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Adaptive Output-Feedback Stabilization of Non-Local Hyperbolic PDEs ⋆

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Abstract: We address the problem of adaptive output-feedback stabilization of general first-order hyperbolic partial integro-differential equations (PIDE). Such systems are also referred to as PDEs with non-local (in space) terms. We apply control at one boundary, take measurements on the other boundary, and allow the system's functional coefficients to be unknown. To deal with the absence of both full-state measurement and parameter knowledge, we introduce a pre-transformation (which happens to be based on backstepping) of the system into an observer canonical form. In that form, the problem of adaptive observer design becomes tractable. Both the parameter estimator and the control law employ only the input and output signals (and their histories over one unit of time). Prior to presenting the adaptive design, we present the non-adaptive/baseline controller, which is novel in its own right and facilitates the understanding of the more complex, adaptive system. The parameter estimator is of the gradient type, based on a parametric model in the form of an integral equation relating delayed values of the input and output. For the closed-loop system we establish boundedness of all signals, pointwise in space and time, and convergence of the PDE state to zero pointwise in space. We illustrate our result with a simulation.

1. INTRODUCTION

Much attention has been dedicated in recent years to hyperbolic PDEs and to their stabilization (Coron et al. [2007], Coron et al. [2008], Bastin and Coron [2011]). In this paper, we focus on the stabilization of a general first-order hyperbolic PIDE, where the state is controlled at one boundary (input), and measured at the other (output). Our work's novelty is in how little knowledge we require to stabilize the system: the state is measured at only one boundary, and we allow the system's functional coefficients to be unknown. The key to our result is our introduction of an "observer canonical form" for this class of systems, which enables the design of an adaptive observer for stabilization of the system.

Despite a growing number of publications on the topic of boundary control of hyperbolic PDEs, stabilization by adaptive output feedback has never been attempted. The backstepping method, introduced in Smyshlyaev and Krstic [2004] for parabolic systems, has seen use in increasingly complex systems of coupled hyperbolic PDEs (Vazquez et al. [2012],Meglio et al. [2012], and Meglio et al. [2013], Coron et al. [2013]), as well as in Krstic and Smyshlyaev [2008] and Krstic [2009] for the hyperbolic PIDE that we tackle here.

We provide in this paper two novel contributions: a new output-feedback controller to face the absence of full-state measurement and, more importantly, the output-feedback controller's adaptive version for the case of unknown parameters.

The key new ingredient in our approach lies in the use of backstepping to transform the system into an observer canonical form, analogous to the transform used in Smyshlyaev and Krstic [2010] to transform parabolic PIDEs into a parabolic observer canonical form. Unlike the original plant in which a product of unknown coefficients and unmeasured state appears, our pre-transformation leads to a system structure in which only one infinite-dimension parameter is unknown but is multiplied by the measured output, making simultaneous state and parameter estimation feasible.

For parameter estimation we use a gradient-based update law similar to those employed in Smyshlyaev and Krstic [2007], which differ from Lyapunov-based update laws developed in Krstic and Smyshlyaev [2005], and then in Krstic and Bresch-Pietri [2009], Bresch-Pietri and Krstic [2009], Bresch-Pietri and Krstic [2010] and Bresch-Pietri et al. [2012] to estimate delays or unknown parameters. This gradient update law is obtained via a parametric model in the form of an integral equation relating delayed values of the input and output. The use of projection enables to keep the estimated parameter within an a-priori bound, which we assume known.

As for the problem of state estimation, it was already addressed in Vazquez et al. [2011] for a 2 x 2 hyperbolic linear system, through the design of a collocated boundary observer. In our paper, however, we present an explicit state observer employing the delayed values of both the

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input and the output over one unit of time, which enables us to design an output-feedback controller. Associated with the parameter estimation, and using the certainty equivalence principle, we get an adaptive output-feedback controller which achieves pointwise-in-space convergence of the PDE state to zero. All signals are established to be bounded pointwise in space and time.

