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# A sensorless speed-tacking controller for permanent magnet synchronous motors with uncertain parameters

**M.L. Corradini** * **A. Cristofaro** *

* *Scuola di Scienze e Tecnologie, Università di Camerino, via Madonna delle Carceri, 62032 Camerino (MC), Italy, ( e-mail: {letizia.corradini, andrea.cristofaro}@unicam.it).*

**Abstract:** This paper focuses on the so-called sensorless speed-tracking control for permanent magnet synchronous motors. A control strategy is proposed allowing to extract rotor position using electrical signals and an observer of electrical variables, and to guarantee the robust asymptotical tracking of a reference speed without measuring rotor velocity and position. Bounded parameter variations are supposed to affect the mechanical system, and a (possibly time-varying) uncertain load torque is considered. Numerical simulations are provided supporting the effectiveness of the proposed control policy.

**Keywords** Permanent magnet motors, energy systems, sensorless control

## 1. INTRODUCTION

Control algorithms for electrical motors used in high performance applications require, as well known, feedback information about rotor position and, for speed tracking applications, of rotor speed. On the other hand, just to mention a single application which is attracting an increasing interest in recent years, position or speed sensors of wind turbines equipped with permanent-magnet synchronous generators are usually physically inaccessible, particularly for large size devices. As a consequence, so-called *sensorless control methods* [Hamida et al., 2013], [Paulus et al., 2013], [Qiao et al., 2013], [Kim et al., 2011], [Ortega et al., 2011] avoiding the need of mechanical sensors, attracted the industrial interest, and has induced an intensive research activity in the control community. Indeed, a number of contributions on this topic have recently appeared for different types of electrical machines (induction machines, stepper motors, Permanent Magnet Synchronous Motors (PMSMs)). With specific reference to PMSMs, whose popularity is growing for the reasons discussed in [Ortega et al., 2011], the replacement of speed/position sensors with mathematical algorithms can finally allow low-cost motor drives to fully exploit the inherent properties of Field Oriented Control (FOC), i.e. closed loop current control, high efficiency and performances, and quiet operation.

Classification of sensorless techniques can be made according to the operational domain (see Acarnley and Watson [2006] for a review). Back-EMF-based approaches are largely used in Surface PMSMs, which perform poorly at low-speed regimes [Paulus et al., 2013], [Ortega et al., 2011] (indeed, position looses the property of observability at zero speed [Poulain et al., 2008]). Such behavior, which can empirically explained since the induced voltage vanishes at low speed, has to be attributed to the approximations inherently introduced in the linearization of the model of the PMSM, whose first principles model is highly nonlinear [Ortega et al., 2011]. In the low-speed domain, non-model-based methods based on rotor saliency are used instead, which are realized by a high frequency signal injection [Paulus et al., 2013], [Hamida et al., 2013]. A number of back-EMF-based methods have been reported in the literature. An interconnected scheme of adaptive observers has been very recently proposed [Hamida et al., 2013], and extended Kalman filtering has been applied to estimate the whole state vector [Bolognani et al., 2003], though requiring intensive computing. In contrast, sliding mode observers are being currently largely proposed [Qiao et al., 2013], [Kim et al., 2011] due to their easy implementability, robustness and good dynamic behavior. Moreover, following a number of studies addressing modeling and nonlinear control of electrical machines [Astolfi et al., 2007], [Karagiannis et al., 2009], [Ortega et al., 2007], a nonlinear back-EMF-based observer for rotor position has been proposed recently [Ortega et al., 2011],[Lee et al., 2010], mostly relying on a different state variable representation of motor dynamics.

The above cited approaches share the common methodology consisting in building one or more model-based observers. The need of proving closed-loop convergence, nonetheless, usually provide undesired behaviors in asymptotic performances of the system, or require the introduction of limiting assumptions which are hardly verified in practice. This paper is aimed at providing an answer to the previous problems, proposing a sensorless, back-EMF based, FOC algorithm for a PMSM featuring robustness with respect to motor parameter variations and load torque. More in detail:

- deviations of model parameters with respect to their nominal values, along with the initial angular position, are computed off-line before operation;
- rotor position is extracted from electrical signals;
- the robust tracking is ensured of a reference speed by a FOC scheme without the need of rotor speed measurements;

- the robust algorithm can account for bounded parameter variations affecting the electrical and mechanical system;
- also time-varying load torque affected by bounded uncertainty can be accounted for.

The proposed technical development exploits the particular expression of the classical fixed-frame ($\alpha$,$\beta$) nonlinear model of PMSM [Ichikawa et al., 2006] [Krause, 1986], and makes use of well known techniques based on sliding modes. A simulation study is reported to support the proposed theoretical development.

