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Output Control of Nonlinear Systems with Unmodelled Dynamics[1]

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Abstract: In this paper we consider the problem of output control of nonlinear systems in the presence of structural disturbances caused by so called unmodelled dynamics. This paper develops results published in Bobtsov (2002). In Bobtsov (2002) conditions of efficiency of consecutive compensator were found for the case of output stabilization of linear parametrically uncertain plant under conditions of unmodelled asymptotically stable dynamics. We added disturbances as smooth nonlinear function meeting the conditions of sector restriction to the model and synthesized regulator for this case.

Keywords: singular perturbations, nonlinear systems, feedback control, stability, control laws.

1. INTRODUCTION

In this paper we consider the problem of output control of nonlinear systems in the presence of structural disturbances caused by so called parasite dynamics. A system with unmodelled dynamics can be written in the following form

$$\dot{x} = f(x,t) + g(x,t)\nu \,, \tag{1}$$

$$\mu\dot{z} = \Gamma z + du(x,t), \quad \nu = c^T z \,, \tag{2}$$

where $x \in R^n$ is state vector of the model (1), $f(x,t) \in R^n$ and $g(x,t) \in R^n$ are vector functions, $u(x,t)$ is a scalar function, equation (2) describes unmodelled dynamics, μ is a small constant parameter, $z \in R^m$ is state vector of the model (2). Γ , d and c are constant matrix and vectors of corresponding dimensions. When μ turns into zero the structure (dimension) of the system (1), (2) changes, this kind of perturbations is called singular, and the systems are called singularly perturbed (Fradkov *et. al*. (1999)) or slow-fast systems (Berglung and Gentz, (2006)).

Typical singularly perturbed systems include direct-drive robots, flexible joint robots, flexible space structures, DC motors, flexible mechanical systems, tunnel diode circuits, airplane model, inverted pendulum and nonlinear time-invariant RLC networks see for instance Kokotovich et al (1999), Naidu (2002).

Nowadays there are many papers dedicated to problems of analysis and control of the systems in conditions of structural (singular) perturbations, see for instance Kokotovich et al, (1999), Naidu (2002), Sari and Lobri (2006), Gelig et. al. (1978), Wang and Sontag (2006a,b), (2007), (2008), Huang et. al. (2009), Mastellone et. al. (2007), Nguyen and Gajic (2010), (Fradkov and Andrievsky, 2004a,b), Fradkov (1987), Druzhinina et al. (1996), Gelig et. al. (1978). In the papers Wang and Sontag (2006a), (2006b), (2007), (2008) authors consider problems of stability of singularly perturbed monotonous systems. In the papers [Huang et. al., 2009, Mastellone et. al, 2007, Nguyen and Gajic, 2010) problems of control of linear singular perturbed systems are considered. In the papers (Fradkov and Andrievsky, 2004a,b) the task of control of nonlinear singularly perturbed systems is considered. These systems can be written in the following form

$$\dot{x}_1 = f_1(x_1, x_2, u, t)\,, \tag{3}$$

$$\mu\dot{x}_2 = f_2(x_1, x_2, u, t)\,, \tag{4}$$

where u is the control action, $x_1 \in R^{n_1}$ is the vector of slow variables, $x_2 \in R^{n_2}$ is the vector of fast variables, and $f_1(\cdot)$, $f_2(\cdot)$ are the vector functions of appropriate dimensions.

Despite active development of control of nonlinear singularly perturbed systems, problems of output control of nonlinear uncertain plants under conditions of influence of unmodelled dynamics are still open.

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This paper is dedicated to analysis of efficiency of output control law (method of consecutive compensator) considered in Bobtsov and Nikolaev, (2005), under conditions of unmodelled asymptotically stable dynamics. We will consider a nonlinear plant

$$\begin{cases} \dot{\chi}_1 = A_\chi \chi_1 + d_\chi \varphi(y) + b_\chi v, \\ y = c_\chi^{\mathrm{T}} \chi_1, \end{cases} \tag{5}$$

$$\begin{cases} \mu\dot{\chi}_2 = F_\chi \chi_2 + qu, \\ v = l^{\mathrm{T}} \chi_2, \end{cases} \tag{6}$$

where $\chi_1 \in R^n$ is state vector of the system (5); $\chi_2 \in R^r$ is state vector of the system (6); $y \in R$ is measured output; function $v \in R$ is not measured; $u \in R$ is control; A_χ, F_χ, b_χ, c_χ, d_χ, q and l are matrixes and vectors of proper dimensions; we suppose, as in (Fradkov *et. al*., 1999) that $-F_\chi l = q$; equation (6) defines asymptotically stable dynamics (i. e. matrix F_χ is Hurwitz) unmodelled for design of control law; number μ determines response speed of the system; $\varphi(y)$ is a smooth nonlinear function meeting the conditions of sector restriction of the view $|\varphi(y)| \le C|y|$, where number $C > 0$ is unknown.

