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The Stabilization of Multi-Dimensional Wave Equation with Boundary Control Matched Disturbance ⋆

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Abstract: We consider boundary stabilization for a multi-dimensional wave equation with boundary control matched disturbance that depends on both time and spatial variables. The active disturbance rejection control (ADRC) approach is adopted in investigation. An disturbance estimator is designed to estimate, in real time, the disturbance, and the disturbance is canceled in the feedback loop with its approximation. All subsystems in the closed-loop are shown to be asymptotically stable. The numerical experiments are carried out to illustrate the convergence and effect of peaking value reduction.

Keywords: Wave equation, boundary control, disturbance rejection, stabilization.

1. INTRODUCTION

The active disturbance rejection control (ADRC), as an unconventional design strategy similar to the external model principle ([6]), was first proposed by Han in [4]. One of the remarkable features of ADRC is that the disturbance is estimated in real time through an extended state observer ([1]) and is canceled in the feedback loop which makes the control energy significantly reduced. The generalization of ADRC to the systems described by one-dimensional PDEs are also available in our previous work [2] but there is no study for multi-dimensional PDEs.

In this paper, we are concerned with stabilization for a multi-dimensional wave equation with Neumann boundary control described by the following PDE:

$$\begin{cases} w_{tt}(x,t) = \Delta w(x,t), x \in \Omega, t > 0, \\ w(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial w}{\partial \nu}\Big|_{\Gamma_1} = v(x,t) + d(x,t), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded domain with a smooth C^2-boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$. Γ_0 and Γ_1 ($\mathrm{int}(\Gamma_1) \neq \emptyset$) are disjoint, relatively open in $\partial\Omega$, $\mathrm{int}(\Gamma_0) \neq \emptyset$, and ν is the unit normal vector of $\partial\Omega$ pointing the exterior of Ω; v is the control input, d is the unknown external disturbance which is supposed to satisfy

$$\begin{array}{l} d \in L^\infty(0,\infty; C(\Gamma_1)) \cap C(0,\infty; C(\Gamma_1)), \\ d_t \in L^\infty(0,\infty; C(\Gamma_1)). \end{array} \tag{2}$$

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It is well known that when there is no disturbance, the collocated feedback control

$$v(x,t) = -kw_t(x,t),\ x \in \Gamma_1, k > 0 \tag{3}$$

exponentially stabilizes system (1) provided that there exists a coercive smooth vector field h on Γ, that is, the following condition is satisfied ([5, p. 668]).

$$\begin{cases} \text{(i). } h \cdot \nu \leq 0 \text{ on } \Gamma_0. \\ \text{(ii). } h \text{ is parallel to } \nu \text{ on } \Gamma_1, h(\sigma) = \ell(\sigma)\nu(\sigma) \\ \qquad\qquad \text{for a smooth } \ell, \sigma \in \Gamma_1. \\ \text{(iii). For some constant } \rho > 0, \forall\, y \in (L^2(\Omega))^n : \\ \displaystyle\int_\Omega H(x)y(x) \cdot y(x) dx \geq \rho \int_\Omega |y(x)|^2 dx \\ \quad \text{where } H(x) = \{\partial h_i / \partial x_j\}_{i,j=1}^n. \end{cases} \tag{4}$$

However, the stabilizing controller (3) is not robust to the external disturbance as shown from the following example.

Example 1.1. Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 |\ 1 < x_1^2 + x_2^2 < 4\}$ be a two-dimensional annulus. Let Γ_0 be the unit disk of $\mathbb{R}^2$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, and $d(x,t) \equiv d$ be a constant. Then condition (4) is satisfied with $h(x) = x$. However, system (1) under the feedback (3) admits a solution $(w, w_t) = (d \ln(x_1^2 + x_2^2), 0)$.

We consider system (1) in the energy Hilbert state space $\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, $H_{\Gamma_0} = \{f \in H^1(\Omega) |\ f = 0 \text{ on } \Gamma_0\}$ with the usual inner product given by

$$\langle (f_1, g_1)^\top, (f_2, g_2)^\top \rangle = \int_\Omega [\nabla f_1(x) \overline{\nabla f_2(x)} + g_1(x) \overline{g_2(x)}] dx,$$
$$\forall\, (f, g)^\top \in \mathcal{H},$$

and the control space $U = L^2(\Gamma_1)$. Define the operator $\mathcal{A}$ as follows:

$$\begin{cases} \mathcal{A}(f,g)^\top = (g, \Delta f)^\top, \ \forall \ (f,g)^\top \in D(\mathcal{A}), \\ D(\mathcal{A}) = \Big\{ (f,g)^\top \in \mathcal{H} \cap (H^2(\Omega) \times H^1(\Omega)) \\ \qquad\qquad \Big| \ \dfrac{\partial f}{\partial \nu}\Big|_{\Gamma_1} = g|_{\Gamma_0} = 0 \Big\}. \end{cases} \tag{5}$$

Then it is easy to verify that $\mathcal{A}^* = -\mathcal{A}$ in $\mathcal{H}$.

