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Application of Person-by-Person Equilibrium and Performance-Measure Statistics to Distributed Control of Uncertain Stochastic Large-Scale Systems with Time Delays ⋆

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Abstract: This paper presents a distributed control design for a class of interconnected linear systems with Markovian jumps, parametric uncertainties and time delays in both system states and control actuations. The switching of a countable number of modes associated with constituent systems is encapsulated by appropriate probability transitions. The parameter uncertainties are considered to be unknown but norm bounded. Analysis of time-delay systems is supported by Pade approximations of the first order. A general framework of multi-person nonzero-sum stochastic differential games is leveraged for distributed decision making with performance risk-aversion. Towards performance risk-aversion and reliability, self-directed controllers and/or decision makers are now capable of effectively incorporating risk-averse attitudes via performance-measure statistics into person-by-person equilibrium decisions with decentralized output feedback for distributed interactions.

Keywords: Distributed control, Markovian jumps, parametric uncertainty, time delays, Pade approximation of the first order, nonzero-sum stochastic games, person-by-person equilibrium, decentralized output feedback, risk-averse attitudes, performance-measure statistics

1. INTRODUCTION

The problem of control and coordination of ever larger and more sophisticated systems has received growing attention during the last 40 years, as can be seen from Sandell (1978) and Siljak (1991). Many applied fields have already been concerned with distributed control and coordination of interconnected dynamical systems, with applications in product design Androulakis (1999), manufacturing systems Krothapalli (1999), and computing architectures Shahabi (2002). In fact, decentralization is recommended as a way to speed up product development processes and decrease the computational time and the complexity of the problem Prewitt (1998).

Although many progresses have been made in the development of different frameworks to address analysis, stability and control problems of large-scale systems along with the long-standing challenges due to dimensionality, information structure constraints, parametric uncertainty and delays Mahmoud (2010), there is little emphasis and work on fully integrated approaches that take into account of dynamic interactions and performance riskiness among interconnected stochastic dynamical systems in the literature.

In reaction, it is the aim of this paper to extend the recent results Pham (2010) with the hope that the class of uncertain stochastic large-scale systems will even accommodate both state and control delays. It is also imperative to envision a more effective integration of Pade approximations of the first order and multiperson Nash game-theoretic decision optimization to not only approximate time delayed states and controls but also to distribute person-by-person equilibrium strategies for efficient achievements of performance robustness and reliability requirements that are now characterized by performance average and risks.

As earlier suggested, the specific contributions from the line of research herein are to overcome the limitations of standard trends through developing and utilizing: i) decentralized filtering with estimation interferences imposed by immediate neighbors for each distributed systems; ii) an efficient and tractable paradigm that calculates exactly all the mathematical statistics associated with the generalized chi-squared performance measure for decision making under performance risk aversion; and iii) a synthesis of distributed person-by-person equilibrium decision policies with output feedback for reliable performance that now guarantee performance robustness with something much stronger than ensemble average measures.

2. PRELIMINARIES

In this section, some spaces of random variables and stochastic processes are introduced; e.g., a fixed proba-

⋆ Correspondence to the Air Force Research Laboratory, Space Vehicles Directorate, 3550 Aberdeen Ave, S.E., Kirtland Air Force Base, New Mexico 87117 U.S.A. Email: AFRL.RVSV@kirtland.af.mil bility space $(\Omega, \mathbb{F}, \{\mathbb{F}_{0,t} : t \in [0,t_f]\}, \mathbb{P})$ with filtration satisfying the usual conditions. All the filtrations are right continuous and complete and $\mathbb{F}_{t_f} \triangleq \{\mathbb{F}_{0,t} : t \in [0,t_f]\}$. In addition, let $\mathcal{L}^2_{\mathbb{F}_{t_f}}([0,t_f];\mathbb{R}^n)$ denote the space of $\mathbb{F}_{t_f}$-adapted random processes $\{\hbar(t) : t \in [0,t_f]\}$ such that $E\{\int_0^{t_f} ||\hbar(t)||^2 \, dt\} < \infty\}$.

As for the setting, a class of uncertain large-scale linear Markov jump systems having time delays in both states and controls is considered as an approximate class of real nonlinear systems with Markov jumps in the neighborhood of the operating points. Each constituent system is controlled by a separate controller or decision maker, who belongs to the set $\mathcal{N} \triangleq \{1,2,\ldots,N\}$ and is in charge of implementing control and/or decision policies $u_1,\ldots,u_N$. The decision horizon, on which the interaction dynamics evolves, is $[0,t_f]$.

Also relevant is that the mode switching at each distributed system and decision maker i for $i \in \mathcal{N}$ is governed by a continuous-time Markov process $\{r_t^i, t \geq 0\}$ taking values in the state space $\mathcal{S}_i = \{1,2,\ldots,S_i\}$ and having the following infinitesimal generator $\Lambda_i = (\lambda_{i_1 i_2})$ for all i_1, i_2 in $\mathcal{S}_i$, where $\lambda_{i_1 i_2} \geq 0$, for all $i_1 \neq i_2$ and $\lambda_{i_1 i_1} = -\sum_{i_2 \neq i_1} \lambda_{i_1 i_2}$. Then, the modes of transition probabilities are described as

$$Pr(r^i_{t+\triangle t} = i_2 | r^i_t = i_1) = \begin{cases} \lambda_{i_1 i_2}\triangle t + o(\triangle t) & i_2 \neq i_1 \\ 1 + \lambda_{i_1 i_1}\triangle t + o(\triangle t) & i_2 = i_1 \end{cases}$$

where $\lim_{\triangle t \mapsto 0} \frac{o(\triangle t)}{\triangle t} = 0$. Henceforth, the dynamical system associated with distributed controllers or decision makers i and $i \in \mathcal{N}$ with Markov jumps and time delays is modeled by the time-delay differential equation

$$\begin{aligned} \dot{x}_i(t) &= A_i(r^i_t,t)x_i(t) + A^d_i(r^i_t,t)x_i(t-\tau_i) + B_i(r^i_t,t)u_i(t) \\ &+ B^d_i(r^i_t,t)u_i(t-\sigma_i) + C_i(r^i_t,t)d_i(t) + G_i(r^i_t,t)w_i(t) \quad (1) \end{aligned}$$

where for each $t \in [0,t_f]$, $x_i(t) \in \mathbb{R}^{n_i}$ is the dynamical state, $u_i(t) \in \mathbb{R}^{m_i}$ is the control input, $d_i(t) \in \mathbb{R}^{l_i}$ is the coupling interaction from immediate neighbors, τ_i and σ_i are the state and input time delays, $x_i(0) = x_{i0}$, $x_i(t) = g_i(t)$ for $t \in [-\tau_i, 0]$ and $u_i(t) = h_i(t)$ for $t \in [-\sigma_i, 0]$ are the initial functions, and $w_i(t) \in \mathbb{R}^p$ represents unmodeled nonlinearities via a mutually uncorrelated stationary $\mathbb{F}^i_{t_f}$-adapted Gaussian process with its mean $E\{w_i(t)\} = m_{w_i}$ and covariance $cov\{w_i(t_1), w_i(t_2)\} = W_i\delta(t_1 - t_2)$ for all t_1, $t_2 \in [0,t_f]$ and $W_i > 0$.

In addition, the parameter uncertainties in (1) with time delays are assumed to be unknown but bounded; e.g.,

$$\begin{aligned} A_i(r^i_t,t) &= A_i(r^i_t) + D^i_A(r^i_t)\triangle^i_1(r^i_t,t)E^i_A(r^i_t) \\ A^d_i(r^i_t,t) &= A^d_i(r^i_t) + D^i_{A_d}(r^i_t)\triangle^i_2(r^i_t,t)E^i_{A_d}(r^i_t) \\ B_i(r^i_t,t) &= B_i(r^i_t) + D^i_B(r^i_t)\triangle^i_1(r^i_t,t)E^i_B(r^i_t) \\ B^d_i(r^i_t,t) &= B^d_i(r^i_t) + D^i_{B_d}(r^i_t)\triangle^i_2(r^i_t,t)E^i_{B_d}(r^i_t) \\ C_i(r^i_t,t) &= C_i(r^i_t) + D^i_C(r^i_t)\triangle^i_1(r^i_t,t)E^i_C(r^i_t) \\ G_i(r^i_t,t) &= G_i(r^i_t) + D^i_G(r^i_t)\triangle^i_1(r^i_t,t)E^i_G(r^i_t) \end{aligned}$$

with $A_i(r^i_t)$, $A^d_i(r^i_t)$, $B_i(r^i_t)$, $B^d_i(r^i_t)$, $C_i(r^i_t)$, and $G_i(r^i_t)$ are constant matrices with appropriate dimensions. Admissible uncertainties $\triangle_1(r^i_t,t)$ and $\triangle_2(r^i_t,t)$ are unknown time-varying matrices with appropriate dimensions representing the parameter uncertainties of (1) and satisfying $\triangle^T_1(r^i_t,t)\triangle_1(r^i_t,t) \leq I$ and $\triangle^T_2(r^i_t,t)\triangle_2(r^i_t,t) \leq I$.

Towards distributed decision making, each decision maker i and $i \in \mathcal{N}$ is further endowed with an incomplete information structure which is consisted of a linear transformation $H_i(r^i_t,t) \triangleq H_i(r^i_t) + D^i_H(r^i_t)\triangle^i_1(r^i_t,t)E^i_H(r^i_t)$ of the states $x_i(t)$ through the local online data $\{y_i(\tau) : \tau \in [0,t]\}$

$$y_i(t) = H_i(r^i_t,t)x_i(t) + v_i(t) \quad (2)$$

whereupon $v_i(t)$ is another mutually uncorrelated stationary $\mathbb{F}^i_{t_f}$-adapted Gaussian process with its mean $E\{v_i(t)\} = m_{v_i}$ and covariance $cov\{v_i(t_1), v_i(t_2)\} = V_i\delta(t_1 - t_2)$ for all t_1, $t_2 \in [0,t_f]$ and $V_i > 0$.

