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FO sliding surface for the robust control of integer-order LTI plants

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Abstract: This paper investigates the possible adoption of a fractional order sliding surface for the robust control of perturbed integer-order LTI systems. It is proved that the standard approach used in Sliding Model Control (SMC) cannot be used and a substantial redesign of the control policy is needed. A novel control strategy is discussed, ensuring that the sliding manifold is hit at an infinite succession of time instants becoming denser as time grows. Interesting asymptotic properties are derived relatively to the closed loop response in the presence of a wide class of disturbances. It is also proved that the chattering phenomenon may be remarkably alleviated. A careful simulation study is reported using an electromechanical system taken from the literature, which includes also a comparative analysis of performances with respect to standard SMC and second-order SMC.

Keywords: Fractional-Order Control, Sliding-Mode Control, Robust Control.

1. INTRODUCTION

In recent years, fractional calculus has attracted the interest of the control community in relation to those physical real world phenomena which have been found to be effectively modeled with Fractional Order (FO) dynamics Podlubny [2002], Atanackovic et al. [2007], Caponetto et al. [2010], Ding and Te [2009].Since a relevant number of fractional calculus applications can be found in different areas Magin [2006], Monje et al. [2010], the scientific interest mostly focussed on the control of inherently FO systems, and extensions of many different control approaches have been derived to accommodate fractional order systems Efe [2010], Agrawal [2004], Dadras and Momeni [2010], Oustaloup et al. [1996], Podlubny [1999a], Delavari et al. [2010], Faieghi et al. [2012].

With reference to the well known field of Sliding Mode Control (SMC) Utkin [1992], the adoption of a sliding surface, possibly of fractional order, to deal with FO systems is well established in the literature, this approach being known as *fractional sliding mode control* Dadras and Momeni [2011], El-Khazali [2005], Pisano et al. [2010]. In particular, the use of second-order sliding mode approaches to control and estimation of FO dynamics has been widely investigated by Pisano et al. [2011], Pisano and Caponetto [2013], Pisano et al. [2012b], with applications to fault detection Pisano et al. [2011, 2012a]. In this context, the basic features and control design methods typical of sliding mode control are shown to be mostly extendable, at least for some classes of FO plants, from the classical integer-order setup to the FO framework. To the best of the authors' knowledge, however, the possibility of using a fractional order sliding surface for Linear Time-Invariant (LTI) integer-order systems has been never investigated, although it could be worthwhile to study the possibility that fractional order dynamics are imposed, by forcing the plant onto a FO sliding surface, to the reduced order system of an originally LTI integer-order system. Undoubtedly, it can be foreseen that, due the particular form of such a surface, the concept itself of sliding motion could need to be revisited, since it should be first established if a sliding motion can be achieved and, in this case, which condition can be imposed for its achievement.

To pursue the above goal, the present note investigates the possible adoption of a FO sliding surface $s(t) = 0$ for controlling an LTI integer-order, perturbed plant. After proving the asymptotical stability of the plant restricted onto the surface, it will be shown that the traditional strategy of choosing a discontinuous control input, in addition to the equivalent control, satisfying the classical sliding condition cannot be used for the proposed FO sliding surface. An ad hoc control strategy will be proposed ensuring that the sliding manifold is hit at an infinite succession of time instants becoming denser as t grows to infinity. Moreover, the robust asymptotical achievement of the condition $s(t) = 0$ will be proved for the system restricted onto the surface. Investigating the behavior of the system, interesting asymptotical properties will be derived relatively to the closed loop response to wide classes of disturbances. Moreover, the non uniform distribution of the time instants when the crossing of the sliding manifold occurs, becoming denser asymptotically, coupled with the asymptotical vanishing of the function $s(t)$ suggests that the well known chattering phenomenon may be remarkably alleviated. In order to test the practical feasibility of the proposed approach, a detailed simulation study is reported in this note. Reference has been made to the electromechanical system studied by Eker [2010], Eker [2013], and a careful performance comparison has been made with standard SMC Utkin [1992] and second-order SMC Bartolini et al. [2003], Levant [2005].

2. PRELIMINARIES

2.1 Preliminary definitions

Given a real number $\alpha \in]0,1[$, the Caputo derivative (Podlubny [1999b]) of a function $f(t)$ is defined as

$$D^{\alpha} f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau, \qquad (1)$$

and the Caputo integral is given by

$$D^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \qquad (2)$$

where $\Gamma(\cdot)$ is the Euler-Gamma function.