Let $A = -\Delta$ be the usual Laplacian with $D(A) = \{f| \ f \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega), \frac{\partial f}{\partial \nu}|_{\Gamma_1} = 0\}$, which is a positive definite unbounded operator in $L^2(\Omega)$. It is easily shown (see e.g., [3]) that $D(A^{1/2}) = H^1_{\Gamma_0}(\Omega)$ and $A^{1/2}$ is an canonical isomorphism from $H^1_{\Gamma_0}(\Omega)$ onto $L^2(\Omega)$. An extension $\tilde{A} \in \mathcal{L}(D(A^{1/2}, D(A^{1/2})')$ of A is defined by

$$\langle \tilde{A} f, g\rangle_{D(A^{1/2})' \times D(A^{1/2})} = \langle A^{1/2} f, A^{1/2} g\rangle_{L^2(\Omega)}, \\ \forall \ f, g \in D(A^{1/2}) = H^1_{\Gamma_0}(\Omega).$$

Define the Neumann map $\Upsilon \in \mathcal{L}(L^2(\Gamma_1), H^{3/2}(\Omega))$ ([5, p.668]), i.e., $\Upsilon u = v$ if and only if

$$\begin{cases} \Delta v = 0 \text{ in } \Omega, \\ v|_{\Gamma_0} = 0, \ \dfrac{\partial v}{\partial \nu}|_{\Gamma_1} = u. \end{cases} \tag{6}$$

Using the Neumann map, one can write (1) in $D(A^{1/2})'$ as

$$\ddot{w} = -\tilde{A} w + Bu, \tag{7}$$

where $B \in \mathcal{L}(U, D(A^{1/2})')$ is given by

$$Bu = \tilde{A}\Upsilon u, \ \forall \ u \in U, B^* f = f|_{\Gamma_1}, \forall, f \in D(A^{1/2}). \tag{8}$$

Therefore, system (1) can be written as

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathcal{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathcal{B}[u(x,t) + d(x,t)], \tag{9}$$

where $\mathcal{B} = (0, B^*)^\top$. However, since $\mathcal{B}$ is not admissible for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$ on $\mathcal{H}$ (see [5, p.669]), (9) does not always admit a unique solution in $\mathcal{H}$ for general $v \in L^2_{loc}(0, \infty, U)$. To overcome this difficulty, we first introduce a damping on the control boundary by designing

$$v(x,t) = -k_1 w_t(x,t) + u(x,t), k_1 > 0, \forall \ x \in \Gamma_1, \tag{10}$$

under which, the system (1) becomes

$$\begin{cases} w_{tt}(x,t) = \Delta w(x,t), x \in \Omega, t > 0, \\ w(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial w}{\partial \nu}|_{\Gamma_1} = -k_1 w_t(x,t) + u(x,t) + d(x,t), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x). \end{cases} \tag{11}$$

Similar from (1) to (7), we can write (11) as

$$\ddot{w} = -\tilde{A} w - k_1 BB^* \dot{w} - B(u+d) \text{ in } D(A^{1/2})' \tag{12}$$

or in the first order form

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} = \mathbb{A}\begin{pmatrix} w \\ w_t \end{pmatrix} + \mathbb{B}(u+d) \\ \text{in } D(A^{1/2}) \times D(A^{1/2})', \tag{13}$$

where

$$\begin{cases} \mathbb{A}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ -\tilde{A} f - k_1 BB^* g \end{pmatrix}, \\ D(\mathbb{A}) = \Big\{ (f,g)^\top | \ f, g \in D(A^{1/2}), \tilde{A} f \\ \qquad\qquad + k_1 BB^* g \in L^2(\Omega) \Big\}, \\ \mathbb{B} = (0, -B)^\top. \end{cases} \tag{14}$$

The following result is well known.

Proposition 1.1. The operator $\mathbb{A}$ defined in (14) generates a C_0-semigroup of contractions $e^{\mathbb{A}t}$ on $\mathcal{H}$ and $\mathbb{B}$ is admissible to $e^{\mathbb{A}t}$. Therefore, for any initial value $(w(\cdot,0), \dot{w}(\cdot,0))^\top \in \mathcal{H}$ and control input $u \in L^2_{loc}(0,\infty,U)$, (11) admits a unique solution $(w, \dot{w})^\top \in \mathcal{H}$.

By proposition 1.1, the (weak) solution of (11) satisfies

$$\frac{d}{dt}\left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \mathbb{A}^* \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}} \\ + \int_{\Gamma_1} [u(x,t) + d(x,t)] g(x) dx, \forall \ (f,g)^\top \in D(\mathbb{A}^*). \tag{15}$$

A simple computation shows that

$$\mathbb{A}^*(f,g) = -(g, \Delta f)^\top, \\ D(\mathbb{A}^*) = \Big\{ (f,g)^\top \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega) \\ \qquad | \ \frac{\partial f}{\partial \nu}|_{\Gamma_1} = k_1 g \Big\}. \tag{16}$$

In Section 2, we state the main results. Section 3 gives the proof for the main results. Some numerical simulations for Example 1.1 are presented in Section 4 for illustration.

2. THE MAIN RESULTS

In addition to (2), we suppose further that there exists a positive nondecreasing differentiable continuous function K such that for all $x_i \in \Gamma_1, i = 1, 2, \ t \geq 0$,

$$|d(x_1,t) - d(x_2,t)| \leq K(t)|x_1 - x_2|^\alpha. \tag{17}$$

Let ε be a continuous function such that

$$\varepsilon(t) \in (0,1], \ \dot{\varepsilon}(t) < 0, \ \lim_{t\to\infty} \varepsilon(t) = 0. \tag{18}$$

In addition, we can choose ε appropriately so that

$$\delta(t) = \left(\frac{\varepsilon(t)}{K(t)}\right)^{1/\alpha}, \\ \lim_{t\to\infty} r(t)\delta^{n-1}(t) = \infty \text{ and } \sup_{t>0} |\delta'(t)\delta^{n-2}(t)| < \infty, \tag{19}$$

where $r \in C(\bar{\mathbb{R}}^+, \mathbb{R}^+)$ is a time varying gain satisfying

$$\dot{r}(t) > 0, \ \lim_{t\to\infty} r(t) = \infty, \ \frac{\dot{r}(t)}{r(t)} \leq \overline{M}, \overline{M} > 0. \tag{20}$$