There are many ways of approximating time-delay systems with ordinary differential equations. For instance, Pade approximation of the first order is considered herein as one of different schemes. In the domain of unilateral Laplace transform, the relation between $x_i(t-\tau_i)$ and $x_i(t)$ can be written as $X^d_i(s) = e^{-\tau_i s}X_i(s)$ where $X^d_i(s) \triangleq \mathcal{L}\{x_i(t-\tau_i)\}$ and $X_i(s) \triangleq \mathcal{L}\{x(t)\}$. Representing $e^{-\tau_i s}$ by an Pade approximation of the first order yields

$$e^{-\tau_i s} \approx \frac{1 - \frac{1}{2}\tau_i s}{1 + \frac{1}{2}\tau_i s} \,. \quad (3)$$

Further let $p_i(t) \triangleq x_i(t-\tau_i) + x_i(t)$. Then, in the domain of unilateral Laplace transformation, it follows that

$$P_i(s) \triangleq \mathcal{L}\{p_i(s)\} = \frac{2}{1 + \frac{1}{2}\tau_i s}X_i(s) \,. \quad (4)$$

Or, equivalently in the time domain, it is clear to see that

$$\dot{p}_i(t) = \frac{1}{\tau_i}[4x_i(t) - 2p_i(t)], \quad p_i(0) = g_i(-\tau_i) + g_i(0^-) \quad (5)$$

A delay in the control is handled similarly; e.g.,

$$\dot{q}_i(t) = \frac{1}{\sigma_i}[4u_i(t) - 2q_i(t)], \quad q_i(0) = h_i(-\sigma_i) + h_i(0^-) \quad (6)$$

Given the results (3), (5) and (6), the delayed system (1) is therefore reformulated as follows

$$\begin{aligned} \dot{s}_i(t) &= \overline{A}_i(r^i_t,t)s_i(t) + \overline{B}_i(r^i_t,t)u_i(t) + \overline{C}_i(r^i_t,t)d_i(t) \\ &+ \overline{G}_i(r^i_t,t)w_i(t)\,, \quad s_i(0) \quad (7) \end{aligned}$$

where for each $i \in \mathcal{N}$, $s_i(t) \triangleq \left[\, x^T_i(t) \ p^T_i(t) \ q^T_i(t) \,\right]^T$

$$\overline{A}_i(r^i_t,t) \triangleq$$

$$\begin{bmatrix} A_i(r^i_t,t) - A^d_i(r^i_t,t) & A^d_i(r^i_t,t) & B^d_i(r^i_t,t) \\ \frac{4}{\tau_i}I_{n_i \times n_i} & -\frac{2}{\tau_i}I_{n_i \times n_i} & 0 \\ 0 & 0 & -\frac{2}{\sigma_i}I_{m_i \times m_i} \end{bmatrix},$$

$$\overline{B}_i(r^i_t,t) \triangleq \begin{bmatrix} B_i(r^i_t,t) - B^d_i(r^i_t,t) \\ 0 \\ \frac{4}{\sigma_i}I_{m_i \times m_i} \end{bmatrix}, \overline{C}_i(r^i_t,t) \triangleq \begin{bmatrix} C_i(r^i_t,t) \\ 0 \\ 0 \end{bmatrix}$$

$$\overline{G}_i(r^i_t,t) \triangleq \begin{bmatrix} G_i(r^i_t,t) \\ 0 \\ 0 \end{bmatrix}, \quad \overline{H}^T_i(r^i_t,t) \triangleq \begin{bmatrix} H^T_i(r^i_t,t) \\ 0 \\ 0 \end{bmatrix}.$$

When interpreting the mathematical model (1)-(2) at decision maker i in Ito stochastic differentials, one shows

$$\begin{aligned} ds_i(t) &= (\overline{A}_i(r^i_t,t)s_i(t) + \overline{B}_i(r^i_t,t)u_i(t) + \overline{C}_i(r^i_t,t)d_i(t) \\ &+ \overline{G}_i(r^i_t,t)m_{w_i})dt + \overline{G}_i(r^i_t,t)U_{W_i}\Lambda^{1/2}_{W_i}d\xi_i(t) \quad (8) \end{aligned}$$

$$dy_i(t) = (\overline{H}_i(r^i_t,t)s_i(t) + m_{v_i})dt + d\zeta_i(t) \quad (9)$$

where for each $t \in [0,t_f]$, U_{W_i} and Λ_{W_i} correspond to the eigen-decomposition of W_i such that $W_i = U_{W_i}\Lambda_{W_i}U^T_{W_i}$.

The incremental Wiener processes $d\xi_i(t)$ and $d\zeta_i(t)$ are defined as $d\xi_i(t) \triangleq [w_i(t) - m_{w_i}]dt$ and $d\zeta_i(t) \triangleq [v_i(t) - m_{v_i}]dt$, respectively.

Viewed from the mutual influence of one decision maker to those of others, decision maker i is now capable of observing all best responses from the immediate neighbors denoted by $\mathcal{N}_i$; but subject to its current sensor accuracy and confident factors via an uncorrelated stationary Wiener measurement process. Specifically, the observations are locally available at decision maker i

$$du_{-i}(t) \triangleq \overline{C}_i(r_t^i, t) d_i(t) dt\,, \quad i \in \mathcal{N}$$
$$= \overline{C}_i(r_t^i, t) \sum_{j=1, j\neq i}^{\mathcal{N}_i} H_{ij}(t) u_j(t) dt + d\eta_i(t)\,. \quad (10)$$

Notice that all decision makers must operate within their respective local environments, which are now modeled by the uncorrelated stationary Wiener processes adapted for $[0, t_f]$ and have the correlations of independent increments

$$E\left\{[\xi_i(\tau_1) - \xi_i(\tau_2)][\xi_i(\tau_1) - \xi_i(\tau_2)]^T\right\} = W_i|\tau_1 - \tau_2|$$
$$E\left\{[\eta_i(\tau_1) - \eta_i(\tau_2)][\eta_i(\tau_1) - \eta_i(\tau_2)]^T\right\} = N_i|\tau_1 - \tau_2|$$
$$E\left\{[\zeta_i(\tau_1) - \zeta_i(\tau_2)][\zeta_i(\tau_1) - \zeta_i(\tau_2)]^T\right\} = V_i|\tau_1 - \tau_2|$$

where $W_i > 0$, $N_i > 0$ and $V_i > 0$ are also assumed known.

Closely related to the continuing quest for N distributed state estimators is the development of σ-algebra

$$\mathbb{F}_{0,t}^i \triangleq \sigma\{(w_i(\tau), v_i(\tau), \eta_i(\tau)) : 0 \le \tau \le t\}\,, \quad i \in \mathcal{N}$$
$$\mathcal{G}_{0,t}^{y_i} \triangleq \sigma\{y_i(\tau) : 0 \le \tau \le t\}\,, \quad t \in [0, t_f]\,.$$

As the result, Kalman-like estimators that later form part of the estimate-based decisions preserve, even earlier, the inherent linear Gaussian structures of (8)-(9), but these, too, take into account of the information available $\mathcal{G}_{t_f}^{y_i} \triangleq \{\mathcal{G}_{0,t}^{y_i} : t \in [0, t_f]\} \subset \{\mathbb{F}_{0,t}^i : t \in [0, t_f]\}$; e.g.,

$$d\hat{s}_i(t) = (\overline{A}_i(r_t^i, t)\hat{s}_i(t) + \overline{B}_i(r_t^i, t) u_i(t) + \overline{G}_i(r_t^i, t) m_{w_i}$$
$$+ u_{-i}(t))dt + L_i(t)(dy_i(t) - (\overline{H}_i(r_t^i, t)\hat{s}_i(t) + m_{v_i})dt) \quad (11)$$

where $\hat{s}_i(0) = s_i(0)$ and distributed estimation gains $L_i(t)$

$$L_i(t) = \Sigma_i(t) \overline{H}_i^T(r_t^i, t) V_i^{-1}\,, \quad \Sigma_i(0) = 0 \quad (12)$$
$$\dot{\Sigma}_i(t) = \overline{A}_i(r_t^i, t)\Sigma_i(t) + \Sigma_i(t)\overline{A}_i^T(r_t^i, t) + N_i \quad (13)$$
$$+ \overline{G}_i(r_t^i, t) W_i \overline{G}_i^T(r_t^i, t) - \Sigma_i(t)\overline{H}_i^T(r_t^i, t) V_i^{-1} \overline{H}_i(r_t^i, t)\Sigma_i(t).$$

Under the definition $\tilde{s}_i(t) \triangleq s_i(t) - \hat{s}_i(t)$, it follows that

$$d\tilde{s}_i(t) = (\overline{A}_i(r_t^i, t) - L_i(t)\overline{H}_i(r_t^i, t))\tilde{s}_i(t)dt - L_i(t)d\zeta_i(t)$$
$$+ \overline{G}_i(r_t^i, t) U_{W_i} \Lambda_{W_i}^{1/2} d\xi_i(t) - d\eta_i(t),\ \tilde{s}_i(0) = 0\,. \quad (14)$$

Beyond this, decision maker i for $i \in \mathcal{N}$, however, attempts to make risk-bearing decisions u_i caused by $\mathbb{F}_{0,t}^i$ and $\mathcal{G}_{0,t}^{y_i}$ from its admissible feedback policy set $\mathbb{U}^{y_i, u_i}[0, t_f]$, which is a closed subset of $\mathcal{L}^2_{\mathbb{F}^i_{t_f}}([0, t_f], \mathbb{R}^{m_i})$.