Proposition 1. Caputo derivative (1) is only right invertible.

Proposition 2. If $f(t)$ is regular enough, then

$$\frac{d}{dt}\left(D^{\alpha} f(t)\right) = D^{\alpha}\left(\frac{d}{dt} f(t)\right) + \frac{f(0)}{\Gamma(1-\alpha) t^{\alpha}}. \qquad (3)$$

In the following, the notation $f(t) \approx g(t)$ means that functions $f(t)$ and $g(t)$ have the same asymptotic behaviour. Bold type symbols (i.e. $\mathbf{x}$) denote vectors or matrices depending on the context, while scalars are denoted by non-bold characters (i.e. x).

2.2 The plant

Consider an LTI system with matched uncertainties

$$\dot{\overline{\mathbf{x}}}(t) = \overline{\mathbf{A}}\overline{\mathbf{x}}(t) + \overline{\mathbf{B}}(u(t) + d(t)) \qquad (4)$$

where $\overline{\mathbf{x}}(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the input vector and $d(t)$ is an unknown signal representing possible disturbances and/or perturbations.

Assumption 1. The couple $(\overline{\mathbf{A}}, \overline{\mathbf{B}})$ is controllable. Moreover, the disturbance term $d(t)$ belongs to the following general class

$$\mathcal{D}_p := \{d(t) : |d(t)| \leq \rho\, t^p,\, p \geq 0,\, \rho > 0\}. \qquad (5)$$

Due to Assumption 1, there always exists a transformation matrix $\mathbf{T}$ which brings the system into the canonical controllability form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}(u(t) + d(t)) \qquad (6)$$

with

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & A_{22} \end{bmatrix}, \quad \mathbf{B} = [0, \cdots, 1]^T. \qquad (7)$$

where $\mathbf{A_{11}} \in \mathbb{R}^{n-1 \times n-1}$, $\mathbf{A_{12}} \in \mathbb{R}^{n-1 \times 1}$, $\mathbf{A_{21}} \in \mathbb{R}^{1 \times n-1}$, $A_{22} \in \mathbb{R}$. Consequently, the state vector is partitioned as $\mathbf{x}(t) = [\mathbf{x_1}(t), x_2(t)]$ with $\mathbf{x_1}(t) \in \mathbb{R}^{n-1}$ and $x_2(t) \in \mathbb{R}$.

3. A FRACTIONAL ORDER SURFACE

Consider the following function

$$s(t) := D^{\alpha}(\mathbf{C_1}\mathbf{x_1}(t) + \mathbf{C_2} D^{-1}\mathbf{x_1}(t)) =$$
$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathbf{C_1}\dot{\mathbf{x}}_{\mathbf{1}}(\tau) + \mathbf{C_2}\mathbf{x_1}(\tau)}{(t-\tau)^{\alpha}} d\tau \qquad (8)$$

with $\alpha \in]0,1[$, $\mathbf{C_1}, \mathbf{C_2} \in \mathbb{R}^{1 \times (n-1)}$ are row vectors of positive real values and the initial values are $\mathbf{x_1}(0) = x_2(0)$ and $x_2(0) = x_{20} \in \mathbb{R}$ with x_{20} arbitrary chosen. By (8) the sliding surface $s(t) = 0$ is defined.

Proposition 3. The vectors $\mathbf{C_1}, \mathbf{C_2}$ in (8) can be always designed such that the system (4) restricted to the surface $s(t) = 0$ has stable assigned eigenvalues.

Proof. When the plant is restricted onto the surface $s(t) = 0$ for any t, the integrand of (8) is null, i.e. it holds $\forall t$

$$\mathbf{C_1}\dot{\mathbf{x}}_{\mathbf{1}}(t) + \mathbf{C_2}\mathbf{x_1}(t) = 0$$

hence

$$x_2 = -(\mathbf{C_1}\mathbf{A_{12}})^{-1}(\mathbf{C_1}\mathbf{A_{11}} + \mathbf{C_2})\mathbf{x_1}$$

and the system restricted to $s(t) = 0$ verifies $\dot{\mathbf{x}}_{\mathbf{1}} = \mathbf{H}\mathbf{x_1}$, with $\mathbf{H} := (\mathbf{I}_{n-1} - (\mathbf{C_1}\mathbf{A_{12}})^{-1}\mathbf{A_{12}}\mathbf{C_1})\mathbf{A_{11}} - (\mathbf{C_1}\mathbf{A_{12}})^{-1}\mathbf{A_{12}}\mathbf{C_2}$. By construction $\mathbf{H}$ is a companion matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n-1} \\ -h_1 \ \ldots & -h_{n-1} \end{bmatrix},$$

so $\mathbf{C_1}, \mathbf{C_2}$ can always be designed such that the roots of the polynomial $\lambda^{n-1} + h_{n-1}\lambda^{n-2} + \ldots + h_{j+1}\lambda^j + \ldots + h_1 = 0$ lie strictly in the open left half of the complex plane. ■

