bar{\mathcal{Q}}) \\ \bullet & \bar{R}_T^{-1} - (S - \bar{\mathcal{Q}}) \end{bmatrix} > 0 \tag{41}$$

hold.

Proof: For the sufficiency, let us assume that (22) and (41) are valid. Then, analogously to the $\mathcal{H}_2$ case, we set $K = MW^{-1}$ and, thus, (22) yields

$$W^{-1} > \left[\ I \ \ K'\ \right] S^{-1} \begin{bmatrix} I \\ K \end{bmatrix} \tag{42}$$

On the other hand, applying the Schur Complement with respect to the second row and column of inequality (41) it follows that $\bar{R}_T^{-1} + \bar{\mathcal{Q}} > S > 0$ and

$$W - \bar{Q} > \left[\ \bar{A}_T \ \ \bar{B}_T\ \right]\left((S - \bar{\mathcal{Q}})^{-1} - \bar{R}_T\right)^{-1}\begin{bmatrix} \bar{A}_T' \\ \bar{B}_T' \end{bmatrix} \tag{43}$$

However, from (39) we can factorise

$$\bar{Q} = \left[\ \bar{A}_T \ \ \bar{B}_T\ \right] e^{-\bar{\mathcal{A}}T}\bar{\mathcal{Q}}e^{-\bar{\mathcal{A}}'T}\begin{bmatrix} \bar{A}_T' \\ \bar{B}_T' \end{bmatrix} \tag{44}$$

which plugged into (43) and using (37) and (42) gives the constraint appearing in problem (40) therefore, sufficiency follows. Conversely, now we assume that the optimisation problem (40) is feasible for some feedback gain $K \in \mathbb{R}^{m\times n}$ fixed and $S > 0$ such that all indicated inverses exist. From these assumptions it is seen that (43) and consequently (41) hold provided that we choose $W > 0$ such that $W = \Phi^{-1} - \epsilon I > 0$ with $\epsilon > 0$ sufficiently small and $\Phi = [I\ K']\mathcal{S}^{-1}[I\ K']' > 0$. Finally, using the fact that $W^{-1} > \Phi > 0$ and taking $M = KW$, inequality (22) follows. Therefore, the necessity part follows and that completes the proof. □

The results stated in Theorem 4 are the $\mathcal{H}_\infty$ counterpart of the ones presented in Theorem 3. As before, it is remarkable that the feasible set for the optimal control problem is convex. Finally, putting all these relations together, the $\mathcal{H}_\infty$ optimal control problem **(P2)**

$$\inf_{S,W,M,\gamma} \left\{\gamma^2 \ : \ (22) - (41)\right\} \tag{45}$$

is formulated in terms of LMIs. It is clear that, for each $T > 0$ given the minimum value of γ must be determined by line search since all matrix data in the LMI (41) depends nonlinearly on this parameter. Similarly to the $\mathcal{H}_2$ case, it can be shown that, when $T \to 0^+$, the classical solution of the $\mathcal{H}_\infty$ continuous-time problem is recovered.

Now we generalise these results to cope with uneven intersampling time intervals $T_k = t_{k+1} - t_k > 0$, which are time-varying and bounded to the interval $[T_\star, T^\star]$, where the positive numbers $T_\star$ and $T^\star$ are provided by the designer. Thus, we seek for a robust state-feedback gain $K \in \mathbb{R}^{m\times n}$ and a symmetric matrix $S > 0$ optimal solution of the problem

$$\inf_{S,W,M,\gamma} \left\{\gamma^2 \ : \ (22) - (41), \ \forall T \in [T_\star, T^\star]\right\} \tag{46}$$

which, adopting the same reasoning we did before, can be handled computationally by selecting N evenly spaced points $T_i \in [T_\star, T^\star], i = 1, \ldots, N$, and imposing the constraint (41) to each $T = T_i$ simultaneously. Once again, if N is taken large enough, continuity of the constraints on the sampling period assures the validation of this strategy.