Integral equations play prominent roles in our development. One of the plant's representations, the parametric model for the parameter estimator design, the control law, and the control gain kernel are all governed by integral equations. The relation between hyperbolic PDE systems and integral delay equations was recently thoroughly studied in Karafyllis and Krstic [To appear].

Outline. After introducing our system in Section 2, we transform it to the observer canonical form in Section 3. Once this step is accomplished, we start by presenting the non-adaptive controller in Section 4 in order to facilitate the understanding of the more complex, adaptive design, which follows in Section 5 with the statement of the main stability theorem. Section 6 then consists of its proof. We finally end our paper with an illustration of our result through a simulation in Section 7.

Notations. For any functions f and g defined on $[0,1]$, we use the convolution notation

$$f * g(x) = \int_0^x f(x-y)g(y)dy = \int_0^x f(y)g(x-y)dy$$

where $x \in [0,1]$ and for any function f defined on $[0,1] \times [0,\infty)$, we denote the L^2-norm as

$$\|f\|(t) = \sqrt{\int_0^1 f(x,t)^2 dx}.$$

2. GENERAL FIRST-ORDER HYPERBOLIC PIDE

We consider the following class of first-order hyperbolic PIDE:

$$\bar{u}_t(x,t) = \bar{u}_x(x,t) + \lambda(x)\bar{u}(x,t) + \bar{g}(x)\bar{u}(0,t) + \int_0^x \bar{f}(x,y)\bar{u}(y,t)dy \quad (1)$$

$$\bar{u}(1,t) = U(t) \quad (2)$$

$$Y(t) = \bar{u}(0,t), \quad (3)$$

where λ, $\bar{g}$ and $\bar{f}$ are unknown, continuous functions. The goal is to regulate $\bar{u}(x,t)$ to zero for all $x \in [0,1]$ using the measurement of only $Y(t) = \bar{u}(0,t)$ and using boundary control $U(t)$.

PIDEs in the form (1) arise in chemical process systems, as a result of coupling of transport dynamics with faster thermal dynamics.

We first remove the reaction term in $\lambda\bar{u}$ by introducing the scaled state

$$u(x,t) = \exp\left(\int_0^x \lambda(\xi)d\xi\right)\bar{u}(x,t) \quad (4)$$

which is governed by

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)dy \quad (5)$$

$$u(1,t) = \rho\, U(t) \quad (6)$$

$$Y(t) = u(0,t), \quad (7)$$

where

$$g(x) = \exp\left(\int_0^x \lambda(\xi)d\xi\right)\bar{g}(x) \quad (8)$$

$$f(x,y) = \exp\left(\int_y^x \lambda(\xi)d\xi\right)\bar{f}(x,y) \quad (9)$$

$$\rho = \exp\left(\int_0^1 \lambda(\xi)d\xi\right) \quad (10)$$

Hypothesis 1. ρ is known and, without loss of generality, we set it to $\rho = 1$ (by absorbing any non-unity ρ into U).

Hypothesis 2. Constants M_g and M_f are known such that, for all $0 \le y \le x \le 1$, $|g(x)| \le M_g$ and $|f(x,y)| \le M_f$.

3. OBSERVER CANONICAL FORM

The key challenge for feedback design for the plant (5)–(6) is that the term $\int_0^x f(x,y)u(y,t)dy$ is a product of the unmeasured state $u(x,t)$ and of the unknown parameter $f(x,y)$. We overcome this challenge by transforming the system into a form in which an unknown parameter multiplies only the measured output $Y(t) = u(0,t)$.