Associated with each admissible 2-tuple $(u_i(\cdot), u_{-i}(\cdot))$ is the generalized chi-squared random performance

$$J_i(u_i, u_{-i}) = s_i^T(t_f) Q_i^f s_i(t_f) + \int_0^{t_f} [s_i^T(\tau) Q_i(\tau) s_i(\tau)$$
$$+ u_i^T(\tau) R_i(\tau) u_i(\tau) - u_{-i}^T(\tau) M_i(\tau) u_{-i}(\tau)] d\tau \quad (15)$$

where the design parameters $Q_i^f \in \mathbb{R}^{n_i \times n_i}$, $Q_i \in \mathcal{C}([0, t_f]; \mathbb{R}^{n_i \times n_i})$, $M_i \in \mathcal{C}([0, t_f]; \mathbb{R}^{n_i \times n_i})$ and $R_i \in \mathcal{C}([0, t_f]; \mathbb{R}^{m_i \times m_i})$ representing relative weightings for terminal and transient tradeoffs between the regulatory of responses s_i, the effectiveness of the control and/or decision policy u_i and the coordination of control and/or decision policies from the immediate neighbors u_{-i} are deterministic and positive semidefinite with $R_i(t)$ invertible.

In the case of incomplete information, an admissible feedback policy u_i for a local best response to all other decision makers u_{-i} must be of the form, for some $\eth_i(\cdot, \cdot)$

$$u_i(t) = \eth_i(t, y_i(\tau))\,, \quad \tau \in [0, t]\,. \quad (16)$$

In general, the conditional density $p^i(s_i(t)|\mathcal{G}_{0,t}^{y_i})$, which is the density of $s_i(t)$ conditioned on $\mathcal{G}_{0,t}^{y_i}$ represents the sufficient statistics for describing the conditional stochastic effects of future feedback policy u_i. With regards to the linear-Gaussian structure, the conditional covariance $\Sigma_i(t)$ is independent of feedback policy $u_i(t)$ and observations $\{y_i(\tau) : \tau \in [0, t]\}$. Henceforth, an optimal control and/or decision policy $u_i(t)$ of the form (16) should deduce to

$$u_i(t) = \gamma^i(t, \hat{s}_i(t))\,, \quad t \in [0, t_f]\,.$$

Towards these bases, the search for an optimal feedback solution is productively restricted to a linear time-varying feedback policy generated from the locally accessible $\hat{s}_i(t)$

$$u_i(t) = K_i(t)\hat{s}_i(t) + m_i(t)\,, \quad t \in [0, t_f] \quad (17)$$

where feedback decision parameters $K_i \in C([0, t_f]; \mathbb{R}^{m_i \times n_i})$ and $m_i \in C([0, t_f]; \mathbb{R}^{m_i})$ will be formally defined later.

In effect, the a-priori knowledge about neighboring disturbances $u_{-i}(\cdot)$ and the admissible feedback policy (17), the aggregation of the dynamics (11) and (14) associated with decision maker i is described by the controlled stochastic differential equation together with the initial state $z_i(0) = z_{i0}$

$$dz_i(t) = (A_z^i(r_t^i, t) z_i(t) + b_z^i(r_t^i, t))dt + G_z^i(r_t^i, t) d\varsigma_i(t) \quad (18)$$

and the performance measure (15) is rewritten as follows

$$J_i(u_i, u_{-i}) = z_i^T(t_f) Q_{zf}^i z_i(t_f) + \int_0^{t_f} [z_i^T(\tau) Q_z^i(\tau) z_i(\tau) \quad (19)$$
$$+ 2 z_i^T(\tau) S_z^i(\tau) + m_i^T(\tau) R_i(\tau) m_i(\tau) - u_{-i}^T(\tau) M_i(\tau) u_{-i}(\tau)] d\tau$$

where the aggregate stationary Wiener process noise is denoted by $d\varsigma_i(t) \triangleq \left[\, d\xi_i^T(t)\ d\zeta_i^T(t)\ d\eta_i^T(t) \,\right]^T$ together with $E\left\{[\varsigma_i(\tau_1) - \varsigma_i(\tau_2)][\varsigma_i(\tau_1) - \varsigma_i(\tau_2)]^T\right\} = \Xi_i|\tau_1 - \tau_2|$, $\forall \tau_1, \tau_2 \in [0, t_f]$ and whereas the aggregate system coefficients and state variables are defined by

$$A_z^i(r_t^i, t) \triangleq$$
$$\begin{bmatrix} \overline{A}_i(r_t^i, t) + \overline{B}_i(r_t^i, t) K_i(t) & L_i(t)\overline{H}_i(r_t^i, t) \\ 0 & \overline{A}_i(r_t^i, t) - L_i(t)\overline{H}_i(r_t^i, t) \end{bmatrix}$$

$$b_z^i(r_t^i, t) \triangleq \begin{bmatrix} \overline{B}_i(r_t^i, t) m_i(t) + u_{-i}(t) + \overline{G}_i(r_t^i, t) m_{w_i} \\ 0 \end{bmatrix}$$

$$G_z^i(r_t^i, t) \triangleq \begin{bmatrix} 0 & L_i(t) & 0 \\ \overline{G}_i(r_t^i, t) U_{W_i} \Lambda_{W_i}^{1/2} & -L_i(t) & -I \end{bmatrix}$$

$$z_i \triangleq \begin{bmatrix} \hat{s}_i \\ \tilde{s}_i \end{bmatrix}, \quad z_{i0} \triangleq \begin{bmatrix} s_i(0) \\ 0 \end{bmatrix}, \quad Q_{zf}^i \triangleq \begin{bmatrix} Q_i^f & Q_i^f \\ Q_i^f & Q_i^f \end{bmatrix}$$

$$Q_z^i(t) \triangleq \begin{bmatrix} Q_i(t) + K_i^T(t) R_i(t) K_i(t) & Q_i(t) \\ Q_i(t) & Q_i(t) \end{bmatrix}$$

$$S_z^i(t) \triangleq \begin{bmatrix} K_i^T(t) R_i(t) m_i(t) \\ 0 \end{bmatrix}, \quad \Xi_i \triangleq \begin{bmatrix} W_i & 0 & 0 \\ 0 & V_i & 0 \\ 0 & 0 & N_i \end{bmatrix}.$$

Such an acknowledgement of the linear dynamics (18) and the integral-quadratic-form performance measure (19) creates the following conclusion: performance measure associated with decision maker i is clearly a random variable of the generalized chi-squared type. More important, perhaps, is the fact that the degrees of uncertainty of the ensemble performance-measure (19) must be assessed via a complete set of higher-order statistics. One productive step involved in extracting information from complex behavior of (19) is modeling and management of all the mathematical statistics (also known as semi-invariants). The methodology below is pursued for its central role in the endeavor of extracting performance-measure statistics pertaining to random distributions.

Theorem 1. Cumulant-Generating Function.
Let distributed controller or decision maker i be associated with the states $z_i(\cdot)$ of the stochastic dynamics (18) and subject to the performance measure (19). Further, let initial states $z_i(\tau) \equiv z_i^\tau$ and $\tau \in [0, t_f]$, the moment-generating function $\varphi^i(\tau, z_i^\tau, \theta) = \varrho^i(\tau,\theta)\exp\{(z_i^\tau)^T \Upsilon^i(\tau,\theta) z_i^\tau + 2(z_i^\tau)^T \ell^i(\tau,\theta)\}$ and $\upsilon^i(\tau,\theta) = \ln\{\varrho^i(\tau,\theta)\}$ for $\theta \in \mathbb{R}_+$ Then, the cumulant-generating function has the form of

$$\psi^i(\tau, z_\tau^i, \theta) = (z_i^\tau)^T \Upsilon^i(\tau,\theta) z_i^\tau + 2(z_i^\tau)^T \ell^i(\tau,\theta) + \upsilon^i(\tau,\theta)$$

where the backward-in-time scalar valued $\upsilon^i(\tau,\theta)$ satisfies

$$\begin{aligned} \frac{d}{d\tau}\upsilon^i(\tau,\theta) &= -\mathrm{Tr}\{\Upsilon^i(\tau,\theta) G_z^i(r_\tau^i,\tau)\Xi_i (G_z^i)^T(r_\tau^i,\tau)\} \\ &\quad - 2(\ell^i)^T(\tau,\theta) b_z^i(r_\tau^i,\tau) - \theta m_i^T(\tau) R_i(\tau) m_i(\tau) \\ &\quad + \theta u_{-i}^T(\tau) M_i(\tau) u_{-i}(\tau), \quad \upsilon^i(t_f,\theta) = 0 \end{aligned} \quad (20)$$

whereas the backward-in-time matrix valued $\Upsilon^i(\tau,\theta)$ with $\Upsilon^i(t_f,\theta) = \theta Q_{zf}^i$ and vector valued $\ell^i(\tau,\theta)$ satisfy

$$\begin{aligned} \frac{d}{d\tau}\Upsilon^i(\tau,\theta) &= -(A_z^i)^T(r_\tau^i,\tau)\Upsilon^i(\tau,\theta) - \Upsilon^i(\tau,\theta) A_z^i(r_\tau^i,\tau) \\ &\quad -2\Upsilon^i(\tau,\theta) G_z^i(r_\tau^i,\tau)\Xi_i (G_z^i)^T(r_\tau^i,\tau)\Upsilon^i(\tau,\theta) - \theta Q_z^i(\tau) \quad (21) \\ \frac{d}{d\tau}\ell^i(\tau,\theta) &= -(A_z^i)^T(r_\tau^i,\tau)\ell^i(\tau,\theta) - \Upsilon^i(\tau,\theta) b_z^i(r_\tau^i,\tau) \\ &\quad - \theta S_z^i(\tau), \quad \ell^i(t_f,\theta) = 0. \quad (22) \end{aligned}$$