Example 2. We consider again the continuous-time linear system of Example 1, also analysed in [Souza et al., 2013] in the $\mathcal{H}_\infty$ setting. In that reference, the authors only provide optimal $\mathcal{H}_\infty$ conditions for piecewise constant exogenous disturbances, since they consider all input channels have limited bandwidth in their networked control scenario. In order to compare both $\mathcal{H}_\infty$ design strategies, Figure 1 shows the behaviour of three different $\mathcal{H}_\infty$ performance indexes with respect to the sampling period $T \in (0, 3]$. The dashed green curve reproduces the one presented in [Souza et al., 2013], which shows the optimal constrained closed-loop $\mathcal{H}_\infty$ norm, considering that only piecewise constant inputs are admissible. However, if we take the whole $\mathcal{L}_2$ space as the acceptable exogenous inputs, then the feedback gain provided by the conditions stated in [Souza et al., 2013] can be imposed in our design conditions and now yields the closed-loop $\mathcal{H}_\infty$ norm represented by the dot-dashed blue curve, which is clearly worse than its constrained counterpart. Moreover, one should note that the points in which the green curve attains its local minima are the ones in which the blue line attains its local maxima. Clearly these points are the ones in which the special structure of the inputs is exploited to its maximum, and, thus, these solutions are not robust to

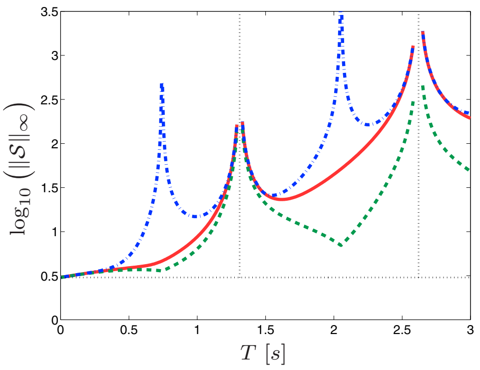

Fig. 1. $\mathcal{H}_\infty$ closed-loop performance.

cope with other classes of external disturbances. Finally, the red continuous line illustrates the optimal closed-loop $\mathcal{H}_\infty$ performance, obtained by solving the optimisation problem (45).

For the sake of illustration, we also consider $T = 0.5\,\mathrm{s}$, for which the optimal state feedback provided in [Souza et al., 2013] is $K_s = [1.1351 \;\; -2.9486]$, which can be evaluated with the conditions developed in this section, yielding a closed-loop performance of $\|\mathcal{S}\|_\infty = 5.2775$. We apply the design conditions stated in the optimisation problem (45) and obtain the optimal state-feedback gain $K_{opt} = [1.5614 \;\; -2.8168]$ with the associated closed-loop $\mathcal{H}_\infty$ performance $\|\mathcal{S}\|_\infty = 3.8751$. We also design a robust controller that provides a guaranteed $\mathcal{H}_\infty$ performance for any intersampling interval in $[T_\star, T^\star] = [200, 800]$ ms by solving problem (46), with $N = 200$ points in this interval. We obtain the robust feedback gain $K_{rob} = [4.6301 \;\; -1.1686]$, which ensures the guaranteed cost $\|\mathcal{S}\|_\infty < 14.3727$ for the closed-loop system.

6. CONCLUSION

This paper has addressed the optimal state-feedback control design problem for sampled-data systems. To this end, the closed-loop sampled-data system has been recast into a structured hybrid linear system, which is the basis for the statement of two theorems that allow us to evaluate the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ performances of hybrid linear systems. Both theorems have then been adapted to yield optimal control conditions, that have been reformulated as convex optimisation problems, whose solutions can be derived efficiently. Finally, both performance indexes have been considered for the design of robust controllers, in which the data-rate is time-varying, but bounded. Numerical examples illustrate the theoretical results.

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