We introduce the backstepping pre-transformation

$$v(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy \quad (11)$$

where q is the solution to the PDE

$$q_y(x,y) + q_x(x,y) = \int_y^x q(x,s)f(s,y)ds - f(x,y) \quad (12)$$

$$q(1,y) = 0 \quad (13)$$

and which maps the system (5)–(6) into

$$v_t(x,t) = v_x(x,t) + \theta(x)v(0,t) \quad (14)$$

$$v(1,t) = U(t), \quad (15)$$

where

$$Y(t) = v(0,t) = u(0,t) = \bar{u}(0,t) \quad (16)$$

is measured and

$$\theta(x) = q(x,0) + g(x) - \int_0^x q(x,y)g(y)dy. \quad (17)$$

We refer to the form (14), (15), (16) as the *observer canonical form* due to its analogy with the eponymous form for finite-dimensional systems. The transformation (11) is not a part of design but of analysis only. The kernel $q(x,y)$ is unknown and so is the new system parameter $\theta(x)$. Unlike the term $\int_0^x f(x,y)u(y,t)dy$ in (5)–(6), which is a product of two unknown quantities, the term $\theta(x)v(0,t) = \theta(x)Y(t)$ in (14) has only $\theta(x)$ as an unknown. This is the key feature with which the observer canonical form (14), (15), (16) enables us to perform adaptive output-feedback design.

In the following theorem, proved in Section 6.1, we show that the PDE (12)-(13) is well posed and thus that the pre-transformation (11) is invertible.

Theorem 3. The PDE (12)-(13) has a unique $C^1([0,1]\times[0,1])$ solution with the bound

$$|q(x,y)| \le M_f(1-x)\; e^{M_f(x-y)(1-x)}, \tag{18}$$

where M_f is a bound for the function f on $[0,1]\times[0,1]$.

4. NON-ADAPTIVE OUTPUT-FEEDBACK CONTROL DESIGN

Our non-adaptive controller is given by

$$U(t) = \int_{t-1}^{t} \kappa(t-\tau)U(\tau)d\tau + \int_{t-1}^{t}\left(\int_{t-\tau}^{1}\kappa(\mu)\theta(1-\mu+t-\tau)d\mu\right)Y(\tau)d\tau, \tag{19}$$

where κ is solution of the Volterra equation

$$\kappa(x) = -\theta(x) + \int_0^x \kappa(y)\theta(x-y)dy\,. \tag{20}$$

Theorem 4. For the system consisting of the plant (5)-(6) and the controller (19)-(20), there exist $M_\mathrm{o}\ge 1$ and $\delta>0$ such that the following holds:

$$\Omega(t) \le M_\mathrm{o}\mathrm{e}^{-\delta t}\Omega(0)\,, \qquad \forall t\ge 0\,, \tag{21}$$

$$\Omega(t) \triangleq \int_0^1 u^2(x,t)dx + \int_{t-1}^{t}\left(U^2(\tau)+Y^2(\tau)\right)d\tau\,. \tag{22}$$

5. ADAPTIVE DESIGN

We apply the certainty equivalence principle and use an adaptive version of controller (19), namely, we replace θ and κ by their estimate $\hat\theta$ and $\hat\kappa$, obtaining the control law

$$U(t) = \int_{t-1}^{t} \hat\kappa(t-\tau,t)U(\tau)d\tau + \int_{t-1}^{t}\left(\int_{t-\tau}^{1}\hat\kappa(\mu,t)\hat\theta(1-\mu+t-\tau,t)d\mu\right)Y(\tau)d\tau, \tag{23}$$

where $\hat\theta$ is generated by an estimator (to be designed) and $\hat\kappa$ is obtained from $\hat\theta$ by real-time solution of the Volterra equation

$$\hat\kappa(x,t) = -\hat\theta(x,t) + \int_0^x \hat\kappa(y,t)\hat\theta(x-y,t)dy\,. \tag{24}$$

For the design of a parameter estimator for $\theta(x)$, we need a parametric model. The observer canonical form (14), (15), (16) serves as our parametric model, however, we use the following alternative representation of the observer canonical form to motivate our choice of the estimator:

$$Y(t) = U(t-1) + \int_{t-1}^{t}\theta(t-\tau)Y(\tau)d\tau + \varepsilon(t), \tag{25}$$

where the function ε is arbitrary for $t\in[0,1]$ and $\varepsilon(t)=0$ for $t>1$.

Our parameter update law will need to employ projection to keep the estimate $\hat\theta(x,t)$ within an a priori known bounded interval for each $x\in[0,1]$. We make an assumption in Hypothesis 2, which enables us to determine an a priori bound on the true $\theta(x)$.