The purpose of this work is to find the conditions which ensure stability of a system with controller (consecutive compensator), published in Bobtsov and Nikolaev (2005).

2. MAIN RESULT

Let us rewrite system (5), (6) according to Bobtsov, (2002) in the input-output form

$$a(p)y(t) = b(p)v(t) + g(p)\varphi(y), \tag{7}$$

$$d(p)v(t) = c(p)u(t), \tag{8}$$

where $p = d/dt$ is differentiation operator; output variable $y = y(t)$ is measured, but its derivatives are not measured; $b(s) = b_m s^m + \ldots + b_1 s + b_0$, $c(p) = d(0)$, $a(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$, $d(s) = d_r s^r + d_{r-1}s^{r-1} + \ldots + d_1 s + d_0$, $g(s) = g_e s^e + g_{e-1}s^{e-1} + \ldots + g_1 s + g_0$ are polynomials with unknown parameters; s is complex variable; $m \le n-1$; transfer function $\frac{b(s)}{a(s)}$ has relative degree of $\rho = n - m$; polynomial $b(s)$ is Hurwitz and coefficient $b_m > 0$.

Let us choose the following control law (Bobtsov and Nikolaev, 2005)

$$u = -(k+\gamma)\alpha(p)\xi_1, \tag{9}$$

$$\begin{cases} \dot{\xi}_1 = \sigma\xi_2, \\ \dot{\xi}_2 = \sigma\xi_3, \\ \ldots \\ \dot{\xi}_{\rho-1} = \sigma(-k_1\xi_1 - k_2\xi_2 - \ldots - k_{\rho-1}\xi_{\rho-1} + k_1 y), \end{cases} \tag{10}$$

where number $k > 0$ and polynomial $\alpha(p)$ of degree $\rho - 1$ are selected for the transfer function $H(s) = \frac{\alpha(s)b(s)}{a(s) + k\alpha(s)b(s)}$ to be strictly positively real, positive parameter γ is used for compensation of nonlinearity $\varphi(y)$, number $\sigma > k$, and coefficients k_i are counted for the system (10) to be asymptotically stable for zero input $y(t)$.

As we proved in Bobtsov and Nikolaev, (2005), technically realizable algorithm (9), (10) ensures exponential stability of the system (5), (6) for the case $\mu = 0$ (i. e. in the absence of unmodelled dynamics). But for $\mu > 0$ analytical conditions of the control law (9), (10) applicability were not considered in the paper Bobtsov and Nikolaev, (2005). So we have to find restrictions on numbers μ and σ for which the system (5)–(10) is exponentially stable. Let us make some transformations. Substituting (9) into (8), we obtain

$$\begin{aligned} v &= \frac{c(p)}{d(p)}(-(k+\gamma)\alpha(p)\xi_1) = -(k+\gamma)\alpha(p)\frac{c(p)}{d(p)}\xi_1 = \\ &= -(k+\gamma)\alpha(p)\hat{y} = -(k+\gamma)\alpha(p)(y - \varepsilon_1), \end{aligned} \tag{11}$$

where $\hat{y} = \frac{c(p)}{d(p)}\xi_1$ and $\varepsilon_1 = y - \hat{y}$.