## 2. SYSTEM MODEL

### 2.1 PMSM modeling

In the ($\alpha$,$\beta$) reference frame, the electrical equations of motion of a PMSM can be written as:

$$\begin{cases} \dfrac{di_\alpha(t)}{dt} &= -\dfrac{R}{L}i_\alpha(t) + \omega_e\dfrac{\lambda_0}{L}\sin(\theta_e(t)) + \dfrac{1}{L}v_\alpha(t) \\ \dfrac{di_\beta(t)}{dt} &= -\dfrac{R}{L}i_\beta(t) - \omega_e\dfrac{\lambda_0}{L}\cos(\theta_e(t)) + \dfrac{1}{L}v_\beta(t) \end{cases} \tag{1}$$

where $i_\alpha(t)$ and $i_\beta(t)$ are the stator currents, respectively; $v_\alpha(t)$ and $v_\beta(t)$ are the stator voltages, respectively; $R$ is the winding resistance and $L$ is the winding inductance, $\lambda_0$ is the flux linkage of the permanent magnet, $\theta_e$ and $\omega_e$ are the electrical angular position and speed, respectively, of the motor rotor.

The electrical torque $T_e$ is given by:

$$T_e(t) = K_t(i_\beta(t)\cos(\theta_e(t)) - i_\alpha(t)\sin(\theta_e(t))) \tag{2}$$

in which $K_t = \frac{3}{2}\lambda_0 N_r$ is the torque constant with $N_r$ the number of pole pairs. For the electrical angular position/speed and the mechanical angular position/speed, the following relations hold: $N_r = \dfrac{\omega_e}{\omega_r} = \dfrac{\theta_e}{\theta_r}$, $\theta_r$ denoting the mechanical angular position of the motor rotor. In the following, the whole development will be made in the electrical frame. The mechanical motion equation is described by:

$$J\dot{\omega}_r(t) + B\omega_r(t) = T_e(t) - \tau(t) \tag{3}$$

$$\dot{\theta}_r(t) = \omega_r(t) \tag{4}$$

where $J$ is the total mechanical inertia of the the PMSM. The torque $\tau$ summarizes the effect of the external torque (generally known with large inaccuracy). It is likely to introduce the following assumptions:

*Assumption 2.1.* The model parameter $B$ and the load torque $\tau$ appearing in (3) are uncertain, with bounded uncertainty:

$$B = \bar{B} + \Delta B; \quad \tau(t) = \bar{\tau} + \Delta\tau(t) \tag{5}$$

$$|\Delta B| \le \rho_B; \quad |\Delta\tau| \le \rho_\tau \tag{6}$$

*Remark 1.* The inertia parameter $J$ has been assumed known only for simplifying the statement of the next results. As it will be clear in the following, a bounded variation for this parameter can be accounted for with straightforward changes.

*Assumption 2.2.* Similarly, slowly varying parameter variations [Hamida et al., 2013] can affect model (1), i.e.

$$R = \bar{R} + \Delta R, \quad \dot{R} \simeq 0 \tag{7}$$

being $\bar{R}$ the nominal values and $\Delta R$ the corresponding uncertainty, assumed bounded a by known constants $\rho_R$.

*Remark 2.* Again, the parameter $L$ has been assumed known only for simplifying the statement of the next results. As it will be clear in the following, its bounded parametric variation can be accounted for with straightforward changes.

In view of the inherent physical limitations of the real device, the following assumption is also introduced:

*Assumption 2.3.* A bound exists and is available on the maximum achievable rotor velocity $-\omega_e^M \le \omega_e \le \omega_e^M$.

## 3. OFF-LINE DETERMINATION OF THE INITIAL ANGULAR POSITION AND MOTOR PARAMETER VARIATIONS

The following section is devoted to show that the unknown resistor variation $\Delta R$ in (1) and the initial angular position of the rotor $\theta(0) = \theta_0$ can be derived exactly from currents measurements under weak conditions. Once such quantities are exactly available, they will be used for building a sensorless robust speed-tracking controller as explained in the next section.

A simple procedure is proposed, to be executed off-line before operation. It may take a not negligible amount of time, but this has no effect in the effectiveness of the control algorithm to be described in the following. Once identified off-line, the sensorless controller can begin working using the resistor deviation $\Delta R$ from the nominal value $\bar{R}$ and the initial angular position $\theta_0$ (just determined) which do not vary with time because of Assumption 2.2.