Lemma 2.1. Let δ be defined by (19). Then, one can construct $\{x^{(i)}\}_{i=1}^\infty \subset \Gamma_1$ so that the time varying covers $\{\Gamma_1 \cap U(x^{(i)}, \delta(t))\}_{i=1}^{N(t)}$ of $\overline{\Gamma_1}$ satisfies

$$\Gamma_1 = \{\Gamma_1 \cap U(x^{(i)}, \delta(t))\}_{i=1}^{N(t)}, \\ \sum_{i=1}^{N(t)} \text{meas}(\Gamma_1 \cap U(x^{(i)}, \delta(t))) \\ \leq C(\Gamma_1)(n-1)^{\frac{n-1}{2}} 2^{n-1} \text{meas}(\Gamma_1), \tag{21}$$

where $U(x^{(i)}, \delta(t))$ denotes the ball of $\mathbb{R}^n$ centered at $x^{(i)} \in \Gamma_1$ with radius $\delta(t)$; the boundary measure is the Lebesgue measure in $\mathbb{R}^{(n-1)}$ space; $C(\Gamma_1)$ is a positive constant; and the time dependent integer N depends on δ directly and $\lim_{t\to\infty} N(t) = +\infty$.

The next step is to construct a disturbance estimator. To this end, let $(f_i^t, g_i^t)^\top \in D(\mathbb{A}^*)$, $i = 1, 2, \cdots$, so that

$$\begin{cases} \Delta f_i^t = 0,\ f_i^t|_{\Gamma_0} = 0,\ \dfrac{\partial f_i^t}{\partial \nu}|_{\Gamma_1} = k_1 g_i^t, \\ g_i^t|_{\Gamma_0} = 0,\ g_i^t|_{\Gamma_1 \setminus U(x^{(i)},\delta(t))} = 0,\ g_i^t|_{\Gamma_1 \cap U(x^{(i)},\frac{1}{2}\delta(t))} = 1, \\ 0 \le g_i^t|_{\Gamma_1 \cap U(x^{(i)},\delta(t))} \le 1,\ |g_{it}^t|_{\Gamma_1 \cap U(x^{(i)},\delta(t))}| \le \pi \left| \dfrac{\delta'(t)}{\delta(t)} \right|, \\ \left| \nabla g_i^t|_{\Gamma_1 \cap (U(x^{(i)},\delta(t)) \setminus U(x^{(i)},\frac{1}{2}\delta(t)))} \right| \le \dfrac{\pi}{\delta(t)}, \\ \left| \nabla g_i^t|_{\Gamma_1 \setminus (U(x^{(i)},\delta(t)) \setminus U(x^{(i)},\frac{1}{2}\delta(t)))} \right| = 0. \end{cases} \tag{22}$$

It is seen from (6) that $f_i^t = k_1 \Upsilon g_i^t$ and hence $f_{it}^t = k_1 \Upsilon g_{it}^t$, and g_i^t can be constructed analytically as

$$g_i^t(x) = \begin{cases} 1, & |x - x^{(i)}| < \dfrac{\delta(t)}{2}, \\ 0, & |x - x^{(i)}| \ge \delta(t), \\ -\dfrac{1}{2} \cos\left(\dfrac{2\pi |x - x^{(i)}|}{\delta(t)} \right) + \dfrac{1}{2}, & \text{others}. \end{cases} \tag{23}$$

It is easy to see that $g_{it}^t(x)$, $\nabla g_i^t(x)$ are continuous. Substitute $(f_i^t, g_i^t)^\top$ into (15) to obtain

$$\begin{aligned} &\frac{d}{dt} \int_\Omega [\nabla w(x,t) \nabla f_i^t(x) + w_t(x,t) g_i^t(x)] dx \\ &= \int_\Omega [\nabla w(x,t) \nabla f_{it}^t(x) + w_t(x,t) g_{it}^t(x)] dx \\ &\quad - \int_\Omega \nabla w(x,t) \nabla g_i^t(x) dx + \int_{\Gamma_1} u(x,t) g_i^t(x) dx \\ &\quad + d(\xi_i(t),t) \int_{\Gamma_1} g_i^t(x) dx, \end{aligned} \tag{24}$$

where $\xi_i : [0,\infty) \to \Gamma_1 \cap U(x^{(i)}, \delta(t))$ satisfies

$$\begin{cases} d(\xi_i(t),t) = \dfrac{\int_{\Gamma_1} d(x,t) g_i^t(x) dx}{\int_{\Gamma_1} g_i^t(x) dx}, \\ \dfrac{d}{dt} \left(d(\xi_i(t),t) \displaystyle\int_{\Gamma_1} g_i^t(x) dx \right) = \displaystyle\int_{\Gamma_1} d_t(x,t) g_i^t(x) dx \\ \qquad + \displaystyle\int_{\Gamma_1} d(x,t) g_{it}^t(x) dx. \end{cases} \tag{25}$$