Then for (7) we have

$$\begin{aligned} y &= \frac{b(p)}{a(p)}v + \frac{g(p)}{a(p)}\varphi(y) = \\ &= -(k+\gamma)\frac{\alpha(p)b(p)}{a(p)}(y - \varepsilon_1) + \frac{g(p)}{a(p)}\varphi(y) = \\ &= (k+\gamma)\frac{\alpha(p)b(p)}{a(p)+k\alpha(p)b(p)}\varepsilon_1 + \frac{g(p)}{a(p)+k\alpha(p)b(p)}\varphi(y) - \\ &\quad - \gamma\frac{\alpha(p)b(p)}{a(p)+k\alpha(p)b(p)}y. \end{aligned} \tag{12}$$

Now let us rewrite SISO model (12) in the form of MIMO model

$$\dot{x} = Ax + (k+\gamma)b\varepsilon_1 + g\varphi(y) - \gamma by, \tag{13}$$

$$y = c^T x, \tag{14}$$

where $x \in R^n$ is state of (13); A, b and c are matrixes of transformation from SISO model into MIMO model, and by virtue of Yakubovitch-Kalman lemma we can find symmetrical positively definite matrix P satisfying the two following matrix equations

$$A^T P + PA = -Q_1 , \quad Pb = c , \tag{15}$$

where $Q_1 = Q_1^T$ is a positively definite matrix.
Let us rewrite (10) and (11) in vector-matrix form

$$\dot{\xi} = \sigma(\Gamma \xi + dk_1 y) , \quad \xi_1 = h^T \xi , \tag{16}$$

$$\mu \dot{z} = Fz + q\xi_1 , \quad \hat{y} = l^T z , \tag{17}$$

where $\xi \in R^{\rho-1}$ and $z \in R^r$ are state vectors of the models (16) and (17); matrix $\Gamma = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & -k_2 & -k_3 & \dots & -k_{\rho-1} \end{bmatrix}$ is Hurwitz in virtue of calculation of parameters k_i of the model (10), $d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$, $h = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$; F, q and l are matrixes of transformation from the SISO model to the MIMO one, and following Fradkov *et. al*. (1999) we suppose that $-Fl = q$.
Let us consider deviation vectors

$$\eta_1 = ly - z , \tag{18}$$

$$\eta_2 = hy - \xi . \tag{19}$$

After differentiation of equations (18), (19), we obtain

$$\begin{aligned} \dot{\eta}_1 &= l\dot{y} - \mu^{-1} Fz - \mu^{-1} q\xi_1 = \\ &= l\dot{y} - \mu^{-1} F(ly - \eta_1) - \mu^{-1} q(y - \varepsilon_2) = \\ &= l\dot{y} + \mu^{-1} F\eta_1 + \mu^{-1} q\varepsilon_2 , \end{aligned} \tag{20}$$

$$\varepsilon_1 = y - \hat{y} = l^{\mathrm{T}} \eta_1 , \tag{21}$$

$$\begin{aligned} \dot{\eta}_2 &= h\dot{y} - \sigma(\Gamma(hy - \eta_2) + dk_1 y) = \\ &= h\dot{y} + \sigma\Gamma\eta_2 - \sigma(dk_1 + \Gamma h)y = \\ &= h\dot{y} + \sigma\Gamma\eta_2 , \end{aligned} \tag{22}$$

$$\varepsilon_2 = y - \xi_1 = h^{\mathrm{T}} \eta_2 , \tag{23}$$

where $dk_1 = -\Gamma h$ and $-Fl = q$.
So we have a system of differential equations

$$\dot{x} = Ax + (k + \gamma) b\varepsilon_1 + g\varphi(y) - \gamma by , \quad y = c^{\mathrm{T}} x , \tag{24}$$

$$\dot{\eta}_1 = l\dot{y} + \mu^{-1} F\eta_1 + \mu^{-1} q\varepsilon_2 , \quad \varepsilon_1 = l^{\mathrm{T}} \eta_1 , \tag{25}$$

$$\dot{\eta}_2 = h\dot{y} + \sigma\Gamma\eta_2 , \quad \varepsilon_2 = h^{\mathrm{T}} \eta_2 . \tag{26}$$

Positively definite matrixes $R = R^T$ and $N = N^T$ satisfy Lyapunov equations

$$F^{\mathrm{T}} R + RF = -Q_2 , \tag{27}$$

$$\Gamma^T N + N\Gamma = -Q_3 , \tag{28}$$

where $Q_2 = Q_2^T$ and $Q_3 = Q_3^T$ are positively definite matrixes.