Let us refer to system (1) with $R$ given by (7); in addition let us set $v_\alpha(t) = v_\beta(t) = 0$. Although the currents $i_\alpha(t)$, $i_\beta(t)$ can be directly measured, consider the following pseudo-observer:

$$\begin{cases} \dfrac{d\hat{i}_\alpha(t)}{dt} = -\dfrac{\bar{R}}{L}i_\alpha \\ \dfrac{d\hat{i}_\beta(t)}{dt} = -\dfrac{\bar{R}}{L}i_\beta \end{cases} \tag{8}$$

whose error dynamics are

$$\begin{cases} \dot{e}_\alpha(t) = -\dfrac{\Delta R}{L}i_\alpha + \omega_e\dfrac{\lambda_0}{L}\sin\theta_e(t); \\ \dot{e}_\beta(t) = -\dfrac{\Delta R}{L}i_\beta - \omega_e\dfrac{\lambda_0}{L}\cos\theta_e(t); \end{cases} \tag{9}$$

with $e_\alpha(t) = i_\alpha(t) - \hat{i}_\alpha(t)$, $e_\beta(t) = i_\beta(t) - \hat{i}_\beta(t)$. Assuming without loss of generality $i_\alpha(0) = \hat{i}_\alpha(0)$, $i_\beta(0) = \hat{i}_\beta(0)$ and setting

$$I_\alpha(t) = \int_0^t i_\alpha(s)ds, \quad I_\beta(t) = \int_0^t i_\beta(s)ds,$$

by integration one gets

$$e_\alpha(t) = -\frac{\Delta R}{L} I_\alpha(t) - \frac{\lambda_0}{L}(\cos\theta_e(t) - \cos\theta_0),$$
$$e_\beta(t) = -\frac{\Delta R}{L} I_\beta(t) - \frac{\lambda_0}{L}(\sin\theta_e(t) - \sin\theta_0), \tag{10}$$

which gives

$$\left(e_\alpha(t) + \frac{\Delta R}{L} I_\alpha(t) - \frac{\lambda_0}{L}\cos\theta_0\right)^2;$$
$$+\left(e_\beta(t) + \frac{\Delta R}{L} I_\beta(t) - \frac{\lambda_0}{L}\sin\theta_0\right)^2 = \frac{\lambda_0^2}{L^2}. \tag{11}$$

Exploiting the left-hand side one gets

$$\frac{I_\alpha^2(t) + I_\beta^2(t)}{L^2}\Delta R^2 + \frac{2}{L}\left(e_\alpha(t)I_\alpha(t) + e_\beta(t)I_\beta(t)\right)\Delta R$$
$$-\frac{\lambda_0}{L}\left(I_\alpha(t)\cos\theta_0 + I_\beta(t)\sin\theta_0\right)\Delta R +$$
$$-\frac{2\lambda_0}{L}(e_\alpha(t)\cos\theta_0 + e_\beta(t)\sin\theta_0) = -e_\alpha^2(t) - e_\beta^2(t). \tag{12}$$

As the time $t$ varies, (12) constitutes an infinite family of equations with unknown variables $\Delta R, \theta_0$. Equation coefficients can be grouped in a time-dependent vector $W(t)$ as follows

$$W(t) = \left[\frac{I_\alpha^2(t) + I_\beta^2(t)}{L^2} \quad \frac{2\left(e_\alpha(t)I_\alpha(t) + e_\beta(t)I_\beta(t)\right)}{L} \quad \frac{2\lambda_0}{L^2}I_\alpha(t) \right.$$
$$\left. \frac{2\lambda_0}{L^2}I_\beta(t) \quad \frac{2\lambda_0}{L}e_\alpha(t) \quad \frac{2\lambda_0}{L}e_\beta(t)\right]; \tag{13}$$

and defining $E(t) := e_\alpha^2(t) + e_\beta^2(t)$, equation (12) is equivalent to

$$W(t)\cdot[\Delta R^2 \quad \Delta R \quad \Delta R\cos\theta_0 \quad \Delta R\sin\theta_0 \quad \cos\theta_0 \quad \sin\theta_0]^T$$
$$= -E(t) \tag{14}$$

Let us consider now two distinct time instants $t_1, t_2$ such that $W(t_2)$ is not a multiple of $W(t_1)$; the triple $(\cos\theta_0, \sin\theta_0, \Delta R) = (x, y, z)$ satisfies the following set of quadratic equations in the variables $(x, y, z)$ :

$$\begin{cases} x^2 + y^2 = 1; \\ W(t_1)\cdot[z^2 \quad z \quad zx \quad zy \quad x \quad y]^T = -E(t_1); \\ W(t_2)\cdot[z^2 \quad z \quad zx \quad zy \quad x \quad y]^T = -E(t_2); \end{cases} \tag{15}$$

By Bézout's theorem [Walker, 1978], the above set (15) admits at most $8 = 2^3$ distinct solutions. Since we are interested in possibly recovering, without ambiguity, the true solution $(\cos\theta_0, \sin\theta_0, \Delta R)$, it is helpful consider a further time instant $t_3$ and the associate equation:

$$W(t_3)\cdot[z^2 \quad z \quad zx \quad zy \quad x \quad y]^T = -E(t_3). \tag{16}$$

The following result is straightforward.