Proof. For notional simplicity, it is convenient to have $\varpi^i(\tau, z_\tau^i, \theta) \triangleq \exp\{\theta J_i(\tau, z_i^\tau)\}$, in which the performance measure (19) is rewritten as the cost-to-go function

$$\begin{aligned} J_i(\tau, z_i^\tau) &= z_i^T(t_f) Q_{zf}^i z_i(t_f) + \int_\tau^{t_f} [z_i^T(\tau) Q_z^i(\tau) z_i(\tau) \quad (23) \\ &\quad + 2 z_i^T(\tau) S_z^i(\tau) + m_i^T(\tau) R_i(\tau) m_i(\tau) - u_{-i}^T(\tau) M_i(\tau) u_{-i}(\tau)] d\tau \end{aligned}$$

subject to the stochastic dynamics (18) with the initial condition $z_i(\tau) = z_i^\tau$. By definition, $\varphi^i(\tau, z_i^\tau, \theta) \triangleq E\{\varpi^i(\tau, z_i^\tau, \theta)\}$. Thus, its total time derivative is of the form

$$\begin{aligned} \frac{d}{d\tau}\varphi^i(\tau, z_i^\tau, \theta) &= -\theta[(z_i^\tau)^T Q_z^i(\tau) z_i^\tau + 2(z_i^\tau)^T S_z^i(\tau) \\ &\quad + m_i^T(\tau) R_i(\tau) m_i(\tau) - (u_{-i})^T(\tau) M_i(\tau) u_{-i}(\tau)]\varphi^i(\tau, z_i^\tau, \theta). \end{aligned}$$

Using the standard Ito's formula, it follows that

$$\begin{aligned} d\varphi^i(\tau, z_i^\tau, \theta) &= \varphi_\tau^i(\tau, z_i^\tau, \theta) d\tau + \varphi_{z_i^\tau}^i(\tau, z_i^\tau, \theta) dz_i^\tau \\ &\quad + \frac{1}{2}\mathrm{Tr}\left\{\varphi_{z_i^\tau z_i^\tau}^i(\tau, z_i^\tau, \theta) G_z^i(r_\tau^i,\tau)\Xi_i (G_z^i)^T(r_\tau^i,\tau)\right\} d\tau. \end{aligned}$$

Given that $\varphi^i(\tau, z_i^\tau, \theta) = \varrho^i(\tau,\theta)\exp\{(z_i^\tau)^T \Upsilon^i(\tau,\theta) z_i^\tau + 2(z_i^\tau)^T \ell^i(\tau,\theta)\}$ and its partial derivatives, it is clear to see

$$\begin{aligned} &- \theta[(z_i^\tau)^T Q_z^i(\tau) z_i^\tau + 2(z_i^\tau)^T S_z^i(\tau) + m_i^T(\tau) R_i(\tau) m_i(\tau) \\ &-(u_{-i})^T(\tau) M_i(\tau) u_{-i}(\tau)]\varphi^i(\tau, z_i^\tau, \theta) = \Big\{(z_i^\tau)^T \frac{d}{d\tau}\Upsilon^i(\tau,\theta) z_i^\tau \\ &+ \frac{\frac{d}{d\tau}\varrho^i(\tau,\theta)}{\varrho^i(\tau,\theta)} + 2(z_i^\tau)^T \frac{d}{d\tau}\ell^i(\tau,\theta) + 2(\ell^i)^T(\tau,\theta) b_z^i(r_\tau^i,\tau) \\ &+ (z_\tau^i)^T[(A_z^i)^T(r_\tau^i,\tau)\Upsilon^i(\tau,\theta) + \Upsilon^i(\tau,\theta) A_z^i(r_\tau^i,\tau)] z_i^\tau \\ &+ 2(z_i^\tau)^T (A_z^i)^T(r_\tau^i,\tau)\ell^i(\tau,\theta) + 2(z_i^\tau)^T \Upsilon^i(\tau,\theta) b_z^i(r_\tau^i,\tau) \\ &+ 2(z_i^\tau)^T \Upsilon^i(\tau,\theta) G_z^i(r_\tau^i,\tau)\Xi_i (G_z^i)^T(r_\tau^i,\tau)\Upsilon^i(\tau,\theta) z_i^\tau \\ &+ \mathrm{Tr}\{\Upsilon^i(\tau,\theta) G_z^i(r_\tau^i,\tau)\Xi_i (G_z^i)^T(r_\tau^i,\tau)\}\Big\}\varphi^i(\tau, z_i^\tau, \theta). \end{aligned}$$

To have constant and quadratic terms be independent of arbitrary z_i^τ, it requires that the results (20)-(22) hold. Finally, at $\tau = t_f$, it follows that $\varphi^i(t_f, z_i^{t_f}, \theta) = \exp\{\theta (z_i^{t_f})^T Q_{zf}^i z_i^{t_f}\}$. Consequently, the terminal-value conditions $\Upsilon^i(t_f,\theta) = \theta Q_{zf}^i$, $\ell^i(t_f,\theta) = 0$, $\varrho^i(t_f,\theta) = 1$ and hence, $\upsilon^i(t_f,\theta) = 0$ which complete the proof.

In addition, it is reasonable to employ a MacLaurin series expansion of the cumulant-generating function to capture performance variations of (19) through the knowledge representation of all the mathematical statistics, e.g.,

$$\psi^i(\tau, z_i^\tau, \theta) = \sum_{r=1}^{\infty} \frac{\partial^{(r)}}{\partial\theta^{(r)}}\psi^i(\tau, z_i^\tau, \theta)\bigg|_{\theta=0} \frac{\theta^r}{r!} \quad (24)$$

where $\kappa_r^i \triangleq \frac{\partial^{(r)}}{\partial\theta^{(r)}}\psi^i(\tau, z_i^\tau, \theta)\Big|_{\theta=0}$ are the performance-measure statistics available at decision maker i and $i \in \mathcal{N}$

$$\begin{aligned} \kappa_r^i &= \frac{\partial^{(r)}}{\partial\theta^{(r)}}\psi^i(\tau, z_i^\tau, \theta)\bigg|_{\theta=0} = (z_i^\tau)^T \frac{\partial^{(r)}}{\partial\theta^{(r)}}\Upsilon^i(\tau,\theta)\bigg|_{\theta=0} z_i^\tau \\ &\quad + 2(z_i^\tau)^T \frac{\partial^{(r)}}{\partial\theta^{(r)}}\ell^i(\tau,\theta)\bigg|_{\theta=0} + \frac{\partial^{(r)}}{\partial\theta^{(r)}}\upsilon^i(\tau,\theta)\bigg|_{\theta=0}. \end{aligned} \quad (25)$$

For notational convenience, the change of variables

$$\begin{aligned} H_r^i(\tau) &\triangleq \frac{\partial^{(r)}\Upsilon^i(\tau,\theta)}{\partial\theta^{(r)}}\bigg|_{\theta=0}; \quad \breve{D}_r^i(\tau) \triangleq \frac{\partial^{(r)}\ell^i(\tau,\theta)}{\partial\theta^{(r)}}\bigg|_{\theta=0} \\ D_r^i(\tau) &\triangleq \frac{\partial^{(r)}\upsilon^i(\tau,\theta)}{\partial\theta^{(r)}}\bigg|_{\theta=0}; \quad \tau \in [0, t_f]; \quad r \in \mathbb{N} \end{aligned} \quad (26)$$

is introduced so that the next theorem provides an effective and accurate capability for forecasting all the higher-order characteristics associated with performance uncertainty.

Theorem 2. Performance-Measure Statistics.
Associate with distributed stochastic systems governed by (18)-(19), where the pairs $(\overline{A}_i(r_t^i,t), \overline{B}_i(r_t^i,t))$ and $(\overline{A}_i(r_t^i,t), \overline{H}_i(r_t,t))$ are uniformly stabilizable and detectable. For $k^i \in \mathbb{N}$ fixed, the k^ith-cumulant of performance measure (19) concerned by decision maker i is

$$\kappa_k^i = z_{i0}^T H_{k^i}^i(0) z_{i0} + 2 z_{i0}^T \breve{D}_{k^i}^i(0) + D_{k^i}^i(0), \quad i \in \mathcal{N} \quad (27)$$

where the supporting variables $\{H_r^i(\tau)\}_{r=1}^{k^i}$, $\{\breve{D}_r^i(\tau)\}_{r=1}^{k^i}$ and $\{D_r^i(\tau)\}_{r=1}^{k^i}$ evaluated at $\tau = 0$ satisfy the differential equations (with the dependence of $H_r^i(\tau)$, $\breve{D}_r^i(\tau)$ and $D_r(\tau)$ upon the admissible feedback parameters $K_i(\tau)$ and $m_i(\tau)$ in connection of other interactions $u_{-i}(\tau)$ suppressed)

$$\frac{d}{d\tau}H_1^i(\tau) = -(A_z^i)^T(r_\tau^i,\tau)H_1^i(\tau) - H_1^i(\tau)A_z^i(r_\tau^i,\tau) - Q_z^i(\tau)\,, \quad H_1^i(t_f) = Q_{zf}^i \tag{28}$$

$$\frac{d}{d\tau}H_r^i(\tau) = -(A_z^i)^T(r_\tau^i,\tau)H_r^i(\tau) - H_r^i(\tau)A_z^i(r_\tau^i,\tau) - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}H_s^i(\tau)G_z^i(r_\tau^i,\tau)\Xi_i(G_z^i)^T(r_\tau^i,\tau)H_{r-s}^i(\tau) \tag{29}$$

$$\frac{d}{d\tau}\breve{D}_1^i(\tau) = -(A_z^i)^T(r_\tau^i,\tau)\breve{D}_1^i(\tau) - H_1^i(\tau)b_z^i(r_\tau^i,\tau) - S_z^i(\tau)\,, \quad \breve{D}_1^i(t_f) = 0 \tag{30}$$