Let M be a constant such that $|d(x,t)| \le M$, $|d_t(x,t)| \le M$ for all $x \in \Gamma_1$ and $t \ge 0$. By (19),

$$\begin{cases} |d(\xi_i(t),t)| \le M, \\ \left| \dfrac{d}{dt} \left(d(\xi_i(t),t) \displaystyle\int_{\Gamma_1} g_i^t(x) dx \right) \right| \le M \\ \quad + C'(\Gamma_1) \|d\|_{L^\infty(0,\infty;C(\Gamma_1))} \sup\limits_{t>0} (|\delta'(t)| \delta^{n-2}(t)) < \infty, \end{cases} \tag{26}$$

where $C'(\Gamma_1) > 0$ is a constant. Let

$$\begin{cases} y_i(t) = \displaystyle\int_\Omega [\nabla w(x,t) \nabla f_i^t(x) + w_t(x,t) g_i^t(x)] dx, \\ y_{2i}(t) = \displaystyle\int_\Omega [\nabla w(x,t) \nabla f_{it}^t(x) + w_t(x,t) g_{it}^t(x)] dx \\ \qquad - \displaystyle\int_\Omega \nabla w(x,t) \nabla g_i^t(x) dx,\ i = 1,2,\ldots. \end{cases} \tag{27}$$

Then

$$\begin{aligned} \dot{y}_i(t) &= y_{2i}(t) + \int_{\Gamma_1} u(x,t) g_i^t(x) dx \\ &\quad + d(\xi_i(t),t) \int_{\Gamma_1} g_i^t(x) dx, i = 1,2,\ldots. \end{aligned} \tag{28}$$

We design a time varying high gain extended state observer for system (27) as

$$\begin{cases} \dot{\hat{y}}_i(t) = y_{2i}(t) + \displaystyle\int_{\Gamma_1} u(x,t) g_i^t(x) dx \\ \qquad + \hat{d}_i(t) \displaystyle\int_{\Gamma_1} g_i^t(x) dx - r(t)[\hat{y}_i(t) - y_i(t)], \\ \dfrac{d}{dt} \left(\hat{d}_i(t) \displaystyle\int_{\Gamma_1} g_i^t(x) dx \right) = -r^2(t)[\hat{y}_i(t) - y_i(t)], \\ \qquad i = 1,2,\ldots, \end{cases} \tag{29}$$

which is served as a disturbance estimator.

Lemma 2.2. Let $\{x^{(i)}\}$ be defined in Lemma 2.1, g_i^t be defined by (23), ξ_i and $d(\xi_i(t),t)$ by (25), and y_i and y_{2i} by (27). Then under conditions (18), (19), and (20), the solution of (29) satisfies

$$\lim_{t\to\infty} |\hat{d}_i(t) - d(\xi_i(t),t)| = 0,\ \lim_{t\to\infty} |\hat{y}_i(t) - y_i(t)| = 0 \tag{30}$$

uniformly for all $i = 1,2,\cdots$.

Now we define

$$\hat{d}(x,t) = \begin{cases} \hat{d}_1(t), x \in \Gamma_1 \cap U(x^{(1)}, \delta(t)), \\ \hat{d}_2(t), x \in \Gamma_1 \cap (U(x^{(2)}, \delta(t)) \setminus U(x^{(1)}, \delta(t))), \\ \cdots, \\ \hat{d}_i(t), x \in \Gamma_1 \cap (U(x^{(i)}, \delta(t)) \setminus \cup_{j=1}^{i-1} U(x^{(j)}, \delta(t))), \\ \cdots, \\ \hat{d}_{N(t)}(t), x \in \Gamma_1 \cap (U(x^{N(t)}, \delta(t)) \\ \qquad \setminus \cup_{j=1}^{N(t)-1} U(x^{(j)}, \delta(t))), \end{cases} \tag{31}$$

where $x^{(i)}$ is defined in Lemma 2.1.

Lemma 2.3. Let $\hat{d}$ be defined by (31). Then under the conditions of Lemma 2.2, we have

$$\lim_{t\to\infty} \|\hat{d}(\cdot,t) - d(\cdot,t)\|_{L^2(\Gamma_1)} = 0. \tag{32}$$

By (32), we design naturally a collocated like state feedback control law to (11) as follows:

$$u(x,t) = -k_2 w_t(x,t) - \hat{d}(x,t), k_2 > 0. \tag{33}$$

Under the feedbacks (10) and (33), the closed-loop system of (11) becomes, for all $i \ge 1$, that

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) = 0, x \in \Omega, t > 0, \\ w(x,t)|_{\Gamma_0} = 0, \\ \dfrac{\partial w}{\partial \nu}(x,t)\big|_{\Gamma_1} = -kw_t(x,t) - \hat{d}(x,t) + d(x,t), \\ \qquad\qquad\qquad\qquad k = k_1 + k_2, \\ \dot{\hat{y}}_i(t) = y_{2i}(t) \\ -k\displaystyle\int_{\Gamma_1} g_i^t(x) w_t(x,t)dx - r(t)[\hat{y}_i(t) - y_i(t)], \\ \dfrac{d}{dt}\left(\hat{d}_i(t)\displaystyle\int_{\Gamma_1} g_i^t(x)dx\right) = -r^2(t)[\hat{y}_i(t) - y_i(t)]. \end{cases} \tag{34}$$

We state our main result.

Theorem 2.1. Let $\{x^{(i)}\}$ be defined in Lemma 2.1, g_i^t by (23), $\hat{d}$ by (31), and y_i and y_{2i} by (27). Then for any initial value $(w(\cdot,0), w_t(\cdot,0))^\top \in \mathcal{H}$, the closed-loop system (34) admits a unique solution $(w(\cdot,t), w_t(\cdot,t))^\top \in C(0,\infty,\mathcal{H})$; $\{\hat{y}_i, \hat{d}_i\}_{i=1}^\infty \in C(0,\infty)$. Moreover, under conditions (2), (4), (17), (18), (19), and (20), system (34) is asymptotically stable:

$$\lim_{t\to\infty} E_i(t) = 0 \tag{35}$$

uniformly for $i = 1, 2, \dots$, where

$$\begin{aligned} E_i(t) = &\int_\Omega [|\nabla w(x,t)|^2 + |w_t(x,t)|^2]dx + |\hat{y}_i(t)| \\ &+ \int_{\Gamma_1} |\hat{d}(x,t) - d(x,t)|^2 dx. \end{aligned} \tag{36}$$

3. PROOF OF MAIN RESULTS

Proof of Lemma 2.1. Since $\Gamma_1 \in C^2$ is compact in $\mathbb{R}^{n-1}$, we may assume (by finite covering theorem) without loss of generality that Γ_1 can be described by $x_n = \psi(x_1, x_2, \dots, x_{n-1}) \in C^2(\Omega_{n-1})$ for some $\Omega_{n-1} \subset \mathbb{R}^{n-1}$.