Reminding the reader that we have assumed (without loss of generality) that $\rho=1$, from the expression (17) for θ, we get that, for all $x\in[0,1]$,

$$\begin{aligned}|\theta(x)| &\le M_f(1-x)e^{M_f x(1-x)}(1+M_g)+M_g\\ &\le M_f\; e^{M_f}(1+M_g)+M_g \triangleq M\,,\end{aligned} \tag{26}$$

which is a bound that we shall employ to limit the estimate $\hat\theta(x)$ using projection.

Now, guided by the parametric model (25), we introduce the update law

$$\hat\theta_t(x,t) = \frac{\gamma(x)}{1+\int_{t-1}^{t}Y^2(\tau)d\tau}\mathrm{Proj}\left(Y(t-x)\hat e(0,t),\hat\theta(x,t)\right)\,, \tag{27}$$

where γ is a strictly positive-valued adaptation gain function, $Y(t-x)$ is the “regressor”,

$$\hat e(0,t) = Y(t) - \; U(t-1) - \int_{t-1}^{t}\hat\theta(t-\tau,t)Y(\tau)d\tau \tag{28}$$

is the “estimation error,” and the projection is given by

$$\mathrm{Proj}(a,b) = \begin{cases}0, \text{ if } |b|=M \text{ and } ab>0\\ a, \text{ otherwise.}\end{cases} \tag{29}$$

The gain γ must be adapted depending on the desired convergence speed of θ. Our main theorem is stated next.

Theorem 5. Consider the plant (5)-(6) under Hypotheses 1 and 2 with the controller (23)-(24) and the update law (27)-(28). Then, for any initial conditions $\hat\theta(\cdot,0)\in C^1(0,1)$, the solution $(u,\hat\theta)$ and the control U are bounded for all $x\in[0,1]$, $t\ge 0$ and

$$\lim_{t\to\infty} u(x,t)=0, \quad \forall x\in[0,1] \tag{30}$$

$$\lim_{t\to\infty} U(t)=0. \tag{31}$$

6. PROOF OF THEOREM 5

6.1 Well-posedness of the transformation into the observer canonical form

The PDE (12)-(13) is defined on the triangular domain: $\tau=\{(x,y), 0\le y\le x\le 1\}$. The change of variables $\tilde x = 1-y$, $\tilde y = 1-x$, $\tilde f(\tilde x,\tilde y)=f(x,y)$, $\tilde q(\tilde x,\tilde y)=q(x,y)$ leads us to a new PDE, defined on τ:

$$\tilde q_{\tilde y}(\tilde x,\tilde y)+\tilde q_{\tilde x}(\tilde x,\tilde y) = -\int_{\tilde y}^{\tilde x}\tilde q(\tilde x,s)\tilde f(s,\tilde y)ds + \tilde f(\tilde x,\tilde y) \tag{32}$$

$$\tilde q(\tilde x,0)=0. \tag{33}$$

The function $\tilde q(\tilde x,\tilde y)$ satisfies the integral equation

$$\tilde q(\tilde x,\tilde y) = F_0(\tilde x,\tilde y)+F[\tilde q](\tilde x,\tilde y), \tag{34}$$

where

$$F_0(\tilde x,\tilde y) = \int_0^{\tilde y}\tilde f(\tilde x-\tilde y+\xi,\xi)d\xi \tag{35}$$

$$F[\tilde q](\tilde x,\tilde y) = -\int_0^{\tilde y}\int_0^{\tilde x-\tilde y}\tilde q(\tilde x-\tilde y+\eta,\xi+\eta)\tilde f(\xi+\eta,\eta)d\xi d\eta. \tag{36}$$

We solve this equation by the method of successive approximations. We define the sequence

$$\tilde{q}^0(\tilde{x},\tilde{y}) = F_0(\tilde{x},\tilde{y}) \quad (37)$$

$$\tilde{q}^{n+1}(\tilde{x},\tilde{y}) = F_0(\tilde{x},\tilde{y}) + F[\tilde{q}^n](\tilde{x},\tilde{y}) \quad (38)$$

and the differences $\Delta\tilde{q}^n = \tilde{q}^{n+1} - \tilde{q}^n$. Then, we get

$$\Delta\tilde{q}^{n+1}(\tilde{x},\tilde{y}) = F[\Delta\tilde{q}^n](\tilde{x},\tilde{y}). \quad (39)$$