$$\frac{d}{d\tau}\breve{D}_r^i(\tau) = -(A_z^i)^T(r_\tau^i,\tau)\breve{D}_r^i(\tau) - H_r^i(\tau)b_z^i(r_\tau^i,\tau) \tag{31}$$
$$\breve{D}_r^i(t_f) = 0\,, \quad 2 \le r \le k^i$$

$$\frac{d}{d\tau}D_1^i(\tau) = -\mathrm{Tr}\left\{H_1^i(\tau)G_z^i(r_\tau^i,\tau)\Xi_i(G_z^i)^T(r_\tau^i,\tau)\right\} - 2(\breve{D}_1^i)^T(\tau)b_z^i(r_\tau^i,\tau) - m_i^T(\tau)R_i(\tau)m_i(\tau) + u_{-i}^T(\tau)M_i(\tau)u_{-i}(\tau), \quad D_1^i(t_f) = 0 \tag{32}$$

$$\frac{d}{d\tau}D_r^i(\tau) = -\mathrm{Tr}\left\{H_r^i(\tau)G_z^i(r_\tau^i,\tau)\Xi_i(G_z^i)^T(r_\tau^i,\tau)\right\} - 2(\breve{D}_r^i)^T(\tau)b_z^i(r_\tau^i,\tau), \quad D_r^i(t_f) = 0, \quad 2 \le r \le k^i \tag{33}$$

where the terminal conditions $H_r^i(t_f) = 0$ for $2 \le r \le k^i$.

Proof. The expression of (27) is readily justified by using the result (25) and definition (26). What remains is to show that $H_r^i(\tau)$, $\breve{D}_r^i(\tau)$ and $D_r^i(\tau)$ for $1 \le r \le k^i$ indeed satisfy the back-in-time differential equations (28)-(33). Of note, these deterministic differential equations (28)-(33) are then obtained by successively taking derivatives of (20)-(22) with respect to θ and subject to the assumptions of $(\overline{A}_i(r_\tau^i,\tau), \overline{B}_i(r_\tau^i,\tau))$ and $(\overline{A}_i(r_\tau^i,\tau), \overline{H}_i(r_\tau^i,\tau))$ being uniformly stabilizable and detectable on $[0,t_f]$.

3. PROBLEM STATEMENTS

Increased insight into the roles played by performance-measure statistics associated with (19) creates a paradigm shift for robust decision making under performance uncertainty. Particularly, it will affect the core design strategies in distributed control and analysis with performance risk aversion. For such a problem, it is important to have a compact statement of the risk-averse decision and control optimization herein so as to aid the following mathematical manipulations. The approach here is to let $H_r^i(\cdot) \triangleq \begin{bmatrix}(H_r^i)_{11}(\cdot)(H_r^i)_{12}(\cdot)\\(H_r^i)_{21}(\cdot)(H_r^i)_{22}(\cdot)\end{bmatrix}$, $\breve{D}_r^i(\cdot) \triangleq \begin{bmatrix}(\breve{D}_r^i)_{11}(\cdot)\\(\breve{D}_r^i)_{21}(\cdot)\end{bmatrix}$, $\Pi_1(\cdot) \triangleq L_i(\cdot)V_iL_i^T(\cdot)$, $\Pi_2(\cdot) = \Pi_3(\cdot) \triangleq -L_i(\cdot)V_iL_i^T(\cdot)$, $\Pi_4(\cdot) \triangleq \overline{G}_i(r_\cdot^i,\cdot)U_{W_i}\Lambda_{W_i}^{1/2}W_i\Lambda_{W_i}^{1/2}U_{W_i}^T\overline{G}_i^T(r_\cdot^i,\cdot) + L_i(\cdot)V_iL_i^T(\cdot) + N_i$. In effect, the time-backward state evolutions (28)-(33) of which the admissible feedback parameters K_i and m_i are embedded, are further considered as the new dynamical equations with the associated $4k^i$-tuple matrix, vector and scalar state variables

$$\mathcal{H}^i \triangleq (\mathcal{H}_1^i,\ldots,\mathcal{H}_{k^i}^i,\mathcal{H}_{k^i+1}^i,\ldots,\mathcal{H}_{2k^i}^i, \mathcal{H}_{2k^i+1}^i,\ldots,\mathcal{H}_{3k^i}^i,\mathcal{H}_{3k^i+1}^i,\ldots,\mathcal{H}_{4k^i}^i)$$
$$= ((H_1^i)_{11},\ldots,(H_{k^i}^i)_{11},(H_1^i)_{12},\ldots,(H_{k^i}^i)_{12}, (H_1^i)_{21},\ldots,(H_{k^i}^i)_{21},(H_1^i)_{22},\ldots,(H_{k^i}^i)_{22})$$
$$\breve{\mathcal{D}}^i \triangleq (\breve{\mathcal{D}}_1^i,\ldots,\breve{\mathcal{D}}_{k^i}^i,\breve{\mathcal{D}}_{k^i+1}^i,\ldots,\breve{\mathcal{D}}_{2k^i}^i)$$
$$= ((\breve{D}_1^i)_{11},\ldots,(\breve{D}_{k^i}^i)_{11},(\breve{D}_1^i)_{21},\ldots,(\breve{D}_{k^i}^i)_{21})$$
$$\mathcal{D}^i \triangleq (\mathcal{D}_1^i,\ldots,\mathcal{D}_{k^i}^i) = (D_1^i,\ldots,D_{k^i}^i)$$

and the rules of action, for $r=1$

$$\frac{d}{d\tau}\mathcal{H}_1^i(\tau) = \mathcal{F}_1^i(\tau,\mathcal{H}^i,K_i), \quad \mathcal{H}_1^i(t_f) = Q_f^i \tag{34}$$
$$= -(\overline{A}_i(r_\tau^i,\tau) + \overline{B}_i(r_\tau^i,\tau)K_i(\tau))^T\mathcal{H}_1^i(\tau) - Q_i(\tau) - \mathcal{H}_1^i(\tau)(\overline{A}_i(r_\tau^i,\tau)+\overline{B}_i(r_\tau^i,\tau)K_i(\tau)) - K_i^T(\tau)R_i(\tau)K_i(\tau)$$

$$\frac{d}{d\tau}\mathcal{H}_{k^i+1}^i(\tau) = \mathcal{F}_{k^i+1}^i(\tau,\mathcal{H}^i,K_i),\ \mathcal{H}_{k^i+1}^i(t_f) = Q_f^i \tag{35}$$
$$= -(\overline{A}_i(r_\tau^i,\tau) + \overline{B}_i(r_\tau^i,\tau)K_i(\tau))^T\mathcal{H}_{k^i+1}^i(\tau) - \mathcal{H}_{k^i+1}^i(\tau)(\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau)) - \mathcal{H}_1^i(\tau)L_i(\tau)\overline{H}_i(r_\tau^i,\tau) - Q_i(\tau)$$

$$\frac{d}{d\tau}\mathcal{H}_{2k^i+1}^i(\tau) = \mathcal{F}_{2k^i+1}^i(\tau,\mathcal{H}^i,K_i),\mathcal{H}_{2k^i+1}^i(t_f) = Q_f^i \tag{36}$$
$$= -\mathcal{H}_{2k^i+1}^i(\tau)(\overline{A}_i(r_\tau^i,\tau) + \overline{B}_i(r_\tau^i,\tau)K_i(\tau)) - (\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\mathcal{H}_{2k^i+1}^i(\tau) - Q_i(\tau) - (L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\mathcal{H}_1^i(\tau)$$

$$\frac{d}{d\tau}\mathcal{H}_{3k^i+1}^i(\tau) = \mathcal{F}_{3k^i+1}^i(\tau,\mathcal{H}^i,K_i),\mathcal{H}_{3k^i+1}^i(t_f) = Q_f^i \tag{37}$$
$$= -\mathcal{H}_{3k^i+1}^i(\tau)(\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau)) - (\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\mathcal{H}_{3k^i+1}^i(\tau) - Q_i(\tau) - (L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\mathcal{H}_{k^i+1}^i(\tau) - \mathcal{H}_{2k^i+1}^i(\tau)L_i(\tau)\overline{H}_i(r_\tau^i,\tau)$$

$$\frac{d}{d\tau}\breve{\mathcal{D}}_1^i(\tau) = \breve{\mathcal{G}}_1^i(\tau,\mathcal{H}^i,\breve{\mathcal{D}}^i,K_i,m_i), \quad \breve{\mathcal{D}}_1^i(t_f) = 0 \tag{38}$$
$$= -(\overline{A}_i(r_\tau^i,\tau)+\overline{B}_i(r_\tau^i,\tau)K_i(\tau))^T\breve{\mathcal{D}}_1^i(\tau) - K_i^T(\tau)R_i(\tau)m_i(\tau) - \mathcal{H}_1^i(\tau)(\overline{B}_i(r_\tau^i,\tau)m_i(\tau)+u_{-i}(\tau)+\overline{G}_i(r_\tau^i,\tau)m_{w_i})$$

$$\frac{d}{d\tau}\breve{\mathcal{D}}_{k^i+1}^i(\tau) = \breve{\mathcal{G}}_{k^i+1}^i(\tau,\mathcal{H}^i,\breve{\mathcal{D}}^i,m_i),\ \breve{\mathcal{D}}_{k^i+1}^i(t_f) = 0 \tag{39}$$
$$= -(\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\breve{\mathcal{D}}_{k^i+1}^i(\tau) - \mathcal{H}_{2k^i+1}^i(\tau)(\overline{B}_i(r_\tau^i,\tau)m_i(\tau)+u_{-i}(\tau)+\overline{G}_i(r_\tau^i,\tau)m_{w_i}) - (L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\breve{\mathcal{D}}_1^i(\tau)$$