Let Ω_c be a $(n-1)$-hypercube in $\mathbb{R}^{n-1}$ space. Suppose that each side of Ω_c parallels the corresponding orthogonal coordinate axis of $\mathbb{R}^{n-1}$ so that $\Omega_{n-1} \subset \Omega_c$. Let $C(\Gamma) = \left\|\sqrt{1 + \psi_{x_1}^2 + \psi_{x_2}^2 + \dots + \psi_{x_{n-1}}^2}\right\|_{C(\Omega_{n-1})}$, $\delta_1(t) = \frac{\delta(t)}{C(\Gamma)}$. We suppose that $\Omega_c = \cup_{j=1}^{k_0} \overline{U}_{rect}(y_0^{(j)}, \frac{\delta_1(0)}{2\sqrt{n-1}})$ and $U_{rect}(y_0^{(p)}, \frac{\delta_1(0)}{2\sqrt{n-1}}) \cap U_{rect}(y_0^{(q)}, \frac{\delta_1(0)}{2\sqrt{n-1}}) = \emptyset$ for $1 \le p \ne q \le k_0 \ge 1$, where $U_{rect}(y^*, r) = \{y \in \mathbb{R}^{n-1} |\ |y_i - y_i^*| < r, i = 1, 2, \dots, n-1\}$ stands for a hypercube of $\mathbb{R}^{n-1}$ where y_i is the i−th component of y and so is y_i^* for y^*.

We first state a simple fact on geometry of $\mathbb{R}^{n-1}$. For $\rho > 0$, let $S = \{z_h = (\rho i_1, \dots, \rho i_{n-1}) | i_j \in \mathbb{Z}, j = 1, 2 \dots, n-1\}$ be the set of grid points of $\mathbb{R}^{n-1}$. Let $\{U_{rect}(z_h, \rho)|\ z_h \in S\}$ be a set of hypercube of $\mathbb{R}^{n-1}$. Then, the length of boundary of $U_{rect}(z_h, \rho)$ along any parallel direction of the orthogonal coordinate axis of $\mathbb{R}^{n-1}$ is just 2ρ. It is seen that along a fixed orthogonal coordinate direction of $\mathbb{R}^{n-1}$, any point of $\mathbb{R}^{n-1}$ belongs to at most two hypercubes of $\{U_{rect}(z_h, \rho)\}$. Since $\mathbb{R}^{n-1}$ has $n-1$ number of axes, any point of $\mathbb{R}^{n-1}$ belongs to at most 2^{n-1} number of hypercubes of $\{U_{rect}(z_h, \rho)\}$.

Let $\mathscr{X}(0) = \{y_0^{(j)}|\ U_{rect}(y_0^{(j)}, \frac{\delta(0)}{2\sqrt{n-1}}) \cap \Omega_{n-1} \ne \emptyset\}$. Obviously, $\{U_{rect}(y_0^{(j)}, \frac{\delta(0)}{\sqrt{n-1}})|\ y_0^{(j)} \in \mathscr{X}(0)\}$ is a cover of Ω_{n-1}. Taking $\rho = \frac{\delta(0)}{\sqrt{n-1}}$ as that in above paragraph, we see that there are at most 2^{n-1} number of such hypercubes such that

$$y \in \bigcup_{j=1}^{2^{n-1}} U_{rect}\left(y_0^{(j)}, \frac{\delta(0)}{\sqrt{n-1}}\right), y_0^{(j)} \in \mathscr{X}(0), \quad \forall\, y \in \Omega_{n-1} \tag{37}$$

and hence

$$\begin{aligned} &\sum_{j=1}^{N(0)} \text{meas}\left(\Omega_{n-1} \cap U_{rect}\left(y_0^{(j)}, \frac{\delta(0)}{\sqrt{n-1}}\right)\right) \\ &\le 2^{n-1}\text{meas}(\Omega_{n-1}), \end{aligned} \tag{38}$$

where $N(0) = \#\mathscr{X}(0)$. By the continuity of $\delta(t)$, for all sufficiently small $t > 0$,

$$\begin{aligned} &\sum_{j=1}^{N(0)} \text{meas}\left(\Omega_{n-1} \cap U_{rect}\left(y_0^{(j)}, \frac{\delta(t)}{\sqrt{n-1}}\right)\right) \\ &\le 2^{n-1}\text{meas}(\Omega_{n-1}). \end{aligned}$$

Since $\lim_{t\to\infty} \delta(t) = 0$, there exists a $t^* > 0$ such that for all $t > t^*$, $\{\Omega_{n-1} \cap U_{rect}(y_0^{(j)}, \frac{\delta(t)}{\sqrt{n-1}})\}_{j=1}^{N(0)}$ cannot cover Ω_{n-1}. Let

$$t_1 = \inf\left\{t > 0 \middle|\ \left\{\Omega_{n-1} \cap U_{rect}\left(y_0^{(j)}, \frac{\delta(t)}{\sqrt{n-1}}\right)\right\}_{j=1}^{N(0)} \text{cannot cover } \Omega_{n-1}\right\}.$$