By induction, we prove that, for all integers n,

$$|\Delta\tilde{q}^n(\tilde{x},\tilde{y})| \leq \frac{M_f^{n+1}(\tilde{x}-\tilde{y})^n}{n!}\tilde{y}^{n+1}. \quad (40)$$

Therefore, the series

$$\tilde{q}(\tilde{x},\tilde{y}) = \lim_{n\to\infty}\tilde{q}^n(\tilde{x},\tilde{y}) = F_0(\tilde{x},\tilde{y}) + \sum_{n=0}^{\infty}\Delta\tilde{q}^n(\tilde{x},\tilde{y}). \quad (41)$$

uniformly converges in τ to solution of (34) with the bound $|\tilde{q}(\tilde{x},\tilde{y})| \leq M_f\tilde{y}\, e^{M_f(\tilde{x}-\tilde{y})\tilde{y}}$. Thus, we also have that $\tilde{q} \in C^1(\tau)$ since $\tilde{q}^n \in C^1(\tau)$ according to (35) and (36). The bound (18) on q is easily deduced.

If we suppose $\tilde{q}_1$ and $\tilde{q}_2$ are two solutions and we consider their difference $\delta\tilde{q} = \tilde{q}_1 - \tilde{q}_2$, then we get

$$\delta\tilde{q}(\tilde{x},\tilde{y}) = F[\delta\tilde{q}](\tilde{x},\tilde{y}) \quad (42)$$

and for all integer n,

$$|\delta\tilde{q}(\tilde{x},\tilde{y})| \leq \frac{M_f^{n+1}(\tilde{x}-\tilde{y})^n}{n!}\tilde{y}^{n+1} \quad (43)$$

Thus, $\delta\tilde{q} = 0$ and $\tilde{q}_1 = \tilde{q}_2$. Hence, we establish uniqueness of the solution.

6.2 Nonadaptive Observer

We represent the delayed input and output signals with the transport PDEs

$$\phi_t(x,t) = \phi_x(x,t), \qquad \phi(x,0) = \phi_0(x),\ x \in [0,1] \quad (44)$$

$$\phi(1,t) = Y(t) \quad (45)$$

and

$$\psi_t(x,t) = \psi_x(x,t), \qquad \psi(x,0) = \psi_0(x),\ x \in [0,1] \quad (46)$$

$$\psi(1,t) = U(t)\,. \quad (47)$$

where ϕ_0, ψ_0 are arbitrary initial conditions verifying $\phi_0(1) = Y(0)$ and $\psi_0(1) = U(0)$. We can define for $x \in [0,1]$, $Y(x-1) = \phi_0(x)$ and $U(x-1) = \psi_0(x)$. Then, the explicit solutions to the PDE filters, for $x \in [0,1], t \geq 0$, are given by

$$\phi(x,t) = Y(t+x-1) \quad (48)$$

$$\psi(x,t) = U(t+x-1)\,. \quad (49)$$

The non-adaptive observer error

$$e(x,t) = v(x,t) - \psi(x,t) - \int_x^1 \theta(\xi)\phi(1-(\xi-x),t)d\xi \quad (50)$$

satisfies the autonomous PDE

$$e_t(x,t) = e_x(x,t) \quad (51)$$

$$e(1,t) = 0\,. \quad (52)$$

Therefore, for $t \geq 1$, $e(x,t) = 0$ and we get the non-adaptive observer:

$$v(x,t) = \psi(x,t) + \int_x^1 \theta(\xi)\phi(1-(\xi-x),t)d\xi, \quad t \geq 1. \quad (53)$$