Conditions of efficiency of the control law (9), (10) for stabilization of the system (5), (6), (24)–(26) are given by the following theorem.
Theorem. Let the control law (9), (10) is used for stabilization of the system (1), (2). Let number k ensures that the transfer function $H(s) = \dfrac{\alpha(s)b(s)}{a(s) + k\alpha(s)b(s)}$ is strictly positively real. Let positive numbers μ, γ and $0 \le \theta < 1$ meet the conditions

$$\begin{aligned} &-\mu^{-1}\eta_1^{\mathrm{T}} Q_2 \eta_1 + \delta^{-1} k^2 (l^{\mathrm{T}}\eta_1)^2 + (\eta_1^{\mathrm{T}} Rq)^2 + \delta^{-1}(\eta_1^{\mathrm{T}} Rl)^2 + \\ &+ (k+\gamma)(\eta_1^{\mathrm{T}} Rlc^{\mathrm{T}} b)^2 + (k+\gamma)(l^{\mathrm{T}}\eta_1)^2 + \delta(l^{\mathrm{T}}\eta_1)^2 + \\ &+ 4\gamma^{-1} C^2 (\eta_1^{\mathrm{T}} Rlc^{\mathrm{T}} g)^2 + \gamma(\eta_1^{\mathrm{T}} Rlc^{\mathrm{T}} b)^2 < 0 , \end{aligned} \tag{29}$$

$$\begin{aligned} &-x^{\mathrm{T}} Q_1 x + \delta x^{\mathrm{T}} Pbb^{\mathrm{T}} Px + 2\delta(c^{\mathrm{T}} Ax)^2 + 4C^2\gamma^{-1}(x^{\mathrm{T}} Pg)^2 \le \\ &\le -x^{\mathrm{T}} Qx < 0 , \qquad \delta = \delta(\mu) = \mu^\theta , \end{aligned} \tag{30}$$

for all $x \ne 0$ and $\eta_1 \ne 0$.

Then for all σ, meeting the inequality

$$\begin{aligned} &-\sigma\eta_2^{\mathrm{T}} Q_3 \eta_2 + \mu^{-2}(h^{\mathrm{T}}\eta_2)^2 + \\ &+ \delta^{-1}(\eta_2^{\mathrm{T}} Nh)^2 + 4\gamma(\eta_2^{\mathrm{T}} Nhc^{\mathrm{T}} b)^2 + \\ &+ 4C^2\gamma^{-1}(\eta_2^{\mathrm{T}} Nhc^{\mathrm{T}} g)^2 + \\ &+ \delta^{-1}(k+\gamma)^2(\eta_2^{\mathrm{T}} Nhc^{\mathrm{T}} b)^2 \le -\eta_2^{\mathrm{T}} Q\eta_2 < 0 \end{aligned} \tag{27}$$

and for $\eta_2 \ne 0$ the system (5)–(10) is exponentially stable.
Proof of the theorem can be found in Appendix.

Remark. Let us notice that conditions of the theorem are not conflicting. If we pass from inequalities for quadratic forms (29), (30) and (31) on to inequalities for eigenvalues of the corresponding matrixes and making some simple transformations, from (29) and (30) we obtain

$$\begin{aligned} \mu^{-1} > \frac{1}{\lambda_{\min}\{Q_2\}} \Big(& l^{\mathrm{T}} l(k^2\delta^{-1} + k + \gamma + \delta) + l^{\mathrm{T}} R^2 l(\delta^{-1} + \\ & + (k + 2\gamma)(c^{\mathrm{T}} b)^2) + q^{\mathrm{T}} R^2 q + 4\gamma^{-1} C^2 (l^{\mathrm{T}} R^2 l)(c^{\mathrm{T}} g)^2 \Big), \end{aligned} \tag{32}$$

$$\mu^\theta = \delta < \frac{\lambda_{\min}\{Q_1\} - 4C^2\gamma^{-1} g^{\mathrm{T}} P^2 g}{b^{\mathrm{T}} P^2 b + 2c^{\mathrm{T}} AA^{\mathrm{T}} c} , \tag{33}$$

and for (31) we have

$$\begin{aligned} \sigma > \frac{1}{\lambda_{\min}\{Q_3\}} \Big[& \mu^{-2} + \delta^{-1}(h^{\mathrm{T}} N^2 h)(1 + (k+\gamma)^2 (c^{\mathrm{T}} b)^2) + \\ & + (h^{\mathrm{T}} Nh)^2 [4\gamma(c^{\mathrm{T}} b)^2 + 4C^2\gamma^{-1}(c^{\mathrm{T}} g)^2] \Big]. \end{aligned} \tag{34}$$

Indeed, it is easy to see that if we multiply (32) and (33) by μ and trend μ to zero then the inequalities (32) and (33) are correct, and conditions of the theorem are not contradictory.