Define the control input as the sum of an equivalent input and a nonlinear input, i.e

$$u(t) := u_e(t) + u_n(t) \qquad (9)$$

and choose the equivalent control $u_e(t)$ so that $\dot{s}(t)$ vanishes in the disturbance-free case:

$$u_e(t) = -(\mathbf{C_1}\mathbf{A_{12}})^{-1}((\mathbf{C_1}\mathbf{A_{11}}\mathbf{A_{11}} + \mathbf{C_1}\mathbf{A_{12}}\mathbf{A_{21}} + \mathbf{C_2}\mathbf{A_{11}})\mathbf{x_1}(t) +$$
$$+ (\mathbf{C_1}\mathbf{A_{11}}\mathbf{A_{12}} + \mathbf{C_1}\mathbf{A_{12}}A_{22} + \mathbf{C_2}\mathbf{A_{12}})x_2(t)). \qquad (10)$$

Since Caputo derivative is right invertible (Proposition 1), due to Proposition 2 the time derivative of (8) is given by

$$\dot{s}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathbf{C_1}\ddot{\mathbf{x}}_{\mathbf{1}}(\tau) + \mathbf{C_2}\dot{\mathbf{x}}_{\mathbf{1}}(\tau)}{(t-\tau)^{\alpha}} d\tau =$$
$$= K \int_0^t \frac{u_n(\tau) + d(\tau)}{(t-\tau)^{\alpha}} d\tau\,. \qquad (11)$$

with $K := \mathbf{C_1}\mathbf{A_{12}}/\Gamma(1-\alpha) > 0$. As well known, one of the two steps constituting the traditional design of a sliding mode controller is the design of a discontinuous state-feedback control $u_n(t)$ forcing the system state to reach the sliding manifold imposing $s(t)\dot{s}(t) < 0 \ \forall t$.

The following result shows that such usual strategy of choosing $u_n(t)$ satisfying the classical sliding condition cannot be used for the proposed FO sliding surface (8).

Proposition 4. Consider the uncertain LTI system (4) under Assumption 1. There does not exist any control input $u(t)$ able to guarantee the fulfillment of the sliding condition $s(t)\dot{s}(t) < 0 \ \forall t$.

Proof. Due to (11), the sliding condition reads

$$s(t)\,\dot{s}(t) = K s(t) \int_0^t \frac{u_n(\tau) + d(\tau)}{(t-\tau)^{\alpha}} d\tau < 0\,.$$

If $s(t) > 0$, due to (5) and Euler's function properties (Whittaker and Watson [1992]), the sliding condition requires that

$$\int_0^t \frac{u_n(\tau)}{(t-\tau)^{\alpha}} d\tau < -\frac{\Gamma(1-\alpha)\Gamma(p+1)}{\Gamma(-\alpha+p+2)} \rho\, t^{1-\alpha+p}. \qquad (12)$$

Analogously if $s(t) < 0$ the sliding condition requires

$$\int_0^t \frac{u_n(\tau)}{(t-\tau)^{\alpha}} d\tau > \frac{\Gamma(1-\alpha)\Gamma(p+1)}{\Gamma(-\alpha+p+2)} \rho\, t^{1-\alpha+p}. \qquad (13)$$

Since the integral of any function is a continuous function, a control $u_n(t)$ able to guarantee simultaneously (12) for $s(t) > 0$ and (13) for $s(t) < 0$ does not exist. ■

3.1 A relaxed sliding condition

A different condition will be imposed such that the condition $s(t) = 0$ is anyway robustly fulfilled asymptotically. To this purpose, consider a non-uniform partition of the time axis $[0, +\infty[$ by an infinite number of intervals I_i with $I_0 = [0, t_1]$ and $I_i =]t_i, t_{i+1}]$ for $i > 0$, where the infinite succession of instants t_i verifies

$$t_{i+1} = t_i + h/(i + 1)\,, \tag{14}$$

with a real positive constant h. For simplicity we set $h = 1$.