*Proposition 3.1.* A necessary condition for the existence of a unique solution to equations set (15)-(16) is that

$$rank\begin{bmatrix} W(t_1) \\ W(t_2) \\ W(t_3) \end{bmatrix} = 3 \tag{17}$$

for some $0 < t_1 < t_2 < t_3$.

Under the above condition, the maximum number of admissible solutions can be reduced as stated in the following result.

*Theorem 3.1.* Let us consider system (1) with unknowns $\Delta R$ and $\theta_0$. If there exist $t_1, t_2, t_3$ such that rank condition (17) is fulfilled, two cases are admissible:

i) the triple $(\cos\theta_0, \sin\theta_0, \Delta R)$ is the unique solution of equations (15)-(16);
ii) the polynomial system (15)-(16) admits $1 < \nu \leq 4$ solutions.

**Proof.** For any $t \geq 0$, the equation

$$W(t)\cdot[z^2 \quad z \quad zx \quad zy \quad x \quad y]^T = -E(t) \tag{18}$$

can be rewritten in the equivalent form

$$z^2 = zg(t, x, y) + f(t, x, y), \tag{19}$$

where $f(\cdot, x, y)$, $g(\cdot, x, y)$ are polynomial functions obtained from the coefficients of $W(t)$ and $E(t)$. Therefore the system (15)-(16) is equivalent to

$$\begin{cases} x^2 + y^2 = 1; \\ \dfrac{f(t_1, x, y) - f(t_2, x, y)}{g(t_2, x, y) - g(t_1, x, y)} = \dfrac{f(t_1, x, y) - f(t_3, x, y)}{g(t_3, x, y) - g(t_1, x, y)} \\ z(g(t_2, x, y) - g(t_1, x, y)) = f(t_1, x, y) - f(t_2, x, y); \\ z^2 = zg(t_1, x, y) + f(t_1, x, y); \end{cases} \tag{20}$$

By construction the first two equations are both quadratic and depending only on the variables $x, y$; as a consequence, applying again Bézout's theorem and observing that $z$ is completely determined by the third equation, the maximum number of admissible solutions is $4 = 2^2$. We point out that the second equation in (20) is well-posed if and only if (17) is fulfilled. □

It is worth to note that, in many cases, system (20) admits the true solution $(\cos\theta_0, \sin\theta_0, \Delta R)$ only: this is a consequence of the strong constraint given by the fourth equation. On the other hand, if it is not possible to isolate the true solution by means of algebraic methods, one can introduce a decision algorithm based on multiple models.

### *3.1 Multiple solutions: a decision algorithm*

Let us suppose that system (20) (or system (15)) admits $1 < \nu \leq 4$ $(1 < \nu \leq 8)$ distinct solutions $(X_k, Y_k, Z_k)$, $k = 1, ..., \nu$; we define a suitable set of system estimators

$$\begin{cases} \dfrac{d\eta_\alpha^{(k)}(t)}{dt} = -\dfrac{\bar{R} + Z_k}{L} i_\alpha(t) \\ \dfrac{d\eta_\beta^{(k)}(t)}{dt} = -\dfrac{\bar{R} + Z_k}{L} i_\beta(t) \end{cases} \quad k = 1, ..., \nu$$

and we compute the quantities

$$\begin{cases} c_k(t) = -\dfrac{L}{\lambda_0}(i_\alpha(t) - \eta_\alpha^{(k)}(t)) - X_k \\ s_k(t) = -\dfrac{L}{\lambda_0}(i_\beta(t) - \eta_\beta^{(k)}(t)) - Y_k. \end{cases} \quad k = 1, ..., \nu$$