At the same time, analysis of condition (34) allows us to conclude that number σ should be increased to ensure exponential stability of the system (5), (6).

3. EXAMPLE

The designed algorithms can be used for such singularly perturbed plants as pendulum systems, single-link robots, driven with DC motor. In these cases transients of the current circuit of the DC motor are considered as parasite (unmodelled) dynamics (Mastellone et. al. (2007), Fradkov and Andrievsky (2004), Li and Lin (2007), Tsai et. al. (2006), Cao and Hovakimyan (2007), Yang and Zhang (2009), Yu et. al. (2009)).

Let us consider stabilization of unbalanced rotor to show efficiency of compensator under conditions of unmodelled dynamics (Andrievsky et al., 2001). We have the following equation of unbalanced rotor dynamics taking into account optimized current circuit of an electric drive

$$J\ddot{\varphi} + \bar{k}\dot{\varphi} + mg\,\vartheta\sin\varphi = k_M I\ , \tag{35}$$

$$T_T\dot{I} + I = k_e u\ . \tag{36}$$

where J, ϑ are inertia moment and eccentricity of rotor; m is debalance mass; $\bar{k}$ is friction coefficient of bearing (as in (Andrievsky et al., 2001) we suppose that $\bar{k} = 0$); φ is angle of debalance deflection from vertical axes; I is current of rotor circuit; k_M, k_e are construction parameters of the drive; T_T is time constant of optimized current circuit; u is control (voltage).

Let us write system (31), (32) in the form of (1), (2)

$$\dot{\chi}_{11} = \chi_{12}\ , \tag{37}$$

$$\dot{\chi}_{12} = -a_{21}\sin\chi_{11} + a_{23}v\ , \tag{38}$$

$$y = \chi_{11}\ , \tag{39}$$

$$\mu\dot{\chi}_{21} = F_\chi \chi_{21} + qu\ , \tag{40}$$

$$v = \chi_{11}\ , \tag{41}$$

where $a_{21} = mg\,\vartheta / J$, $a_{23} = k_M / J$, $\mu = T_T$, $q = k_e$ and $F_\chi = -1$.

As in (Andrievsky et al., 2001) we assume $a_{21} = 33$ and $a_{23} = 0{,}17$ and simulate system (37)–(38) for $\chi_{11}(0) = 2$, $\chi_{12}(0) = 0$. Results of computer simulation for $u = 0$ are presented in figure 1.

Let us now choose control law in the view (9), (10)

$$u = -(k+\gamma)\alpha(p)\xi_1\ , \tag{42}$$

$$\dot{\xi}_1 = \sigma k_1(y - \xi_1)\ , \tag{43}$$

where by virtue of $\rho = 2$ degree of polynomial $\alpha(p)$ is equal to one ($\alpha(p) = p+1$).

First let us simulate the system (37)–(43) for $\mu = 0$, $q = 1$, and $k = 5$, $\gamma = 10$, $\sigma = 1000$ (figure 2) (i. e. we do not take into account parasite dynamics). Figure 3 shows simulation results for $\mu = 0{,}1$ and $q = 1$ (here we take into account parasite dynamics).

4. CONCLUSIONS

In this paper we analyzed efficiency of control law (9), (10) for stabilization of the system (5), (6). Control method (consecutive compensator) published in (Bobtsov, 2002) was shown to be used successfully for stabilization of nonlinear parametrically and functionally uncertain plant under correctness of conditions (32), (33). Realization of the condition (32) was substantiated in the remark. It is worth to note that conditions (29), (30) or (32), (33) are difficult to check in practice, at least, under assumption of full parametrical uncertainty of the plant. But for some known region of parameter change these evaluations esquire practical sense. We also want to note that this result, like many others in science, may become auxiliary. For instance, this analysis can be used for design of control laws for parameter uncertain systems with inexactly given relative degree. Practical example of unbalanced rotor stabilization was considered to illustrate efficiency of consecutive compensator method under conditions of unmodelled dynamics.