$$\frac{d}{d\tau}\mathcal{D}_1^i(\tau) = \mathcal{G}_1^i(\tau,\mathcal{H}^i,\breve{\mathcal{D}}^i,m_i), \quad \mathcal{D}_1^i(t_f) = 0 \tag{40}$$
$$= -\mathrm{Tr}\{\mathcal{H}_1^i(\tau)\Pi_1(\tau) + \mathcal{H}_{k^i+1}^i(\tau)\Pi_3(\tau)\} - \mathrm{Tr}\{\mathcal{H}_{2k^i+1}^i(\tau)\Pi_2(\tau) + \mathcal{H}_{3k^i+1}^i(\tau)\Pi_4(\tau)\} - 2(\breve{\mathcal{D}}_1^i)^T(\tau)(\overline{B}_i(r_\tau^i,\tau)m_i(\tau) + u_{-i}(\tau)+\overline{G}_i(r_\tau^i,\tau)m_{w_i}) - m_i^T(\tau)R_i(\tau)m_i(\tau) + u_{-i}^T(\tau)M_i(\tau)u_{-i}(\tau)$$

and, for $2 \le r \le k^i$

$$\frac{d}{d\tau}\mathcal{H}_r^i(\tau) = \mathcal{F}_r^i(\tau,\mathcal{H}^i,K_i), \quad \mathcal{H}_r^i(t_f) = 0 \tag{41}$$
$$= -(\overline{A}_i(r_\tau^i,\tau) + \overline{B}_i(r_\tau^i,\tau)K_i(\tau))^T\mathcal{H}_r^i(\tau) - \mathcal{H}_r^i(\tau)(\overline{A}_i(r_\tau^i,\tau) + \overline{B}_i(r_\tau^i,\tau)K_i(\tau))$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}[\mathcal{H}_s^i(\tau)\Pi_1(\tau)+\mathcal{H}_{k^i+s}^i(\tau)\Pi_3(\tau)]\mathcal{H}_{r-s}^i(\tau)$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}[\mathcal{H}_s^i(\tau)\Pi_2(\tau)+\mathcal{H}_{k^i+s}^i(\tau)\Pi_4(\tau)]\mathcal{H}_{2k^i+r-s}^i(\tau)$$

$$\frac{d}{d\tau}\mathcal{H}_{k^i+r}^i(\tau) = \mathcal{F}_{k^i+r}^i(\tau,\mathcal{H}^i,K_i), \quad \mathcal{H}_{k^i+r}^i(t_f) = 0 \tag{42}$$

$$= -(\overline{A}_i(r_\tau^i,\tau) + \overline{B}_i(r_\tau^i,\tau)K_i(\tau))^T\mathcal{H}^i_{k^i+r}(\tau)$$
$$- \mathcal{H}^i_{k^i+r}(\tau)(\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau))$$
$$- \mathcal{H}^i_r(\tau)L_i(\tau)\overline{H}_i(r_\tau^i,\tau)$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}[\mathcal{H}^i_s(\tau)\Pi_1(\tau)+\mathcal{H}^i_{k^i+s}(\tau)\Pi_3(\tau)]\mathcal{H}^i_{k^i+r-s}(\tau)$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}[\mathcal{H}^i_s(\tau)\Pi_2(\tau)+\mathcal{H}^i_{k^i+s}(\tau)\Pi_4(\tau)]\mathcal{H}^i_{3k^i+r-s}(\tau)$$
$$\frac{d}{d\tau}\mathcal{H}^i_{2k^i+r}(\tau) = \mathcal{F}^i_{2k^i+r}(\tau,\mathcal{H}^i,K_i), \quad \mathcal{H}^i_{2k^i+r}(t_f) = 0 \quad (43)$$
$$= -\mathcal{H}^i_{2k^i+r}(\tau)(\overline{A}_i(r_\tau^i,\tau) + \overline{B}_i(r_\tau^i,\tau)K_i(\tau)) - (\overline{A}_i(r_\tau^i,\tau)$$
$$- L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\mathcal{H}^i_{2k^i+r}(\tau) - (L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\mathcal{H}^i_r(\tau)$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}[\mathcal{H}^i_{2k^i+s}(\tau)\Pi_1(\tau)+\mathcal{H}^i_{3k^i+s}(\tau)\Pi_3(\tau)]\mathcal{H}^i_{r-s}(\tau)$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}[\mathcal{H}^i_{2k^i+s}(\tau)\Pi_2(\tau)+\mathcal{H}^i_{3k^i+s}(\tau)\Pi_4]\mathcal{H}^i_{2k^i+r-s}(\tau)$$
$$\frac{d}{d\tau}\mathcal{H}^i_{3k^i+r}(\tau) = \mathcal{F}^i_{3k^i+r}(\tau,\mathcal{H}^i,K_i), \quad \mathcal{H}^i_{3k^i+r}(t_f) = 0 \quad (44)$$
$$= -\mathcal{H}^i_{3k^i+r}(\tau)(\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau))$$
$$- (\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\mathcal{H}^i_{3k^i+r}(\tau)$$
$$- (L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\mathcal{H}^i_{k^i+r}(\tau)-\mathcal{H}^i_{2k^i+r}(\tau)L_i(\tau)\overline{H}_i(r_\tau^i,\tau)$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}[\mathcal{H}^i_{2k^i+s}(\tau)\Pi_1+\mathcal{H}^i_{3k^i+s}(\tau)\Pi_3(\tau)]\mathcal{H}^i_{k^i+r-s}(\tau)$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}[\mathcal{H}^i_{2k^i+s}(\tau)\Pi_2+\mathcal{H}^i_{3k^i+s}(\tau)\Pi_4(\tau)]\mathcal{H}^i_{3k^i+r-s}(\tau)$$
$$\frac{d}{d\tau}\breve{\mathcal{D}}^i_r(\tau) = \breve{\mathcal{G}}^i_r(\tau,\mathcal{H}^i,\breve{\mathcal{D}}^i,K_i,m_i), \quad \breve{\mathcal{D}}^i_r(t_f) = 0 \quad (45)$$
$$= -(\overline{A}_i(r_\tau^i,\tau) + \overline{B}_i(r_\tau^i,\tau)K_i(\tau))^T\breve{\mathcal{D}}^i_r(\tau)$$
$$- \mathcal{H}^i_r(\tau)(\overline{B}_i(r_\tau^i,\tau)m_i(\tau)+u_{-i}(\tau)+\overline{G}_i(r_\tau^i,\tau)m_{w_i})$$
$$\frac{d}{d\tau}\breve{\mathcal{D}}^i_{k^i+r}(\tau) = \breve{\mathcal{G}}^i_{k^i+r}(\tau,\mathcal{H}^i,\breve{\mathcal{D}}^i,m_i),\ \breve{\mathcal{D}}^i_{k^i+r}(t_f) = 0 \quad (46)$$
$$= -(\overline{A}_i(r_\tau^i,\tau) - L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\breve{\mathcal{D}}^i_{k^i+r}(\tau)$$
$$- \mathcal{H}^i_{2k^i+r}(\tau)(\overline{B}_i(r_\tau^i,\tau)m_i(\tau)+u_{-i}(\tau)+\overline{G}_i(r_\tau^i,\tau)m_{w_i})$$
$$- (L_i(\tau)\overline{H}_i(r_\tau^i,\tau))^T\breve{\mathcal{D}}^i_r(\tau)$$
$$\frac{d}{d\tau}\mathcal{D}^i_r(\tau) = \mathcal{G}^i_r(\tau,\mathcal{H}^i,\breve{\mathcal{D}}^i,m_i), \quad \mathcal{D}^i_r(t_f) = 0 \quad (47)$$
$$= -\mathrm{Tr}\{\mathcal{H}^i_r(\tau)\Pi_1(\tau) + \mathcal{H}^i_{k^i+r}(\tau)\Pi_3(\tau)\}$$
$$- \mathrm{Tr}\{\mathcal{H}^i_{2k^i+r}(\tau)\Pi_2(\tau) + \mathcal{H}^i_{3k^i+r}(\tau)\Pi_4(\tau)\}$$
$$- 2(\breve{\mathcal{D}}^i_r)^T(\tau)(\overline{B}_i(r_\tau^i,\tau)m_i(\tau)+u_{-i}(\tau)+\overline{G}_i(r_\tau^i,\tau)m_{w_i})$$

The product system of the equations (34)-(47) becomes

$$\frac{d}{d\tau}\mathcal{H}^i(\tau) = \mathcal{F}^i(\tau,\mathcal{H}^i(\tau),K_i(\tau)), \qquad \mathcal{H}^i(t_f) \quad (48)$$
$$\frac{d}{d\tau}\breve{\mathcal{D}}^i(\tau) = \breve{\mathcal{G}}^i(\tau,\mathcal{H}^i(\tau),\breve{\mathcal{D}}^i(\tau),m_i(\tau)), \quad \breve{\mathcal{D}}^i(t_f) \quad (49)$$
$$\frac{d}{d\tau}\mathcal{D}^i(\tau) = \mathcal{G}^i(\tau,\mathcal{H}^i(\tau),\breve{\mathcal{D}}^i(\tau),m_i(\tau)), \quad \mathcal{D}^i(t_f) \quad (50)$$

where the key point is that $\mathcal{F}^i \triangleq \mathcal{F}^i_1 \times \cdots \times \mathcal{F}^i_{k^i} \times \mathcal{F}^i_{k^i+1} \times \cdots \times \mathcal{F}^i_{2k^i} \times \mathcal{F}^i_{2k^i+1} \times \cdots \times \mathcal{F}^i_{3k^i} \times \mathcal{F}^i_{3k^i+1} \times \cdots \times \mathcal{F}^i_{4k^i}$, $\breve{\mathcal{G}}^i \triangleq \breve{\mathcal{G}}^i_1 \times\cdots\times \breve{\mathcal{G}}^i_{k^i} \times \breve{\mathcal{G}}^i_{k^i+1} \times\cdots\times \breve{\mathcal{G}}^i_{2k^i}$ and $\mathcal{G}^i \triangleq \mathcal{G}^i_1 \times\cdots\times \mathcal{G}^i_{k^i}$.