The terms $c_k(t)$ and $s_k(t)$ are possible candidates for the true values of $\cos\theta(t)$ and $\sin\theta(t)$ respectively. We introduce now the following system models for $k = 1, ..., \nu$ :

$$
\begin{aligned}
i_\alpha^{(k)}(t) &= e^{-\frac{(\bar{R}+Z_k)t}{L}} i_\alpha(0) + \frac{\lambda_0}{L}\left(e^{-\frac{(\bar{R}+Z_k)t}{L}} X_k - c_k(t)\right) \\
&\quad + \lambda_0 \frac{\bar{R}+Z_k}{L^2} \int_0^t e^{-\frac{(\bar{R}+Z_k)(t-s)}{L}} c_k(s) ds, \\
i_\beta^{(k)}(t) &= e^{-\frac{(\bar{R}+Z_k)t}{L}} i_\beta(0) + \frac{\lambda_0}{L}\left(e^{-\frac{(\bar{R}+Z_k)t}{L}} Y_k - s_k(t)\right) \\
&\quad + \lambda_0 \frac{\bar{R}+Z_k}{L^2} \int_0^t e^{-\frac{(\bar{R}+Z_k)(t-s)}{L}} s_k(s) ds.
\end{aligned}
$$

Such models have been obtained by integrating the system of differential equations generated from (1) where the uncertain terms are chosen accordingly to the set of admissible solutions $(X_k, Y_k, Z_k)$.

Fixing arbitrarily the time horizon $T > 0$ and defining the cost functional

$$
\mathcal{J}(k,T) := \int_0^T ||i_\alpha(t) - i_\alpha^{(k)}(t)||^2 + ||i_\beta(t) - i_\beta^{(k)}(t)||^2 dt, \tag{21}
$$

one gets

$$
\min_{k=1,\dots,\nu} \mathcal{J}(k,T) = 0,
$$
$$
(\cos\theta_0, \sin\theta_0, \Delta R) = (X_{k^*}, Y_{k^*}, Z_{k^*}) \tag{22}
$$

with

$$
k^* = \arg \min_{k=1,\dots,\nu} \mathcal{J}(k,T). \tag{23}
$$

We point out that, if for any $T > 0$ the integer $k^*$ is not unique, the associated triple of parameters $(X_k, Y_k, Z_k)$ are *indistinguishable*, i.e. they provide exactly the same effects on the system output.

*Remark 3.* The development discussed so far can account also for constant uncertainties affecting the inductance parameter $L$, at the price of an increase of the number of feasible solutions. In particular, if both $R$ and $L$ are uncertain, these parameters (together with $\theta_0$) have to satisfy a system of 5 quadratic equations, which is the natural extension of (15).

## 4. CONTROL DESIGN

### *4.1 Sensorless Derivation of Rotor Angular Position from Currents Measurements*

In standard drives, rotor position is given by encoder measurements, and rotor speed is usually estimated as the incremental ratio of encoder positions over one sampling period. It will be shown in the following that the angular position of the rotor can be easily derived from measurements of currents $i_\alpha(t)$ and $i_\beta(t)$, making use of previous developments showing that $\Delta R$ and $\theta_0$ can be exactly determined.

*Remark 4.* Resuts reported in the previous section shows that $\Delta R$ and $\theta_0$ can be exactly determined. Accordingly, the quantity $R = \bar{R} + \Delta R$ will be considered exactly known hereafter.

Consider the following pseudo-observer

$$
\begin{cases}
\dfrac{d\hat{i}_\alpha(t)}{dt} &= -\dfrac{R}{L}\hat{i}_\alpha(t) + \dfrac{1}{L} v_\alpha(t) \\[2ex]
\dfrac{d\hat{i}_\beta(t)}{dt} &= -\dfrac{R}{L}\hat{i}_\beta(t) + \dfrac{1}{L} v_\beta(t)
\end{cases} \tag{24}
$$

and define the relative observation errors:

$$
\epsilon_\alpha(t) = i_\alpha(t) - \hat{i}_\alpha(t); \quad \epsilon_\beta(t) = i_\beta(t) - \hat{i}_\beta(t)
$$

The following result can be proved:

*Lemma 5.* With reference to the plant (1), the angular position $\theta_e$ of the rotor can be exactly derived from currents after an arbitrary finite time.

**Proof.** The dynamics of the observation errors are

$$
\begin{cases}
\dot{\epsilon}_\alpha(t) = \dfrac{\lambda_0}{L}\omega_e \sin(\theta_e(t)) \\[2ex]
\dot{\epsilon}_\beta(t) = -\dfrac{\lambda_0}{L}\omega_e \cos(\theta_e(t))
\end{cases} \tag{25}
$$

Integrating one gets:

$$
\epsilon_\alpha(t) = \epsilon_\alpha(0) - \frac{\lambda_0}{L}\left(\cos(\theta_e(t)) - \cos(\theta_0)\right)
$$
$$
\epsilon_\beta(t) = \epsilon_\beta(0) - \frac{\lambda_0}{L}\left(\sin(\theta_e(t)) - \sin(\theta_0)\right)
$$