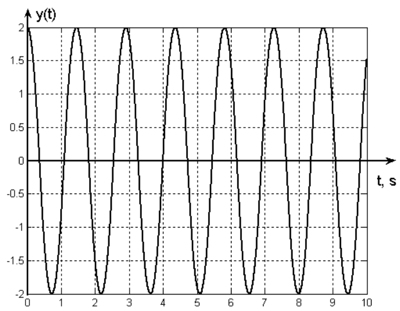

Fig. 1. Simulation results for $u = 0$

Simulation results (figure 3) show that account of unmodelled dynamics reduce quality of transients (response speed and oscillation increase), but the closed-loop system remains stable.

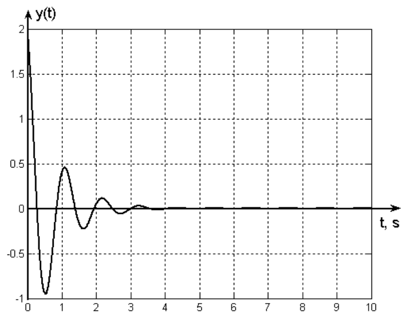

Fig. 2. Simulation results for $\mu = 0$

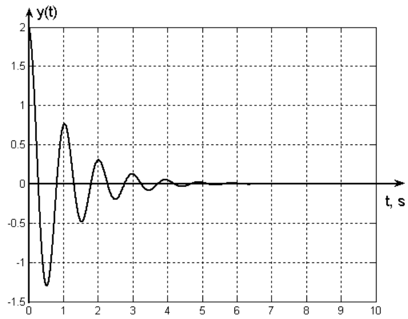

Fig. 3. Simulation results for $\mu = 0{,}1$

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Appendix A

Proof of the theorem. Consider Lyapunov function of the following view

$$V = x^{\mathrm{T}} P x + \eta_1^{\mathrm{T}} R \eta_1 + \eta_2^{\mathrm{T}} N \eta_2 \,. \qquad \text{(A.1)}$$

Differentiation of (A.1) with respect to equations (24) – (26) allows us to write

$$\begin{aligned}\dot V =\ & x^{\mathrm T}(A^{\mathrm T}P+PA)x+2(k+\gamma)x^{\mathrm T}Pbl^{\mathrm T}\eta_1-\\ & -2\gamma x^{\mathrm T}Pby+2x^{\mathrm T}Pg\varphi(y)+\\ & +\mu^{-1}\eta_1^{\mathrm T}(F^{\mathrm T}R+RF)\eta_1+2\mu^{-1}\eta_1^{\mathrm T}Rqh^{\mathrm T}\eta_2+\\ & +2\eta_1^{\mathrm T}Rlc^{\mathrm T}Ax+\\ & +2(k+\gamma)\eta_1^{\mathrm T}Rlc^{\mathrm T}bl^{\mathrm T}\eta_1+\\ & +2\eta_1^{\mathrm T}Rlc^{\mathrm T}g\varphi(y)-2\gamma\eta_1^{\mathrm T}Rlc^{\mathrm T}by+\\ & +\eta_2^{\mathrm T}\sigma(\Gamma^{\mathrm T}N+N\Gamma)\eta_2+2\eta_2^{\mathrm T}Nhc^{\mathrm T}Ax+\\ & +2(k+\gamma)\eta_2^{\mathrm T}Nhc^{\mathrm T}bl^{\mathrm T}\eta_1+\\ & +2\eta_2^{\mathrm T}Nhc^{\mathrm T}g\varphi(y)-2\gamma\eta_2^{\mathrm T}Nhc^{\mathrm T}by\ , \end{aligned}\tag{A.2}$$

where item $\dot y = c^{\mathrm T}(Ax+(k+\gamma)bl^{\mathrm T}\eta_1+g\varphi(y)-\gamma by)$ was used instead of $\dot y$.

Let us substitute equations (15), (27) and (28) into (A.2) and take into account the following expressions