Of note, once the immediate neighbors $j \in \mathcal{N}_i$ of decision maker i fix their person-by-person equilibrium strategies u_j^* and thus the signalling or coordination effects u^*_{-i}, decision maker i will then obtain an optimal stochastic control problem with risk-averse performance considerations. The construction of a person-by-person policy associated with decision maker i now involves the 2-tuple (K_i, m_i). In the sequel and elsewhere, when this dependence is needed to be clear, then the notations $\mathcal{H}^i \equiv \mathcal{H}^i(\cdot, K_i, u^*_{-i})$, $\breve{\mathcal{D}}^i \equiv \breve{\mathcal{D}}^i(\cdot, K_i, m_i, u^*_{-i})$ and $\mathcal{D}^i \equiv \mathcal{D}^i(\cdot, m_i, u^*_{-i})$.

For the terminal data $(t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f)$, the theoretical framework for risk-averse control of distributed stochastic systems with time delays and coordinations u^*_{-i} from immediate neighbors $j \in \mathcal{N}_i$ is next analyzed by admissible feedback policies for decision maker i and $i \in \mathcal{N}$.

Definition 3. Admissible Feedback Policies.
Let compact subsets $\overline{K}^i \subset \mathbb{R}^{m_i\times(2n_i+m_i)}$ and $\overline{M}^i \subset \mathbb{R}^{m_i}$ be the sets of allowable feedback form values available at decision maker i. For the given $k^i \in \mathbb{N}$ and sequence $\mu^i = \{\mu^i_r \geq 0\}_{r=1}^{k^i}$ with $\mu^i_1 > 0$, the sets of feedback parameters $\mathcal{K}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$ and $\mathcal{M}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$ are respectively assumed to be the classes of $\mathcal{C}([0,t_f];\mathbb{R}^{m_i\times(2n_i+m_i)})$ and $\mathcal{C}([0,t_f];\mathbb{R}^{m_i})$ with values $K_i(\cdot) \in \overline{K}^i$ and $m_i(\cdot) \in \overline{M}^i$, for which the solutions to the dynamic equations (48)-(50) with the terminal-value conditions $\mathcal{H}^i(t_f) = \mathcal{H}^i_f$, $\breve{\mathcal{D}}^i(t_f) = \breve{\mathcal{D}}^i_f$ and $\mathcal{D}^i(t_f) = \mathcal{D}^i_f$ exist on $[0, t_f]$.

To minimize the performance vulnerability of (19) against all ensemble realizations from the stochastic environments and mutual influences from $\mathcal{N}_i$, performance risks are henceforth interpreted as worries and fears about certain undesirable characteristics of (19) and thus are managed through a finite set of selective weights. This custom set of design freedoms representing particular uncertainty aversions is hence different from the ones with aversion to risk captured in risk-sensitive optimal control as in Jacobson (1973) and Whittle (1990).

On $\mathcal{K}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i} \times \mathcal{M}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$ the performance index with mean-risk considerations is defined as follows.

Definition 4. Mean-Risk Aware Performance Index.
Let decision maker i select $k^i \in \mathbb{N}$ and the set of scalar coefficients $\mu^i = \{\mu^i_r \geq 0\}_{r=1}^{k^i}$ with $\mu^i_1 > 0$. Then, for the given $s_{i0} \triangleq s_i(0)$, the mean-risk aware performance index

$$\phi^i_0 : (\mathbb{R}^{(2n_i+m_i)\times(2n_i+m_i)})^{k^i} \times (\mathbb{R}^{2n_i})^{k^i} \times \mathbb{R}^{k^i} \mapsto \mathbb{R}_+$$

pertaining to risk-averse decision making over $[0, t_f]$ is

$$\phi^i_0(\mathcal{H}^i(0), \breve{\mathcal{D}}^i(0), \mathcal{D}^i(0)) \triangleq \underbrace{\mu^i_1\kappa^i_1}_{\text{Mean}} + \underbrace{\mu^i_2\kappa^i_2 + \cdots + \mu^i_{k^i}\kappa^i_{k^i}}_{\text{Risk}}$$
$$= \sum_{r=1}^{k^i}\mu^i_r[s^T_{i0}\mathcal{H}^i_r(0)s_{i0} + 2s^T_{i0}\breve{\mathcal{D}}^i_r(0) + \mathcal{D}^i_r(0)] \quad (51)$$

where additional design freedom by means of μ^i_r's utilized by decision maker i are sufficient to meet and exceed different levels of performance-based reliability requirements; e.g., mean (i.e., the average of performance), variance (i.e., the dispersion of performance values), skewness (i.e., the anti-symmetry of performance density), kurtosis (i.e., the heaviness in the performance density tails), etc., pertaining to performance variations while $\{\mathcal{H}^i_r(\tau)\}_{r=1}^{k^i}$, $\{\breve{\mathcal{D}}^i_r(\tau)\}_{r=1}^{k^i}$ and $\{\mathcal{D}^i_r(\tau)\}_{r=1}^{k^i}$ evaluated at $\tau = 0$ satisfy (48)-(50).

To explicitly indicate the dependence of the mean-risk aware performance index (51) expressed in Mayer form on the 2-tuple (K_i, m_i) and the signaling effects u^*_{-i} issued by all immediate neighbors j from $\mathcal{N}_i$, the mean-risk performance index (51) is now rewritten as $\phi^i_0(K_i, m_i; u^*_{-i})$.

Definition 5. Nash Equilibrium.
An N-tuple of control and/or decision policies $\{(K^*_1, m^*_1), \ldots, (K^*_N, m^*_N)\}$ is said to constitute a Nash equilibrium for the N-person stochastic game which is supported by distributed large-scale stochastic systems with time delays if, for all $i \in \mathcal{N}$, the Nash inequalities hold

$$\phi^i_0(K^*_i, m^*_i; u^*_{-i}) \le \phi^i_0(K_i, m_i; u^*_{-i}), \qquad i \in \mathcal{N}\,.$$

For the sake of time consistency and subgame perfection, a Nash equilibrium solution is required to have an additional property that its restriction on $[0, \tau]$ is also a Nash solution to the truncated version of the original problem, defined on $[0, \tau]$. With such a restriction so defined, it is now termed as a feedback Nash equilibrium solution, which is now free of any informational nonuniqueness, and thus whose derivation allows a dynamic programming type argument.

Definition 6. Feedback Nash Equilibrium.
Let the 2-tuple (K^*_i, m^*_i) define a feedback Nash control and/or decision policy for all admissible $(K_i, m_i) \in \mathcal{K}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i} \times \mathcal{M}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$, upon which the solutions to the dynamical systems (48)-(50) exist on $[0, t_f]$.

Then, $\{(K^*_1, m^*_1), \ldots, (K^*_N, m^*_N)\}$ when restricted to $[0, \tau]$ is still a N-tuple feedback Nash equilibrium solution to the N-person Nash decision problem with the appropriate terminal conditions $(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau))$, $\forall\, \tau \in [0, t_f]$.

Now the objective of decision maker i is to minimize (51) over $\mathcal{K}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i} \times \mathcal{M}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$ while subject to the local neighborhood of the Nash decision policies u^*_{-i}.

Definition 7. Optimization of Mayer Problem.
Given the profile of risk-averse attitudes $\mu^i = \{\mu^i_r \ge 0\}_{r=1}^{k^i}$ with $\mu^i_1 > 0$, the decision optimization problem defined by

$$\min_{K_i(\cdot), m_i(\cdot)} \phi^i_0(K_i, m_i; u^*_{-i})\,, \quad i \in \mathcal{N} \tag{52}$$

is subject to the dynamical equations (48)-(50) on $[0, t_f]$.

In conformity with the dynamic programming approach, the terminal time and states $(t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f)$ are parameterized as $(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$, whereby $\mathcal{Y}^i \triangleq \mathcal{H}^i(\varepsilon)$, $\breve{\mathcal{Z}}^i \triangleq \breve{\mathcal{D}}^i(\varepsilon)$ and $\mathcal{Z}^i \triangleq \mathcal{D}^i(\varepsilon)$. Thus, the value function now depends on the parameterization of the terminal-value conditions.

Definition 8. Value Function.
Let $(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \in [0, t_f] \times (\mathbb{R}^{(2n_i+m_i)\times(2n_i+m_i)})^{k^i} \times (\mathbb{R}^{2n_i+m_i})^{k^i} \times \mathbb{R}^{k^i}$. Then, $\mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ is defined by

$$\mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \triangleq \inf_{K^i(\cdot), m_i(\cdot)} \phi^i_0(K_i, m_i; u^*_{-i})\,. \tag{53}$$

Definition 9. Reachable Sets.
Let a reachable set of decision maker i be defined by $\mathcal{Q}^i \triangleq \{(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \in [0, t_f] \times (\mathbb{R}^{(2n_i+m_i)\times(2n_i+m_i)})^{k^i} \times (\mathbb{R}^{2n_i+m_i})^{k^i} \times \mathbb{R}^{k^i} : \mathcal{K}^i_{\varepsilon,\mathcal{Y}^i,\breve{\mathcal{Z}}^i,\mathcal{Z}^i;\mu^i} \times \mathcal{M}^i_{\varepsilon,\mathcal{Y}^i,\breve{\mathcal{Z}}^i,\mathcal{Z}^i;\mu^i} \ne \emptyset\}$.

Moreover, it is shown that the value function associated with decision maker i is satisfying partial differential equation (e.g., Hamilton-Jacobi-Bellman (HJB) equation) at interior points of $\mathcal{Q}^i$, at which it is differentiable.