Condition 3.1. (Relaxed sliding condition). Consider the perturbed plant (4) under Assumption 1, and the surface (8). If it robustly holds that $\lim_{i\to\infty} s(t_i) = 0$ for t_i satisfying (14), then a relaxed sliding condition is said to hold. Such condition, in fact, ensures that the system state is forced onto the sliding surface at infinite time instants becoming infinitely dense as $i \to \infty$.

Remark 5. If in addition to Condition 3.1 it can be proved that $\lim_{i\to\infty} \sup_{t\in I_i} s(t) = 0$ then:

- the system (4) constrained onto the surface $s(t) = 0$ (8) is robustly asymptotically stable;
- commutation at infinitely high frequency between the conditions $s(t) > 0$ and $s(t) < 0$ occurs asymptotically. Since the function (8) vanishes asymptotically, chattering alleviation is achieved.

Considering (11),

$$s(t)=s(0)+K\int_0^t \dot{s}(\tau)d\tau = s(0)+K\int_0^t\int_0^\ell \frac{u_n(\tau)+d(\tau)}{(t-\tau)^\alpha}d\tau d\ell$$

with $K > 0$. By Fubini's theorem (Apostol [1969]) one gets

$$s(t)=s(0) + K\int_0^t (u_n(\ell) + d(\ell))(t - \ell)^{1-\alpha}d\ell. \tag{15}$$

In order to simplify notation, the following definitions are introduced

$$s(t) := s_i(t) \text{ and } u_n(t) := u_{ni}(t) \quad for \quad t \in I_i \tag{16}$$

$$U_i(a, b, c) :=K\int_a^b u_{ni}(\ell)(c - \ell)^{1-\alpha}d\ell\,, \tag{17}$$

$$E_i(t) :=K\int_{t_{i-1}}^{t_i} d(\ell)((t - \ell)^{1-\alpha} - (t_i - \ell)^{1-\alpha})d\ell\,, \tag{18}$$

$$D_i(t) :=K\int_{t_i}^t d(\ell)(t - \ell)^{1-\alpha}d\ell\,, \tag{19}$$

$$\epsilon_i(t) := \begin{cases} E_1(t) & i = 1, \\ \sum_{j=1}^{i-1} E_j(t) - E_j(t_i) + E_i(t) & i > 1. \end{cases} \tag{20}$$

Moreover, consider a control input $u_n(t)$ of the form

$$u_n(t) = \begin{cases} u_{n0}(t) = 0 \quad t \in [0, t_1], \\ u_{ni}(t) = \dfrac{L_i}{t^{2-\alpha}} \quad t \in I_i\,, i > 0 \end{cases} \tag{21}$$

with

$$L_i := \frac{C_i}{K(\mathcal{B}(t_i/t_{i+1}, \alpha - 1, 2 - \alpha) + \pi/\sin(\pi\alpha))}, \tag{22}$$

$$C_i := \begin{cases} s_0(t_1) & i = 1, \\ \sum_{j=1}^{i-1} U_j(t_j, t_{j+1}, t_{i+1}) + \sum_{j=0}^{i-1} s_j(t_{j+1}) & i > 1 \end{cases} \tag{23}$$

where $\mathcal{B}(\cdot, \cdot, \cdot)$ is the incomplete Beta function.

Remark 6. The control input (21) guarantees that for $i \geq 0$

$$U_i(t_i, t_{i+1}, t_{i+1}) = -C_i. \tag{24}$$

In order to prove Proposition 9 the following results need to be addressed.

Remark 7. The quantities (18) and (20) verify the following equation

$$\sum_{j=1}^{i} \epsilon_j(t_{j+1}) = \sum_{j=0}^{i-1} E_{j+1}(t_{i+1}). \tag{25}$$

Lemma 8. Let $s(t)$ be the function defined in (15) and set $s(t) = s_i(t)$ for $t \in I_i =]t_i, t_{i+1}]$ with t_i given by (14). Due to (17)-(23) for $i > 0$

$$s_i(t_{i+1}) = \epsilon_i(t_{i+1}) + D_i(t_{i+1})\,.$$

Proof. The result follows by induction from (15)-(19), (23), Remark 6 and Remark 7. For details see Corradini et al. [2014]. ■

Proposition 9. Consider the uncertain LTI system (4) under Assumption 1. The control input $u(t)$ given by (9), (10), (21) ensures that the relaxed sliding condition 3.1 holds, i.e.

$$\lim_{i\to\infty} s(t_i) = 0$$

for all time instants t_i defined as (14).