Since the initial condition of the observer (24) can be set equal to the initial value of the (measured) currents, it follows that the angular position $\theta_e(t)$ can be computed as

$$
\theta_e(t) = \arctan \frac{\epsilon_\beta(t) + \dfrac{\lambda_0}{L}\sin(\theta_0)}{\epsilon_\alpha(t) + \dfrac{\lambda_0}{L}\cos(\theta_0)}. \tag{26}
$$

□

### *4.2 The sensorless speed-tracking controller*

The tracking problem is here considered, i.e. the variable $\omega_e$ is required to track a known reference $\omega^*$. To guarantee the robust tracking of the assigned reference in the framework of the FOC scheme, consider the following sliding surface:

$$
s(t) = J(\theta_e(t) - \theta^*(t)) = 0 \tag{27}
$$

whose derivative is

$$
\begin{aligned}
\dot{s}(t) &= J(\omega_e(t) - \omega^*(t)) = -(\bar{B} + \Delta B)(\theta_e(t) - \theta_0) \\
&+ K_t \int_0^t \left[i_\beta(p)\cos(\theta_e(p)) - i_\alpha(p)\sin(\theta_e(p))\right] dp \\
&- \int_0^t \left[\bar{\tau}(p) + \Delta\tau(p)\right] dp - J\omega^*(t)
\end{aligned} \tag{28}
$$

In the previous expression, one can consider two reference currents:

$$
i_\beta^*(t) = I^*(t)\cos(\theta_e(t)); \quad i_\alpha^*(t) = -I^*(t)\sin(\theta_e(t)),
$$

whose tracking will be ensured later and where $I^*(t)$ is to be determined. When $i_\alpha(t) = i_\alpha^*(t)$, $i_\beta(t) = i_\beta^*(t)$, one has:

$$
\begin{aligned}
\dot{s}(t) &= J(\omega_e(t) - \omega^*(t)) = -(\bar{B} + \Delta B)(\theta_e(t) - \theta_0) \\
&+ v(t) - \int_0^t \left[\bar{\tau}(p) + \Delta\tau(p)\right] dp - J\omega^*(t)
\end{aligned} \tag{29}
$$

being

$$
v(t) = K_t \int_0^t I^*(p) dp. \tag{30}
$$

For $\epsilon > 0$ arbitrary small, let us define the following continuously differentiable approximation of the function $sign(z)$ :

$$sign_\epsilon(z) = \begin{cases} sign(z) & |z| \geq \epsilon \\ \frac{2\epsilon - |z|}{\epsilon} z & |z| \in (\epsilon/2, \epsilon) \\ 2\epsilon^{-1} z & |z| \leq \epsilon/2 \end{cases} \tag{31}$$

*Lemma 6.* Assume $w^*(0) = 0$. When $i_\alpha(t) = i_\alpha^*(t)$, $i_\beta(t) = i_\beta^*(t)$, for any $\epsilon > 0$, the following control input $I_\epsilon^*(t)$ ensures the uniform asymptotic boundedness condition $|s(t)| \leq \epsilon$ for the sliding surface (27):

$$K_t I_\epsilon^*(t) = J\dot{\omega}^*(t) + \bar{\tau}(t) - \zeta J[(\bar{B} + \rho_B)\omega_e^M + \rho_\tau] sign_\epsilon(s(t))$$
$$-\zeta J[(\bar{B} + \rho_B)\omega_e^M + \rho_\tau] t\chi_\epsilon(s(t))(D^- \theta_e(t) - \omega^*(t)) \tag{32}$$

with $\zeta > 1$ and where

$$\chi_\epsilon(z) = \begin{cases} 0 & |z| \geq \epsilon \\ \frac{2(\epsilon - |z|)}{\epsilon^2} & |z| \in (\epsilon/2, \epsilon) \\ 2\epsilon^{-1} & |z| \leq \epsilon/2 \end{cases}$$

$$D^- \theta_e(t) = \lim_{|h| \to 0} \frac{\theta_e(t) - \theta_e(t - |h|)}{|h|}.$$

**Proof.** Following standard techniques, a sliding motion on (27) is enforced by $v(t) = v_{eq}(t) + v_n(t)$, where $v_{eq}(t)$ is the control input guaranteeing that $\dot{s}(t) = 0$ in the nominal case, i.e.