6.3 Properties of the Update Law

Our update law for the estimate $\hat{\theta}(x,t)$ is based on the parametric model

$$e(0,t) = v(0,t) - \psi(0,t) - \int_0^1 \theta(\xi)\phi(1-\xi,t)d\xi. \quad (54)$$

The estimation error (28) is alternatively written as

$$\hat{e}(0,t) = v(0,t) - \psi(0,t) - \int_0^1 \hat{\theta}(\xi,t)\phi(1-\xi,t)d\xi \quad (55)$$

and the parameter estimation error $\tilde{\theta}(x,t) = \theta(x) - \hat{\theta}(x,t)$ satisfies

$$\hat{e}(0,t) = e(0,t) + \int_0^1 \tilde{\theta}(\xi,t)\phi(1-\xi,t)d\xi. \quad (56)$$

With the filters, we rewrite the update law as

$$\hat{\theta}_t(x) = \frac{\gamma(x)}{1+\|\phi\|^2}\mathrm{Proj}(\hat{e}(0)\phi(1-x),\hat{\theta}(x)) \quad (57)$$

(we remove the time dependance for clarity).

Lemma 6. With $\|\cdot\|$ denoting the L^2-norm in $x \in [0,1]$, and with $\mathcal{L}_2$ and $\mathcal{L}_\infty$ denoting the usual function spaces in $t \in [0,\infty)$, the adaptive law (57) guarantees that

$$|\hat{\theta}(x)| \leq M, \quad \textit{for all } (x,t) \in [0,1]\times[0,\infty) \quad (58)$$

$$\|\tilde{\theta}\| \in \mathcal{L}_\infty \quad (59)$$

$$\|\hat{\theta}_t\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty \quad (60)$$

$$\frac{\hat{e}(0)}{\sqrt{1+\|\phi\|^2}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty \quad (61)$$

Proof. Omitted due to space constraints.

6.4 Backstepping transformation

Based on (50), we introduce the adaptive state estimate

$$\hat{v}(x) = \psi(x) + \int_x^1 \hat{\theta}(\xi)\phi(1-(\xi-x))d\xi \quad (62)$$

and apply the following backstepping transformation:

$$w(x) = \hat{v}(x) - \hat{\kappa} * \hat{v}(x) \triangleq T[\hat{v}](x), \quad (63)$$

where κ is the solution to the Volterra equation

$$\hat{\kappa}(x) = -\hat{\theta}(x) + \hat{\kappa} * \hat{\theta}(x) = -T[\hat{\theta}](x). \quad (64)$$

Transformation (63) is invertible,

$$\hat{v}(x) = w(x) - \hat{\theta} * w(x), \quad (65)$$

and leads to the target system

$$\begin{aligned} w_t &= w_x - \hat{\kappa}(x)\hat{e}(0) \\ &\quad + w * T[\hat{\theta}_t](x) \\ &\quad + T\left[\int_x^1 \hat{\theta}_t(\xi)\phi(1-(\xi-x))d\xi\right] \end{aligned} \quad (66)$$

$$w(1) = 0 \quad (67)$$

This leads to the controller

$$U(t) = \hat{v}(1) = \int_0^1 \hat{\kappa}(1-y)\hat{v}(y,t)dy, \tag{68}$$

i.e.,

$$U(t) = \int_0^1 \hat{\kappa}(1-y)\left[\psi(y,t) + \int_y^1 \hat{\theta}(\xi)\phi(1-(\xi-y),t)d\xi\right]dy, \tag{69}$$

which corresponds to the controller (23) presented in Theorem 5. The ϕ system can be rewritten as

$$\phi_t = \phi_x \tag{70}$$

$$\phi(1) = w(0) + \hat{e}(0) \tag{71}$$

We now have two interconnected systems, ϕ and w, given by (70)-(71) and (66)-(67).