Proof. Let $s(t) = s_i(t)$ for $t \in I_i$. By Lemma 8 we get that for $t \in I_i$ with $i > 0$

$$s_i(t_{i+1}) = \epsilon_i(t_{i+1}) + D_i(t_{i+1}). \tag{26}$$

In order to guarantee the asymptotically vanishing of $s_i(t_{i+1})$ we will show that each term on the right of (26) is smaller than a vanishing quantity.

Under Assumption 1, by (20), (14), Lagrange theorem, Beta function's properties and McLaurin-Cauchy test (Whittaker and Watson [1992])

$$|\epsilon_i(t_{i+1})| \leq \frac{(1-\alpha)K\rho}{i+1}\frac{\Gamma(1-\alpha)\Gamma(p+1)}{\Gamma(p-\alpha+2)}t_i^{1-\alpha+p} \approx \frac{(\log i)^{1-\alpha+p}}{i+1}.$$

By (19) and the Mean value theorem (Apostol [1967]),

$$|D_i(t_{i+1})| \leq K\frac{(1 + \log(i + 1))^p}{(i + 1)^{2-\alpha}} \approx \frac{(\log(i + 1))^p}{(i + 1)^{2-\alpha}}\,. \tag{27}$$

In conclusion for $i > 1$

$$s_i(t_{i+1}) \approx \frac{(\log(i + 1))^{1-\alpha+p}}{i + 1}, \tag{28}$$

so it vanishes when i tends to infinity. ■

Lemma 10. Set a constant value $\alpha \in]0, 1[$. Under Assumption 1 the coefficient L_i defined in (22) asymptotically develops like

$$L_i \approx \begin{cases} \log(\log i)\,(\log i)^{2-\alpha} & \text{if } p = 0, \\ (\log i)^{2-\alpha} & \text{if } p > 0. \end{cases}$$

Proof. At first note that since $\alpha \in]0,1[$ and t_i are verifies (14), it holds that $B(t_1/t_2, \alpha-1, 2-\alpha) \leq B(t_i/t_{i+1}, \alpha-1, 2-\alpha) \leq \Gamma(\alpha-1)\Gamma(2-\alpha)$. Let $W := |\mathcal{B}(t_1/t_2, \alpha-1, 2-\alpha) + \pi/\sin(\pi\alpha)|$.

Due to (17) and (21), (22) can be written as

$$L_i = \sigma_i + \sum_{j=1}^{i-1} \beta_{i,j} L_j \tag{29}$$

where by (28) and McLaurin-Cauchy test

$$|\sigma_i| \leq \frac{1}{W}\left(|s_0(t_1)| + |s_1(t_2)| + \sum_{j=2}^{i} \frac{(\log j)^{1-\alpha+p}}{j}\right) \approx (\log i)^{2-\alpha+p};$$

in addition due to the Mean value integration theorem and McLaurin-Cauchy test

$$|\beta_{i,j}| \leq \frac{(\log(i+1))^{1-\alpha}}{(j+1)(\log(j+1))^{2-\alpha}} \approx \frac{(\log i)^{1-\alpha}}{j(\log j)^{2-\alpha}}.$$

Collecting all the estimates obtained we have that

$$L_i = (\log i)^{2-\alpha+p} + (\log i)^{1-\alpha} \sum_{j=0}^{i-1} \frac{L_j}{j(\log j)^{2-\alpha}}.$$

Solving the corresponding continuous-time ordinary differential equation and rewriting the solution in discrete-time we have that the asymptotic behaviour of L_i is given by

$$L_i \approx \begin{cases} \log(\log i)(\log i)^{2-\alpha} & p = 0 \\ (\log i)^{2-\alpha} & p > 0 \end{cases}$$

■

Corollary 11. Under Assumption 1, the input signal $u_n(t)$ defined in (21) has the following asymptotic behaviour

$$u_n(t) \approx \begin{cases} \log(\log i) & p = 0, \\ 1 & p > 0. \end{cases}$$

Proof. Due to Lemma 10 and the Integral criterion of convergence (Whittaker and Watson [1992]), if $p = 0$, then the input signal $u_{ni}(t)$ defined in (21) verifies

$$u_{ni}(t) \leq \frac{L_i}{(\log i)^{2-\alpha}} \approx \log(\log i). \tag{30}$$

Otherwise if $p > 0$, $u_{ni}(t) \approx k_u$ where k_u is a constant less than 1. ■

Theorem 3.1. Consider the perturbed plant (4) under Assumption 1, driven by the input signal $u(t)$ given by (9), (10), (16) and (21). It can be proved that $s(t)$ defined in (8) verifies $\lim_{i\to\infty} \sup_{t\in I_i} s(t) = 0$.