$$2x^{\mathrm T}Pg\varphi(y)\le 4C^2\gamma^{-1}(x^{\mathrm T}Pg)^2+\frac14 C^{-2}\gamma(\varphi(y))^2\le$$
$$\le 4C^2\gamma^{-1}(x^{\mathrm T}Pg)^2+\frac14\gamma y^2,$$
$$2(k+\gamma)x^{\mathrm T}Pbl^{\mathrm T}\eta_1\le \delta x^{\mathrm T}Pbb^{\mathrm T}Px+\delta^{-1}(k+\gamma)^2(l^{\mathrm T}\eta_1)^2,$$
$$2\mu^{-1}\eta_1^{\mathrm T}Rqh^{\mathrm T}\eta_2\le(\eta_1^{\mathrm T}Rq)^2+\mu^{-2}(h^{\mathrm T}\eta_2)^2,$$
$$2\eta_1^{\mathrm T}Rlc^{\mathrm T}Ax\le\delta^{-1}(\eta_1^{\mathrm T}Rl)^2+\delta(c^{\mathrm T}Ax)^2,$$
$$2(k+\gamma)\eta_1^{\mathrm T}Rlc^{\mathrm T}bl^{\mathrm T}\eta_1\le(k+\gamma)(\eta_1^{\mathrm T}Rlc^{\mathrm T}b)^2+(k+\gamma)(l^{\mathrm T}\eta_1)^2,$$
$$2\eta_2^{\mathrm T}Nhc^{\mathrm T}Ax\le\delta(c^{\mathrm T}Ax)^2+\delta^{-1}(\eta_2^{\mathrm T}Nh)^2,$$
$$2(k+\gamma)\eta_2^{\mathrm T}Nhc^{\mathrm T}bl^{\mathrm T}\eta_1\le\delta^{-1}(k+\gamma)^2(\eta_2^{\mathrm T}Nhc^{\mathrm T}b)^2+\delta(l^{\mathrm T}\eta_1)^2,$$
$$2\eta_2^{\mathrm T}Nhc^{\mathrm T}g\varphi(y)\le 4C^2\gamma^{-1}(\eta_2^{\mathrm T}Nhc^{\mathrm T}g)^2+\frac14\gamma C^{-2}(\varphi(y))^2\le$$
$$\le 4C^2\gamma^{-1}(\eta_2^{\mathrm T}Nhc^{\mathrm T}g)^2+\frac14\gamma y^2,$$
$$-2\gamma\eta_2^{\mathrm T}Nhc^{\mathrm T}by\le 4\gamma(\eta_2^{\mathrm T}Nhc^{\mathrm T}b)^2+\frac14\gamma y^2,$$
$$2\eta_1^{\mathrm T}Rlc^{\mathrm T}g\varphi(y)\le$$
$$\le 4\gamma^{-1}C^2(\eta_1^{\mathrm T}Rlc^{\mathrm T}g)^2+\frac14\gamma C^{-2}(\varphi(y))^2\le$$
$$\le 4\gamma^{-1}C^2(\eta_1^{\mathrm T}Rlc^{\mathrm T}g)^2+\frac14\gamma y^2,$$
$$-2\gamma\eta_1^{\mathrm T}Rlc^{\mathrm T}by\le\gamma(\eta_1^{\mathrm T}Rlc^{\mathrm T}b)^2+\gamma y^2,$$

where $|\varphi(y)|\le C|y|$.

Then for derivative of Lyapunov function (A.1) we obtain

$$\begin{aligned}\dot V\le\ & -x^{\mathrm T}Q_1x-\mu^{-1}\eta_1^{\mathrm T}Q_2\eta_1-\\ & -\sigma\eta_2^{\mathrm T}Q_3\eta_2-2\gamma y^2+\\ & +\delta x^{\mathrm T}Pbb^{\mathrm T}Px+\delta^{-1}k^2(l^{\mathrm T}\eta_1)^2+\\ & +(\eta_1^{\mathrm T}Rq)^2+\mu^{-2}(h^{\mathrm T}\eta_2)^2+\\ & +\delta^{-1}(\eta_1^{\mathrm T}Rl)^2+\delta(c^{\mathrm T}Ax)^2+\\ & +(k+\gamma)(\eta_1^{\mathrm T}Rlc^{\mathrm T}b)^2+(k+\gamma)(l^{\mathrm T}\eta_1)^2+\\ & +\delta(c^{\mathrm T}Ax)^2+\delta^{-1}(\eta_2^{\mathrm T}Nh)^2+\\ & +\delta^{-1}(k+\gamma)^2(\eta_2^{\mathrm T}Nhc^{\mathrm T}b)^2+\delta(l^{\mathrm T}\eta_1)^2+\\ & +4C^2\gamma^{-1}(x^{\mathrm T}Pg)^2+\\ & +4C^2\gamma^{-1}(\eta_2^{\mathrm T}Nhc^{\mathrm T}g)^2+\\ & +4\gamma(\eta_2^{\mathrm T}Nhc^{\mathrm T}b)^2+2\gamma y^2+\\ & +4\gamma^{-1}C^2(\eta_1^{\mathrm T}Rlc^{\mathrm T}g)^2+\gamma(\eta_1^{\mathrm T}Rlc^{\mathrm T}b)^2,\end{aligned}\tag{A.3}$$

where $\delta>0$ is a number.