$$K_t v_{eq}(t) = J\omega^*(t) + \int_0^t \bar{\tau}(p)dp \tag{33}$$

and $v_n(t)$ is designed to ensure such that $s(t)\dot{s}(t) < 0 \ \forall t$. In particular, noticing that $|\theta_e(t) - \theta_0| < \omega_e^M t$, one gets

$$K_t v_n(t) = -\zeta_1 \left[ (\bar{B} + \rho_B)\omega_e^M t + \rho_\tau t \right] sign(s(t)) \tag{34}$$

with $\zeta_1 > 1$. On the other hand, since in the presented scenario the control $v(t)$ is assigned by the integral identity (30), it is not allowed to be discontinuous and a sliding-mode cannot be properly enforced. In order to satisfy the requested regularity conditions, the function $sign(s(t))$ can be replaced by its continuous and differentiable approximation $sign_\epsilon(s(t))$ given by (31).
Let us denote by $v_{n,\epsilon}(t)$ such regularized controller. By construction, since $|\theta_e(t) - \theta_0| \leq \omega_e^M t$, the control law $v(t) = v_e q(t) + v_{n,\epsilon}(t)$ guarantees the uniform boundedness condition

$$|s(t)| \leq \epsilon \ \forall t \geq T_\epsilon > 0.$$

In order to conclude, it is straightforward to verify that

$$sign_\epsilon(s(t)) = J \int_0^t \chi_\epsilon(s(p))(D^- \theta_e(p) - \omega^*(p))dp,$$

this showing that $\dot{v}_{n,\epsilon}(t) = K_t I_\epsilon^*(t)$. □

For any $\epsilon > 0$ the derivative $\frac{d\chi_\epsilon(z)}{dz}$ exists almost everywhere and it satisfies

$$\left| \frac{d\chi_\epsilon(z)}{dz} \right| \leq \frac{2}{\epsilon^2} \quad \forall z \in \mathbb{R};$$

as a consequence the left-derivative $D^- I_\epsilon^*(t)$ is well defined and it verifies

$$K_t D^- I_\epsilon^*(t) = J\ddot{\omega}^*(t) + \dot{\bar{\tau}}(t)$$
$$-\zeta J \rho_\tau \frac{d\chi_\epsilon(s(t))}{dz} (D^- \theta_e(t) - \omega^*(t))^2$$
$$-\zeta J \rho_\tau \chi_\epsilon(s(t))(D^{2,-} \theta_e(t) - \dot{\omega}^*(t)),$$

where

$$D^{2,-} \theta_e(t) := \lim_{h \to 0} \frac{\theta_e(t) - 2\theta_e(t - |h|) + \theta_e(t - 2|h|)}{|h|^2}.$$

The following result summarizes the control strategy guaranteeing the robust tracking of an assigned reference velocity by the rotor, in the presence of bounded perturbations affecting the motor parameters and the load.

*Theorem 4.1.* With reference to the plant (1) under Assumption 2.1, the following control inputs $v_\alpha(t)$, $v_\beta(t)$ ensure the (practical) robust tracking of an assigned reference velocity $\omega^*(t)$ using only measurements of the electrical variables of the motor (i.e. sensorless control):

$$\begin{aligned} \frac{1}{L} v_\alpha(t) = &\frac{R}{L} i_\alpha(t) - D^- I_\epsilon^*(t) \sin(\theta_e(t)) \\ &- \omega_e^M \zeta \left[ \frac{\lambda_0}{L} |\sin(\theta_e(t))| + |I_\epsilon^*(t) \cos(\theta_e(t))| \right] \cdot \\ &sign(i_\alpha(t) + I_\epsilon^*(t) \sin(\theta_e(t))) \end{aligned} \tag{35}$$

$$\begin{aligned} \frac{1}{L} v_\beta(t) = &\frac{R}{L} i_\beta(t) + D^- I_\epsilon^*(t) \cos(\theta_e(t)) \\ &- \omega_e^M \zeta \left[ \frac{\lambda_0}{L} |\cos(\theta_e(t))| + |I_\epsilon^*(t) \sin(\theta_e(t))| \right] \cdot \\ &sign(i_\beta(t) - I_\epsilon^*(t) \cos(\theta_e(t))) \end{aligned} \tag{36}$$

with $\zeta > 1$.

**Proof.** The complete proof is omitted for sake of brevity. The stability of the overall closed-loop stability can be proved using standard Lyapunov techniques. The imposition of the conditions $i_\alpha(t) = i_\alpha^*(t)$ and $i_\beta(t) = i_\beta^*(t)$ can be easily carried out by considering the following Lyapunov function

$$Y(t) = \frac{1}{2}(i_\alpha(t) - i_\alpha^*(t))^2 + \frac{1}{2}(i_\beta(t) - i_\beta^*(t))^2 \tag{37}$$

and observing that, if $v_\alpha(t)$ and $v_\beta(t)$ are chosen according to (35)-(36), then the inequality $\dot{Y}(t) < 0$ holds. □