Proof. In the hypothesis, due to Proposition 9, the relaxed sliding condition 3.1 holds so it is sufficient to prove that for $t \in I_i$ the difference $s_i(t) - s_{i-1}(t_i)$ vanishes when i increases.
By (15) we get $s_i(t) - s_{i-1}(t_i) = Q_1 + Q_2 + Q_3$ with

$$Q_1 = K\int_0^{t_i} u_n(\ell)((t-\ell)^{1-\alpha} - (t_i-\ell)^{1-\alpha})d\ell, \tag{31}$$

$$Q_2 = K\int_{t_i}^{t} u_{ni}(\ell)(t-\ell)^{1-\alpha}d\ell, \tag{32}$$

$$Q_3 = K\int_0^{t_i} d(\ell)((t-\ell)^{1-\alpha} - (t_i-\ell)^{1-\alpha})d\ell + K\int_{t_i}^{t} d(\ell)(t-\ell)^{1-\alpha}d\ell. \tag{33}$$

The cases $p = 0$ and $p > 0$ will be addressed separately. Consider first $p = 0$. By Assumption 1 $|d(t)| \leq \rho$ and due to Corollary 11 $|u_n(t)| \leq \log(\log i)$, therefore all the terms (31)-(33) vanish asymptotically, indeed

$$|Q_1| \leq K\frac{\log(\log i)}{i+1}(1+\log i)^{1-\alpha} \approx \frac{(\log i)^{2-\alpha}}{i+1},$$

$$|Q_2| \leq K\frac{\log(\log i)}{(2-\alpha)(i+1)^{2-\alpha}},$$

$$|Q_3| \leq \frac{\rho K}{2-\alpha}(t^{2-\alpha} - t_i^{2-\alpha}) \approx \frac{\rho K(\log(i+1))^{1-\alpha}}{(i+1)}.$$

Consider next the case $p > 0$. By Corollary 11 $|u_n(t)| \approx 1$. As in the previous case, terms (31)-(33) vanish asymptotically. ■

4. A SIMULATION STUDY

To test the feasibility of the presented approach and evaluate achievable performances, the electromechanical system studied by Eker [2010] has been considered. Denoting by $x_1(t)$ the output measured shaft speed $\omega_L(t)$, $x_2(t)$ the time derivative $\dot{\omega}_L(t)$ and $u(t)$ the control input (armature voltage, whose norm is limited by 10 V), the control objective is the tracking of a desired set-point $\omega_r(t) = 1000$ [rpm]. The linearised system with zero initials is given by

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = a_1x_1(t) + a_2x_2(t) + b\,u(t) + d(t) \end{cases} \tag{34}$$

where $a_1 = -783.5763$, $a_2 = -118.1663$, $b = 644.0997$ and $d(t)$ denotes the uncertainty. We assumed that the disturbance signal belongs to the class $\mathcal{D}_1$ (5) so in particular $|d(t)| \leq \rho t$ with $\rho = 5000$. Note that the eigenvalues of the state matrix are -7.05 and -111.1.

To evaluate the performances achievable with the control policy described above, a systematic comparison with the first-order sliding mode controller (SMC) and the second-order sliding mode controller (2-SMC) reported by Eker have been performed. The two control laws are here shortly reported for the reader's convenience.The sampling period for both the SMC and the 2-SMC approach is 3 ms.