Let $\delta=\delta(\mu)=\mu^{\theta}$ and $0\le\theta<1$, then for some small $\mu>0$ and large γ there can be found a positively definite matrix $Q=Q^{\mathrm T}$ such that

$$\begin{aligned}& -x^{\mathrm T}Q_1x+\delta x^{\mathrm T}Pbb^{\mathrm T}Px+\\ & +2\delta(c^{\mathrm T}Ax)^2+4C^2\gamma^{-1}(x^{\mathrm T}Pg)^2\le\\ & \le -x^{\mathrm T}Qx<0\ .\end{aligned}\tag{A.4}$$

Let us choose σ such a way that inequality

$$\begin{aligned}& -\sigma\eta_2^{\mathrm T}Q_3\eta_2+\mu^{-2}(h^{\mathrm T}\eta_2)^2+\\ & +\delta^{-1}(\eta_2^{\mathrm T}Nh)^2+4\gamma(\eta_2^{\mathrm T}Nhc^{\mathrm T}b)^2+\\ & +4C^2\gamma^{-1}(\eta_2^{\mathrm T}Nhc^{\mathrm T}g)^2+\\ & +\delta^{-1}(k+\gamma)^2(\eta_2^{\mathrm T}Nhc^{\mathrm T}b)^2\le-\eta_2^{\mathrm T}Q\eta_2<0\end{aligned}\tag{A.5}$$

holds. Then for inequality (A.3) we obtain

$$\begin{aligned}\dot V\le\ & -x^{\mathrm T}Qx-\mu^{-1}\eta_1^{\mathrm T}Q_2\eta_1-\eta_2^{\mathrm T}Q\eta_2+\\ & +\delta^{-1}k^2(l^{\mathrm T}\eta_1)^2+(\eta_1^{\mathrm T}Rq)^2+\delta^{-1}(\eta_1^{\mathrm T}Rl)^2+\\ & +(k+\gamma)(\eta_1^{\mathrm T}Rlc^{\mathrm T}b)^2+(k+\gamma)(l^{\mathrm T}\eta_1)^2+\delta(l^{\mathrm T}\eta_1)^2+\\ & +4\gamma^{-1}C^2(\eta_1^{\mathrm T}Rlc^{\mathrm T}g)^2+\gamma(\eta_1^{\mathrm T}Rlc^{\mathrm T}b)^2\ .\end{aligned}\tag{A.6}$$

Let the number $\mu>0$ be such that

$$\begin{aligned}& -\mu^{-1}\eta_1^{\mathrm T}Q_2\eta_1+\delta^{-1}k^2(l^{\mathrm T}\eta_1)^2+(\eta_1^{\mathrm T}Rq)^2+\delta^{-1}(\eta_1^{\mathrm T}Rl)^2+\\ & +(k+\gamma)(\eta_1^{\mathrm T}Rlc^{\mathrm T}b)^2+(k+\gamma)(l^{\mathrm T}\eta_1)^2+\delta(l^{\mathrm T}\eta_1)^2+\\ & +4\gamma^{-1}C^2(\eta_1^{\mathrm T}Rlc^{\mathrm T}g)^2+\gamma(\eta_1^{\mathrm T}Rlc^{\mathrm T}b)^2\\ & \le-\eta_1^{\mathrm T}Q\eta_1<0\ .\end{aligned}\tag{A.7}$$

Then inequality (A.6) takes the form

$$\begin{aligned}\dot V & \le -x^{\mathrm T}Qx-\eta_1^{\mathrm T}Q\eta_1-\eta_2^{\mathrm T}Q\eta_2\le\\ & \le-\lambda_{\min}\{Q\}(|x|^2+|\eta_1|^2+|\eta_2|^2)\ .\end{aligned}\tag{A.8}$$

From (A.8) we conclude that system (24) – (26) is exponentially stable, and hence, system (5) – (10) is exponentially stable which was to be proved.