Then there are a finite number of $\{y_1^{(j)}\}_{j=N(0)+1}^{N(t_1)} \in \Omega_{n-1} \setminus \{\Omega_{n-1} \cap U_{rect}(y_0^{(j)}, \frac{\delta(t_1)}{\sqrt{n-1}})\}_{j=1}^{N(0)}$ such that $\Gamma_1 \subset \{U_{rect}(y_1^{(j)}, \frac{\delta(t_1)}{\sqrt{n-1}})\}_{j=N(0)+1}^{N(t_1)} \cup \{U_{rect}(y_0^{(j)}, \frac{\delta(t_1)}{\sqrt{n-1}})\}_{j=1}^{N(0)}$. For notation simplicity, we still denote by $\{y^{(j)}\}_{j=1}^{N(t_1)} = \{y_0^{(j)}\}_{j=1}^{N(0)} \cup \{y_1^{(j)}\}_{j=N(0)+1}^{N(t_1)}$. Same to (37) and (38), we have

$$y \in \bigcup_{j=1}^{2^{n-1}} U_{rect}\left(y^{(j)}, \frac{\delta(t_1)}{\sqrt{n-1}}\right), y^{(j)} \in \mathscr{X}(t_1), \forall\, y \in \Omega_{n-1},$$

$$\begin{aligned} &\sum_{j=1}^{N(t_1)} \text{meas}\left(\Omega_{n-1} \cap U_{rect}\left(y^{(j)}, \frac{\delta(t_1)}{\sqrt{n-1}}\right)\right) \\ &\le 2^{n-1}\text{meas}(\Omega_{n-1}), \end{aligned} \tag{39}$$

where $\mathscr{X}(t_1) = \mathscr{X}(0) \cup \{y^{(j)}\}_{j=N(0)+1}^{N(t_1)}$, $N(t_1) = \#\mathscr{X}(t_1)$. By induction, there exist $\{t_i\}_{i=2}^\infty$ and $\{y^{(j)}\}_{j=1}^{N(t_i)}$ such that

$$y \in \bigcup_{j=1}^{2^{n-1}} U_{rect}\left(y^{(j)}, \frac{\delta(t_i)}{\sqrt{n-1}}\right), y^{(j)} \in \mathscr{X}(t_i), \forall\, y \in \Omega_{n-1}$$

$$\begin{aligned} &\sum_{j=1}^{N(t_i)} \text{meas}\left(\Omega_{n-1} \cap U_{rect}\left(y^{(j)}, \frac{\delta(t_i)}{\sqrt{n-1}}\right)\right) \\ &\le 2^{n-1}\text{meas}(\Omega_{n-1}), \end{aligned} \tag{40}$$

where $\mathscr{X}(t_i)$ is defined iteratively by

$$\mathscr{X}(t_{i+1}) = \mathscr{X}(t_i) \cup \{y^{(j)}\}_{j=N(t_i)+1}^{N(t_{i+1})}, t_0 = 0,\ i \ge 0. \tag{41}$$

By this construction, we see that the bounded measure cover

$$\bigcup_{j=1}^{N(t_i)}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(j)},\frac{\delta(t_i)}{\sqrt{n-1}}\right)\right)=\Omega_{n-1} \tag{42}$$

is a discrete series of cover which is independent of time t. Now we relate this cover with time t by setting

$$\begin{aligned}&\mathscr{X}(t):=\mathscr{X}(t_i), t\in[t_i,t_{i+1}),\\ &N(t)=\#\mathscr{X}(t), \lim_{t\to\infty}N(t)=\infty, i\geq 0.\end{aligned} \tag{43}$$

Then we get from (40) that for all $t\geq 0$,

$$\bigcup_{j=1}^{N(t)}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(j)},\frac{\delta(t)}{\sqrt{n-1}}\right)\right)=\Omega_{n-1}, \tag{44}$$

$$\begin{aligned}&\sum_{j=1}^{N(t)}\text{meas}\left(\Omega_{n-1}\cap U_{rect}\left(y^{(j)},\frac{\delta(t)}{\sqrt{n-1}}\right)\right)\\ &\leq 2^{n-1}\text{meas}(\Omega_{n-1}).\end{aligned} \tag{45}$$

Let $x^{(i)}=(y^{(i)},\psi(y^{(i)}))\in\Gamma_1$ for $i=1,2,\ldots$. Then

$$x\in\bigcup_{i=1}^{N(t)}U\left(x^{(i)},\delta(t)\right), \forall\, x\in\Gamma_1, t\geq 0, \tag{46}$$

and by (45),

$$\begin{aligned}&\sum_{i=1}^{N(t)}\text{meas}(\Gamma_1\cap U(x^{(i)},\delta(t)))\\ &=\sum_{i=1}^{N(t)}\int_{\Omega_{n-1}\cap U((y^{(i)},0),\delta(t))}\\ &\quad\sqrt{1+\psi_{x_1}^2+\psi_{x_2}^2+\ldots+\psi_{x_{n-1}}^2}\,dx_1dx_2\ldots dx_{n-1}\\ &\leq\sum_{i=1}^{N(t)}\int_{\Omega_{n-1}\cap U_{rect}(y^{(i)},\delta(t))}\\ &\quad\sqrt{1+\psi_{x_1}^2+\psi_{x_2}^2+\ldots+\psi_{x_{n-1}}^2}\,dx_1dx_2\ldots dx_{n-1}\\ &\leq\sum_{i=1}^{N(t)}C(\Gamma_1)\text{meas}(\Omega_{n-1}\cap U_{rect}(y^{(i)},\delta(t)))\\ &\leq C(\Gamma_1)(n-1)^{\frac{n-1}{2}}2^{n-1}\text{meas}(\Gamma_1), \forall\, t\geq 0,\end{aligned} \tag{47}$$

where

$$C(\Gamma_1)=\left\|\sqrt{1+\psi_{x_1}^2+\psi_{x_2}^2+\ldots+\psi_{x_{n-1}}^2}\right\|_{C(\Omega_{n-1})}.$$