6.5 $\mathcal{L}_2$ Boundedness

From (64) and the Gronwall inequality, we get the following bound:

$$|\hat{\kappa}(x)| \leq Me^M \triangleq K \tag{72}$$

We also know from the previous section, that $\|\hat{\theta}_t\|$ is bounded. Let's now consider the Lyapunov functions:

$$V_1 = \frac{1}{2}\int_0^1 (1+x)\phi^2(x)dx \tag{73}$$

$$V_2 = \frac{1}{2}\int_0^1 (1+x)w^2(x)dx \tag{74}$$

Using the PDEs (70)-(71), (66)-(67), and the Young inequality, we get the following upper bounds:

$$\dot{V}_1 \leq \frac{3}{2}w^2(0) + \frac{3}{2}\hat{e}^2(0) - \frac{1}{2}\phi^2(0) - \frac{1}{2}\|\phi\|^2 \tag{75}$$

$$\begin{aligned}\dot{V}_2 \leq & -\frac{1}{2}w^2(0) - \left(\frac{1}{2} - c_1 - c_2 - c_3 - c_4\right)\|w\|^2 \\ & + \frac{K}{c_1}\hat{e}^2(0) + l_1\|w\|^2 + l_2\|\phi\|^2 \end{aligned}\tag{76}$$

where the c_i are arbitrary positive constants and l_i are integrable bounded nonnegative functions.
We consider next the Lyapunov function

$$V = V_1 + 4V_2. \tag{77}$$

Taking $c_1 = c_2 = c_3 = c_4 = \frac{1}{16}$, we get, from (75)-(77), the inequality

$$\dot{V} \leq -\frac{1}{4}V + l\,V + l_4 - \frac{1}{2}\phi^2(0) - \frac{1}{2}w^2(0) \leq -\frac{1}{4}V + l\,V + l_4 \tag{78}$$

since $\frac{1}{2}\|\phi\|^2 \leq V_1 \leq \|\phi\|^2$, $\frac{1}{2}\|w\|^2 \leq V_2 \leq \|w\|^2$.
Therefore, V is bounded and integrable (Lemma D.3 in Smyshlyaev and Krstic [2010]), and $\|\phi\|, \|w\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. The transformation (65) gives $\hat{v} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and with (62), $\|\psi\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.
Then, from (53), we get that $\|v\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and from (11) that $\|u\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

6.6 Pointwise Boundedness

The $\mathcal{L}_2$ boundedness of $\|\phi\|$ and $\|\psi\|$ gives $U \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ (see (69)). Therefore, (49) ensures that for all x, $\psi(x,.) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

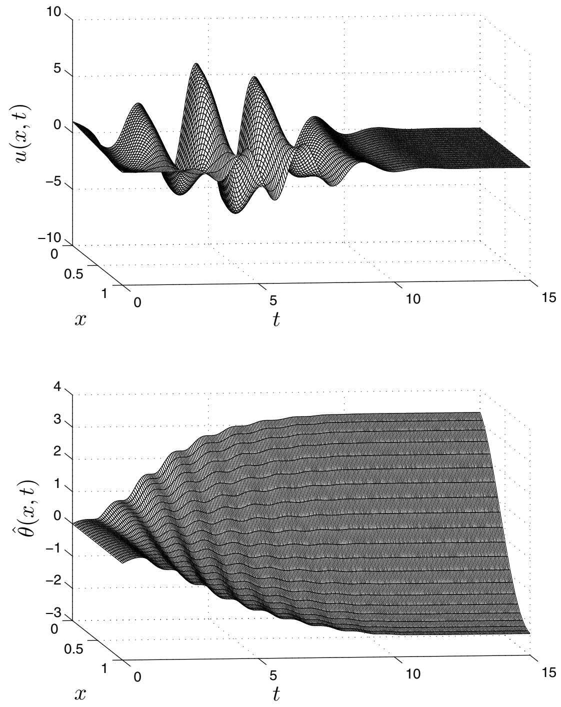

Fig. 1. Response of system (81) to the adaptive control (23)-(24) : evolution of state u (top), and of the estimate $\hat{\theta}$ of the unknown infinite-dimension parameter θ (bottom).