Define the tracking error as $e(t) = \omega_r(t) - x_1(t)$. According to Eker [2010] the SMC sliding surface is given by

$$s_{SMC}(t) = \lambda_c e(t) + \dot{e}(t) \tag{35}$$

with $\lambda_c = 20$ and the control $u_{SMC}(t) = u_{eSMC}(t) + u_{nSMC}(t)$ is defined as

$$u_{eSMC}(t) = -(a_1x_1 + (\lambda_c + a_2)x_2(t))/b, \tag{36}$$

$$u_{nSMC}(t) = k_{sc}\tanh(s_{SMC}(t)/\Omega_c)/b, \tag{37}$$

with $k_{sc} = 15000$ and $\Omega_c = 20$. In Eker [2010] the PID sliding surface for the 2-SMC method is

$$\dot{s}_{2SMC}(t) + \beta s_{2SMC}(t) = k_pe(t) + k_i\int_0^t e(\tau)d\tau + k_d\dot{e}(t) \tag{38}$$

where $\beta = 160$, $k_p = 32$, $k_i = 2$, $k_d = 1$ and the control $u_{2SMC}(t) = u_{e2SMC}(t) + u_{n2SMC}(t)$ is

$$u_{e2SMC}(t) = -(k_ie(t) - k_da_1x_1 - (k_da_2 + k_p)x_2(t) + \beta\dot{s}_{2SMC}(t))/(b\,k_d), \tag{39}$$

$$u_{n2SMC}(t) = \lambda_1 s(t) + k_s\tanh(s_{2SMC}(t)/\Omega), \tag{40}$$

with $\lambda_1 = 200$ and $\Omega = 20$. As remarked by Eker Eker [2013] in his reply to Naderi and Faieghi [2013] the control

(39)-(40) can be referred as second-order sliding mode Levant [2007], since $u_{2SMC}(t)$ is discontinuous, (38) is a function of (39)-(40) and therefore s_{2SMC} and all its derivatives are discontinuous Levant [2005].

With reference to the presented approach, the considered sliding surface has the form

$$s_{FOS}(t) = D^{-\alpha}(c_1 e(t) + c_2 D^{-1} e(t)) \quad (41)$$

with $\alpha = 0.47$, $c_1 = 1$, $c_2 = 13$ and the control signal is given by (10) and (16). Moreover we considered a succession of time instants $t_i = t_{i-1} + h/i$ with $h = 0.05$. From the implementation viewpoint, it may be useful to point out that the succession (14) is expected to produce numerical problems as time grows. The solution adopted was to make a translation in time and reset the initial condition when the difference $t_i - t_{i-1}$ was found smaller than a given threshold (0.001).

All the three approaches guarantee the rapid reaching of the set-point speed $\omega_r = 1000$ rpm. The output controlled with the FOS framework is faster than the SMC output but slower than the 2-SMC output (Figure 1).

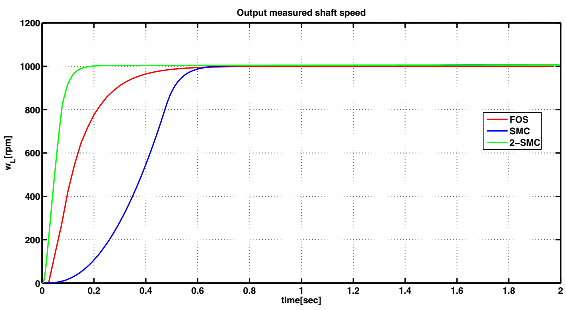

Fig. 1. Output signal $\omega_L(t)$ controlled by (9) in red, (36)-(37) in blue and (39)-(40) in green.

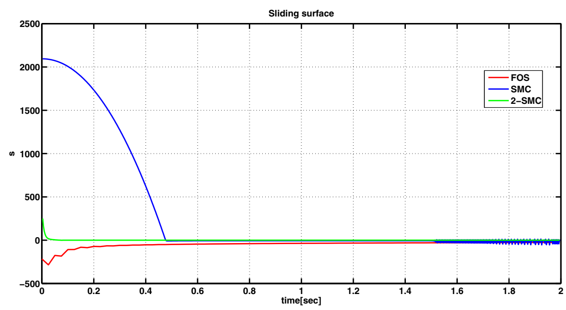

Fig. 2. Sliding surfaces (41), (35), (38) in red, blue and green respectively.

Figure 2 displays that the sliding surfaces (41), (35) and (38) vanishes and therefore the tracking error $e(t)$ tends to zero. The phase plane $e(t)$ and $\dot{e}(t)$ of the FOS, SMC and 2-SMC system are depicted in Figures 3-5. Figure 6 shows the control signal $u(t)$ of the three different frameworks. It's evident that due to the chosen disturbance signal the SMC is effected by a consistent chattering which is not alleviated by the saturation function. On the other hand both the FOS control (9) and the 2-SMC (39)-(40) are not affected by chattering. Figures 7-8 show that after 53 seconds the 2-SMC control input saturates and as a consequence the output speed ω_L does not chase the