Combining (46) and (47) gives the required result. ∎

Proof of Lemma 2.2. Let

$$\tilde{y}_i(t)=r(t)[\hat{y}_i(t)-y_i(t)],\ \tilde{d}_i(t)=\hat{d}_i(t)-d(\xi_i(t),t) \tag{48}$$

be the errors, and we denote by

$$\begin{aligned}V_i(t):=&\left(\tilde{y}_i(t),\tilde{d}_i(t)\int_{\Gamma_1}g_i^t(x)dx\right)\\ &\times P\left(\tilde{y}_i(t),\tilde{d}_i(t)\int_{\Gamma_1}g_i^t(x)dx\right)^{\top},\ i=1,2,\ldots,\end{aligned} \tag{49}$$

where the positive definite matrix P is the solution of the following Lyapunov equation:

$$F^{\top}P+PF=-I_{2\times 2}, F=\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}.$$

Since

$$\lambda_{\min}(P)\|(y_1,y_2)\|^2\leq V(y_1,y_2)\leq\lambda_{\max}(P)\|(y_1,y_2)\|^2, \tag{50}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of P, respectively. By (28) and (29), finding the derivative of V_i along the $(\tilde{y}_i,\tilde{d}_i)$ to yield

$$\begin{aligned}\dot{V}_i(t)\leq&-r(t)\left\|\left(\tilde{y}_i(t),\tilde{d}_i(t)\int_{\Gamma_1}g_i^t(x)dx\right)\right\|^2\\ &+N_1\left\|\left(\tilde{y}_i(t),\tilde{d}_i(t)\int_{\Gamma_1}g_i^t(x)dx\right)\right\|^2\\ &+N_2\left\|\left(\tilde{y}_i(t),\tilde{d}_i(t)\int_{\Gamma_1}g_i^t(x)dx\right)\right\|,\end{aligned} \tag{51}$$

where N_1 and N_2 are two positive constants. In the last step of (51), (26) was used. This together with (50) gives

$$\begin{aligned}\frac{dV_i(t)}{dt}\leq&-\frac{r(t)}{\lambda_{\max}(P)}V_i(t)+\frac{N_1}{\lambda_{\min}(P)}V_i(t)\\ &+\frac{N_2}{\lambda_{\min}(P)}\sqrt{V_i(t)}.\end{aligned} \tag{52}$$

Since $\lim_{t\to\infty}r(t)=+\infty$, there exists $t_0>0$ such that $r(t)>\frac{2\lambda_{\max}(P)}{\lambda_{\min}(P)}N_1$ for all $t\geq t_0$. This together with (52) shows that for all $t\geq t_0$,

$$\frac{d\sqrt{V_i(t)}}{dt}\leq-\frac{1}{4\lambda_{\max}(P)}r(t)\sqrt{V_i(t)}+\frac{N_2}{2\lambda_{\min}(P)}. \tag{53}$$

Applying the L'Hospital rule and assumption (20), we obtain

$$\lim_{t\to\infty}\sqrt{V_i(t)}=0. \tag{54}$$

The left part of the proof needs property of g_i^t. The details are omitted. ∎

Proof of Lemma 2.3. Define

$$\tilde{d}(x,t)=\hat{d}(x,t)-d(x,t),\forall\, x\in\Gamma_1, t\geq 0. \tag{55}$$

Since $d(\cdot,t)$ is Hölder continuous with index $\alpha\in(0,1]$ and satisfies $|d(x',t)-d(x'',t)|\leq K(t)|x'-x''|^{\alpha}$. For the given $\varepsilon(t)>0$, since $\delta(t)=(\frac{\varepsilon(t)}{K(t)})^{\frac{1}{\alpha}}>0$, we have $|d(x',t)-d(x'',t)|\leq\varepsilon(t)$ as long as $|x'-x''|\leq\delta(t)$. Moreover, since $\xi_i:[0,\infty)\to\Gamma_1\cap U(x^{(i)},\delta(t))$ which is defined in (25), we have

$$\begin{aligned}&\|d(\xi_i(t),t)-d(\cdot,t)\|_{L^2(\Gamma_1\cap U(x_i,\delta(t)))}\\ &\leq\text{meas}(\Gamma_1\cap U(x^{(i)},\delta(t)))\varepsilon(t),\forall\, t\geq 0.\end{aligned} \tag{56}$$

From this we can prove

$$\lim_{t\to\infty}\|\tilde{d}(\cdot,t)\|_{L^2(\Gamma_1)}=0. \tag{57}$$

The details are omitted. ∎

Proof of Theorem 2.1. Using the error variables $(\tilde{y}_i,\tilde{d}_i)$ defined in (48), we can write the equivalent system of (34) as follows:

$$
\begin{cases}
w_{tt}(x,t) - \Delta w(x,t) = 0, x \in \Omega, t > 0, \\
w(x,t)|_{\Gamma_0} = 0, \\
\frac{\partial w}{\partial \nu}(x,t)\big|_{\Gamma_1} = -kw_t(x,t) - \hat{d}(x,t) + d(x,t), \\
\dot{\tilde{y}}_i(t) = -r(t)\tilde{y}_i(t) + \tilde{d}_i(t)\int_{\Gamma_1} g_i(x)dx + \frac{\dot{r}(t)\tilde{y}_i(t)}{r(t)}, \\
\frac{d}{dt}\left(\tilde{d}_i(t)\int_{\Gamma_1} g_i(x)dx\right) = -r(t)\tilde{y}_i(t) \\
\qquad -\frac{d}{dt}d\left(\xi_i(t),t\right)\int_{\Gamma_1} g_i(x)dx\right).
\end{cases}
\tag{58}
$$