The following equalities hold:

$$\hat{e}(x) = e(x) - \int_x^1 \tilde{\theta}(\xi)\phi(1-(\xi-x))d\xi \tag{79}$$

$$\hat{e}(x) = v(x) - \psi(x) - \int_x^1 \hat{\theta}(\xi)\phi(1-(\xi-x))d\xi \tag{80}$$

Therefore, using the facts that $e(x) = 0$ for $t \geq 1$, (59) and (79), $\hat{e}(x,.) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and then with (80), $v(x,.) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Then, (11) gives $u(x,.) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ for all $x \in [0,1]$. With (48), we finally get $\phi(x,.) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.
In summary, the solution $(u, \phi, \psi, \hat{\theta})$ is pointwise bounded.

6.7 Convergence

With (78), $\dot{V}$ is bounded from above. As V is also positive and integrable, we obtain that $V \to 0$, that is, $\|w\| \to 0$ and $\|\phi\| \to 0$. From (65), we get $\|\hat{v}\| \to 0$, and from (62), $\|\psi\| \to 0$ follows. (50) and then (11), lead to $\|v\| \to 0$ and $\|u\| \to 0$.
Moreover, with (69), we get $U(t) \to 0$. Therefore, $\psi(x,.)$ tends to 0 (from (49)) and, with (50), we get $v(x,.) \to 0$. Finally, with (11) we get the convergence of $u(x,.)$ to zero. This completes the proof of Theorem 5.

7. SIMULATIONS

We take the example of the Korteweg-de Vries-like equations used in Krstic and Smyshlyaev [2008]. The system

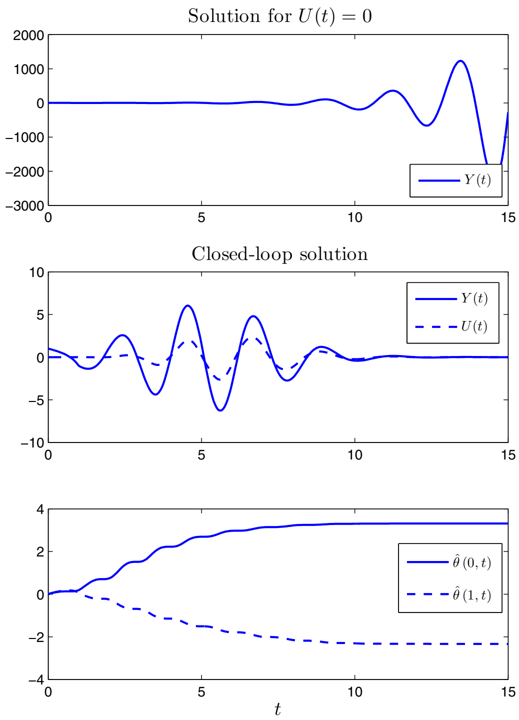

Fig. 2. Comparison of the open and closed loops. For the closed loop, we represent the input U and the output Y (middle), and the boundary values of the estimation $\hat{\theta}$ (bottom).

is determined by three coefficients, a, δ, and ε and a transformation leads to the following PIDE ($b = \sqrt{\frac{a}{\varepsilon}}$):

$$u_t(x,t) = \varepsilon u_x(x,t) - \delta b \sinh(bx)\ u(0,t) + \delta b^2 \int_0^x \cosh(b(x-y))\ u(y,t) dy. \quad (81)$$

Taking $\varepsilon = 1$ and assuming we want to control PIDE (81) without knowing a and δ, we apply the adaptive output-feedback presented in Section 5. The results of the simulation for $a = 1$, $\delta = 4$, and a constant gain function $\gamma(x) = 1$ in the update law, are given in Figure 1.
We see on the first graph of Figure 2 that the open-loop is unstable and oscillatory; the two other graphs in Figure 2 describe how the adaptive control works. $\hat{\theta}$ is initialized at zero, which makes the start of control slow (very small for at least 2 time units); this slow start of control allows u to grow, which excites the update law, enabling $\hat{\theta}$ to converge towards (but not exactly to) $\theta(t)$. Control then catches up and ensures the convergence of u to zero for all $x \in [0, 1]$. A higher gain function would induce a faster convergence: for instance, $\gamma(x) = 10$ doubles the convergence speed.

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