*Remark 7.* We point out that, for control implementation purposes, the left-derivative $D^- \theta(t)$ can be replaced by a discrete derivative defined by the incremental ratio

$$D_\delta^- \theta_e(t) := \frac{\theta_e(t) - \theta_e(t - \delta)}{\delta}$$

for $\delta > 0$ sufficiently small. In a similar way one can discretize the second derivative:

$$D_\delta^{2,-} \theta_e(t) := \frac{\theta_e(t) - 2\theta_e(t - \delta) + \theta_e(t - 2\delta)}{\delta^2}.$$

## 5. SIMULATION TESTS

Simulations have been performed using technical data taken from [Kim et al., 2011], as a preliminary step before experimental implementation. Some of the performed speed-tracking simulations are shown in Figs 2-3. The following conditions have been used for simulations:

- a reference speed trajectory of the form $\omega^*(t) = 100 * \sin(\frac{2\pi}{0.1} t)$;
- a nominal load torque of $0.5 + 0.5 * \sin(\frac{\pi}{8} t)$ $Kg$, with a 20% variation;
- 20% parameter variation applied to the nominal values of parameters $B, R$;
- boundary layers have been used to avoid chattering.

The development described in Section 3 provided the following values: $\Delta R = 0.0416 \ \Omega$, $\theta_0 = 0.0001 \ rad$, very close to true values $\Delta R = 0.05 \ \Omega$, $\theta_0 = 0 \ rad$. The small inaccuracy has to be ascribed to the numerical solver. In particular Figure 1 shows that, in the considered scenario, the measurements taken at three distinct time instants are sufficient for identifying the correct values $\theta_0, \Delta_R$: indeed the three depicted curves, corresponding to the locus of solutions of Eq. (12) for the selected time steps, have a single common intersection point $(\theta_0, \Delta_R) \simeq (0, 0.05)$.

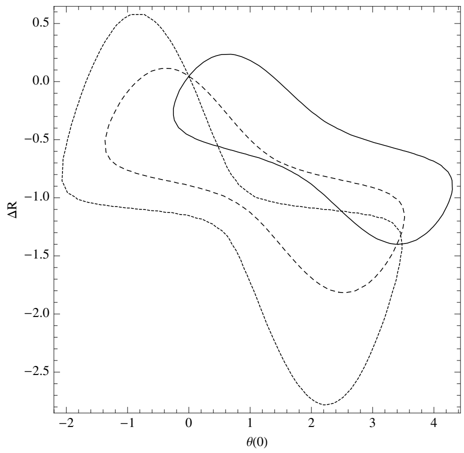


Fig. 1. Solutions of Eq. (12) for $t_1 = 0.05s$ (dotted line), $t_2 = 0.07s$ (dashed line) and $t_3 = 0.08s$ (solid line).

Fig. 2 shows the tracking performance obtained considering the sinusoidal velocity profile with respect to speed in the case of the time-varying load. Furthermore, Fig.3 shows the currents $i_\alpha(t)$, $i_\beta(t)$. It should be noted that the inevitable presence of boundary layers for the implementation of (35), (36) has the effect of constraining speed and positions in a thin layer near the tracked variable.

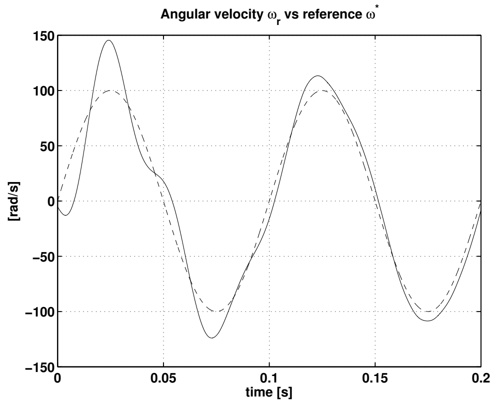


Fig. 2. Reference (dotted line) vs. actual speed for a time-varying uncertain load torque.

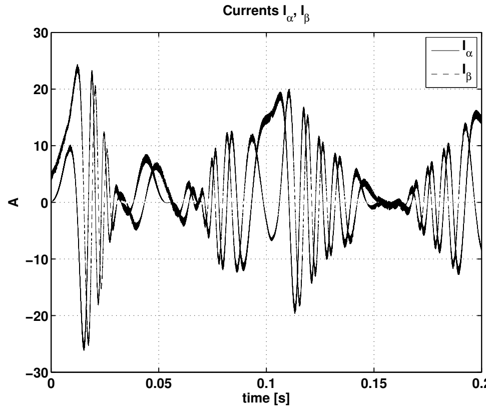


Fig. 3. Currents $i_\alpha(t)$ (continous-line) and $i_\beta(t)$ (dashed line) for a time-varying uncertain load torque.

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