The "ODE part" in (58) has been shown in (30) through (48) to tend zero as $t \to \infty$. Now we only need to consider the "w part" of system (58) which is rewritten as

$$
\begin{cases}
w_{tt}(x,t) - \Delta w(x,t) = 0, x \in \Omega, t > 0, \\
w(x,t)|_{\Gamma_0} = 0, t \geq 0, \\
\frac{\partial w}{\partial \nu}(x,t)\big|_{\Gamma_1} = -kw_t(x,t) - \tilde{d}(x,t), t \geq 0.
\end{cases}
\tag{59}
$$

Exactly to (12), we write (59) as

$$
\ddot{w} = -\tilde{A}w - kB^*B\dot{w} - B\tilde{d} \text{ in } [D(A^{1/2})]' \tag{60}
$$

Let $\mathbb{A}_k$ and $\mathbb{B}$ be defined in (14) with replacement of k_1 by k only. It is known that $e^{\mathbb{A}_k t}$ is exponentially stable ([5, p.668]). By the admissibility of $\mathbb{B}$ proved in Proposition 1.1, along the same line as that presented in [2], we can prove that the solution of (59) satisfies

$$
\lim_{t\to\infty} \left\| \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\|_{\mathcal{H}} = 0. \tag{61}
$$

The remaining part of the proof is straightforward. ∎

4. NUMERICAL SIMULATION

In this section, we present some numerical simulations for Example 1.1 for illustration. The purpose are twofold. The first is to verify the theoretical results and the second is to look at the peaking value reduction by the time varying gain approach. Now, $n = 2$, the closed-loop system is (34), $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | \ 1 < x_1^2 + x_2^2 < 4\}$, $\Gamma_0 = \{(x_1, x_2) \in \mathbb{R}^2 | \ x_1^2 + x_2^2 = 1\}, \Gamma_1 = \partial\Omega \setminus \Gamma_0$, g_i^t is defined by (23), y_i and y_{2i} are defined by (27), ξ_i is define dy (25), and $\hat{d}$ is defined by (31), r is the time varying gain in (20). For numerical computations, we take parameter $k = 3$, disturbance $d(x,t) = \sin(x_1 t)$, and the following initial values in the polar coordinate form:

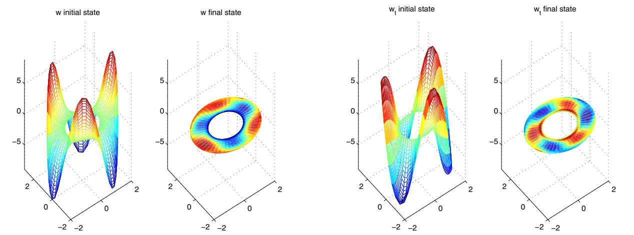
$$
\begin{cases}
w(\gamma,\theta,0) = (\gamma^2-1)^2\cos(3\theta), 1 < \gamma < 2, 0 < \theta < 2\pi, \\
w_t(\gamma,\theta,0) = 9\sin(2\gamma-2)\sin(3\theta), 1 < \gamma < 2, 0 < \theta < 2\pi, \\
\tilde{y}_i(0) = 2r(0), \ \tilde{d}_i(0) = 1.5, i = 1, 2, \ldots.
\end{cases}
\tag{62}
$$

The backward Euler method in time and the Chebvshev spectral method for polar variables are used to discretize system (34) under the polar coordinates. Here we take the grid size $r_N = 30$ for γ, the grid size $\theta_N = 50$ for θ, and the time step $dt = 5 \times 10^{-4}$. The time varying gain function r is taken as

$$
r(t) = \min\{e^{5t}, 30\}. \tag{63}
$$

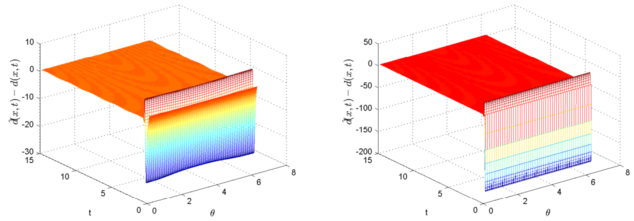
Figures 1(a) and 1(b) display the displacement w and the velocity w_t at the initial time $t = 0$ and the time $t = 15$, respectively.

Figure 2 plots the tracking errors for the disturbance where Figure 2(a) is with the time varying gain (63) and Figure 2(b) is with the constant gain $r = 30$. It is clearly seen from these figures that the peaking value from Figure 2(b) is dramatically reduced by the time varying gain in Figure 2(a).



(a) Displacement w at initial time $t = 0$ and time $t = 15$.
(b) Velocity w_t at initial time $t = 0$ and final time $t = 15$.

Fig. 1. The initial state and state at $t = 15$ of system (34) with $d(x,t) = \sin(x_1 t)$.



(a)
(b)

Fig. 2. (a) The error $\hat{d} - d$ under the time varying high gain; (b) The error $\hat{d} - d$ under the constant high gain.

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