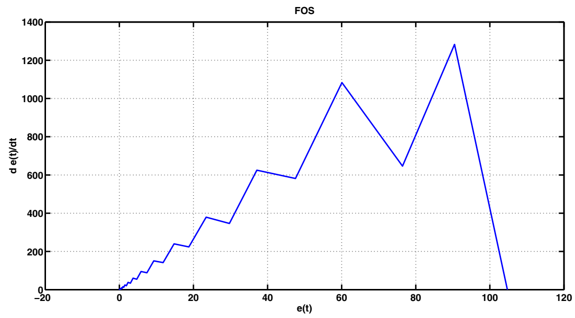

Fig. 3. Phase plane $e(t)$ and $\dot{e}(t)$ of the FOS system.

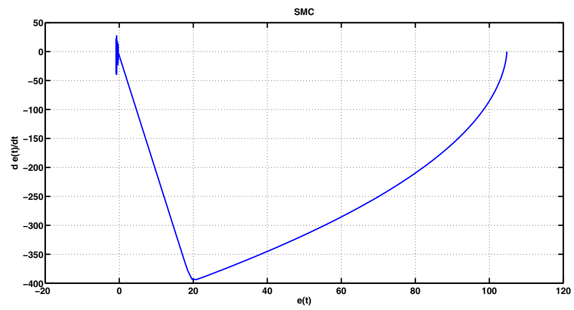

Fig. 4. Phase plane $e(t)$ and $\dot{e}(t)$ of the SMC system.

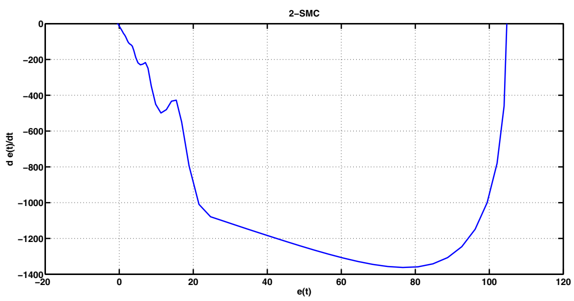

Fig. 5. Phase plane $e(t)$ and $\dot{e}(t)$ of the 2-SMC system.

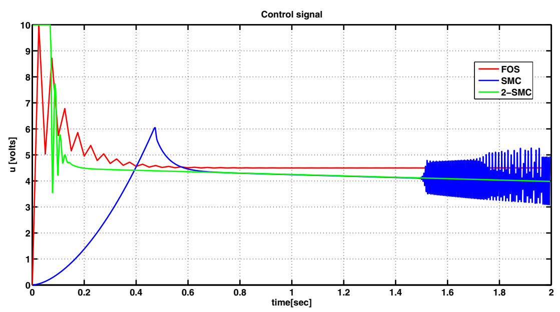

Fig. 6. Control signal (9) of the FOS (red), SMC (blue), 2-SMC (green) framework.

command input 1000 rpm anymore. On the other hand the output speed of the presented FOS framework continues to follow ω_r even after 53 seconds.

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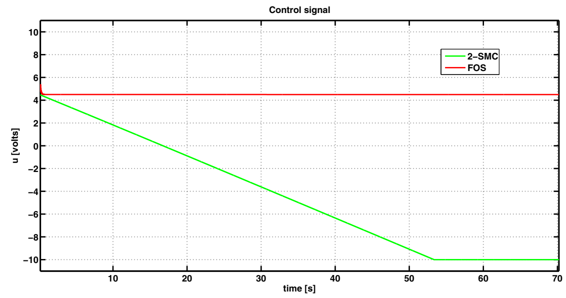

Fig. 7. FOS control signal (red) and 2-SMC signal (green).

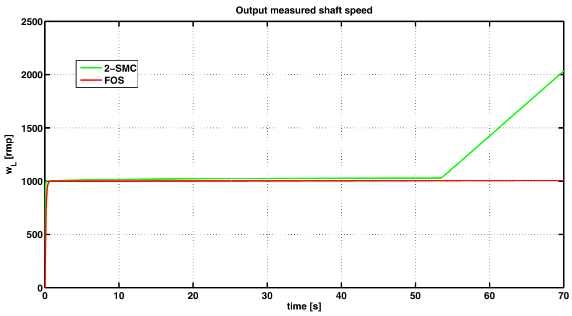

Fig. 8. Performances of the FOS output speed (red) and the 2-SMC output speed (green).

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