Theorem 10. HJB Equation-Mayer Problem.
Let $(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ be any interior point of $\mathcal{Q}^i$, at which the value function $\mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ is differentiable. If there exists a feedback Nash decision $(K^*_i, m^*_i) \in \mathcal{K}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i} \times \mathcal{M}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$, then the partial differential equation

$$\begin{aligned} 0 = \min_{K_i \in \overline{K}^i, m_i \in \overline{M}^i} \Big\{ & \frac{\partial}{\partial \varepsilon} \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \\ & + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y}^i)} \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \operatorname{vec}(\mathcal{F}^i(\varepsilon, \mathcal{Y}^i, K^i)) \\ & + \frac{\partial}{\partial \operatorname{vec}(\breve{\mathcal{Z}}^i)} \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \operatorname{vec}(\breve{\mathcal{G}}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, m_i) \\ & + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z}^i)} \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \operatorname{vec}(\mathcal{G}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, m_i) \Big\} \end{aligned} \tag{54}$$

is satisfied and subject to $\mathcal{V}^i(0, \mathcal{Y}^i(0), \breve{\mathcal{Z}}^i(0), \mathcal{Z}^i(0)) = \phi^i_0(\mathcal{H}^i(0), \breve{\mathcal{D}}^i(0), \mathcal{D}^i(0))$.

Proof. By what have been shown in Pham (2011), the proof for the result herein is readily proven.

Finally, the sufficient conditions used to verify a feedback Nash strategy for decision maker i are given as follows.

Theorem 11. Verification Theorem.
Let $\mathcal{W}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ be a differentiable solution to (54) with $\mathcal{W}^i(0, \mathcal{H}^i(0), \breve{\mathcal{D}}^i(0), \mathcal{D}^i(t_0)) = \phi^i_0(\mathcal{H}^i(0), \breve{\mathcal{D}}^i(0), \mathcal{D}^i(0))$. Let $(t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f) \in \mathcal{Q}^i$, $(K_i, m_i) \in \mathcal{K}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i} \times \mathcal{M}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$ and $(\mathcal{H}^i(\cdot), \breve{\mathcal{D}}^i(\cdot), \mathcal{D}^i(\cdot))$ be the trajectories of (48)-(50). Then, $\mathcal{W}^i(\tau, \mathcal{H}^i(\tau), \breve{\mathcal{D}}^i(\tau), \mathcal{D}^i(\tau))$ is therefore time-backward increasing.

If (K^*_i, m^*_i) is in $\mathcal{K}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i} \times \mathcal{M}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$ with the corresponding solutions $(\mathcal{H}^i_*(\cdot), \breve{\mathcal{D}}^i_*(\cdot), \mathcal{D}^i_*(\cdot))$ of the dynamical equations (48)-(50) such that, for $\tau \in [0, t_f]$

$$\begin{aligned} 0 = {} & \frac{\partial}{\partial \varepsilon} \mathcal{W}^i(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau)) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y}^i)} \mathcal{W}^i(\tau, \\ & \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau)) \operatorname{vec}(\mathcal{F}^i(\tau, \mathcal{H}^i_*(\tau), K^*_i(\tau))) \\ & + \frac{\partial}{\partial \operatorname{vec}(\breve{\mathcal{Z}}^i)} \mathcal{W}^i(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau)) \\ & \operatorname{vec}(\breve{\mathcal{G}}^i(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), m^*_i(\tau))) \\ & + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z}^i)} \mathcal{W}_{is}(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau)) \\ & \operatorname{vec}(\mathcal{G}^i(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), m^*_i(\tau))) \end{aligned} \tag{55}$$

then, the 2-tuple (K^*_i, m^*_i) is a feedback Nash policy and

$$\mathcal{W}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) = \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \tag{56}$$

Proof. It follows the same manner as in Pham (2011).

4. DISTRIBUTED PERSON-BY-PERSON CONTROLS

As already recognized, the initial state s_{i0} represents both quadratic and linear contributions to the mean-risk aware performance index (51) of Mayer type. Therefore, it is suggested that the value function is also linear and quadratic in s_{i0}. Subsequently, a candidate function $\mathcal{W}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ associated with (53) is expected to be

$$\mathcal{W}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) = s_{i0}^T \sum_{r=1}^{k^i} \mu_r^i (\mathcal{Y}_r^i + \mathcal{E}_r^i(\varepsilon))\, s_{i0} + 2 s_{i0}^T \sum_{r=1}^{k^i} \mu_r^i (\breve{\mathcal{Z}}_r^i + \breve{\mathcal{T}}_r^i(\varepsilon)) + \sum_{r=1}^{k^i} \mu_r^i (\mathcal{Z}_r^i + \mathcal{T}_r^i(\varepsilon)) \quad (57)$$

where the functions $\mathcal{E}_r^i \in \mathcal{C}^1([0,t_f]; \mathbb{R}^{(2n_i+m_i)\times(2n_i+m_i)})$, $\breve{\mathcal{T}}_r^i \in \mathcal{C}^1([0,t_f]; \mathbb{R}^{2n_i+m_i})$, and $\mathcal{T}_r^i \in \mathcal{C}^1([0,t_f]; \mathbb{R})$ are yet to be determined. Next, the time derivative of $\mathcal{W}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ is obtained

$$\frac{d}{d\varepsilon}\mathcal{W}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) = \sum_{r=1}^{k^i} \mu_r^i [\mathcal{G}_r^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, m_i) + \frac{d}{d\varepsilon}\mathcal{T}_r^i(\varepsilon)] + s_{i0}^T \sum_{r=1}^{k^i} \mu_r^i [\mathcal{F}_r^i(\varepsilon, \mathcal{Y}^i, K_i) + \frac{d}{d\varepsilon}\mathcal{E}_r^i(\varepsilon)] s_{i0} + 2 s_{i0}^T \sum_{r=1}^{k^i} \mu_r^i [\breve{\mathcal{G}}_r^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, m_i) + \frac{d}{d\varepsilon}\breve{\mathcal{T}}_r^i(\varepsilon)]\,. \quad (58)$$

The substitution of (57) for the value function into the HJB equation (54) and making use of (58) yield

$$0 = \min_{K_i \in \overline{K}^i, m_i \in \overline{M}^i} \Big\{ s_{i0}^T \sum_{r=1}^{k^i} \mu_r^i [\mathcal{F}_r^i(\varepsilon, \mathcal{Y}^i, K_i) + \frac{d}{d\varepsilon}\mathcal{E}_r^i(\varepsilon)] s_{i0} + 2 s_{i0}^T \sum_{r=1}^{k^i} \mu_r^i [\breve{\mathcal{G}}_r^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, m_i) + \frac{d}{d\varepsilon}\breve{\mathcal{T}}_r^i(\varepsilon)] + \sum_{r=1}^{k^i} \mu_r^i [\mathcal{G}_r^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, m_i) + \frac{d}{d\varepsilon}\mathcal{T}_r^i(\varepsilon)] \Big\}. \quad (59)$$

Taking the gradients with respect to K_i and m_i of the expression within the bracket of (59) yields the necessary conditions for an extremum of (51) on $[0, \varepsilon]$

$$K_i^*(\varepsilon) = -R_i^{-1}(\varepsilon) \overline{B}_i^T (r_\varepsilon^i, \varepsilon) \sum_{r=1}^{k^i} \hat{\mu}_r^i \mathcal{H}_{*r}^i(\varepsilon) \quad (60)$$

$$m_i^*(\varepsilon) = -R_i^{-1}(\varepsilon) \overline{B}_i^T (r_\varepsilon^i, \varepsilon) \sum_{r=1}^{k^i} \hat{\mu}_r^i \breve{\mathcal{D}}_{*r}^i(\varepsilon) \quad (61)$$

where $\hat{\mu}_r^i \triangleq \mu_l^i / \mu_1^i$ for $\mu_1^i > 0$.

With the feedback person-by-person decision policy (60)-(61) being replaced in the expression of the bracket (59) and having $\{\mathcal{Y}_r^i\}_{r=1}^{k^i}$, $\{\breve{\mathcal{Z}}_r^i\}_{r=1}^{k^i}$, and $\{\mathcal{Z}_r^i\}_{r=1}^{k^i}$ evaluated on the optimal solution trajectories of (48)-(50), $\mathcal{E}_r^i(\varepsilon)$, $\breve{\mathcal{T}}_r^i(\varepsilon)$ and $\mathcal{T}_r^i(\varepsilon)$ are thus chosen so the sufficient condition (55) in the verification theorem is satisfied regardless the arbitrary values s_{i0}; for example,

$$\mathcal{E}_r^i(\varepsilon) = \mathcal{H}_{*r}^i(0) - \mathcal{H}_{*r}^i(\varepsilon)\,, \quad \breve{\mathcal{T}}_r^i(\varepsilon) = \breve{\mathcal{D}}_{*r}^i(0) - \breve{\mathcal{D}}_{*r}^i(\varepsilon)$$
$$\mathcal{T}_r^i(\varepsilon) = \mathcal{D}_{*r}^i(0) - \mathcal{D}_{*r}^i(\varepsilon)\,, \quad i \in \mathcal{N}\,, \quad 1 \le r \le k^i\,.$$

At last, the sufficient condition (55) of the verification theorem is satisfied so that the extremizing feedback person-by-person decision and/or control policy (60)-(61) associated with decision maker i therefore becomes optimal. And yet it is not too soon to offer a closing summary of the analysis and its procedural mechanism to compute the risk-averse decisions with two degrees of freedom as follows

$$u_i^*(t) = K_i^*(t)\hat{s}^*(t) + m_i^*(t)\,, \quad t = t_f - \varepsilon\,, \quad i \in \mathcal{N} \quad (62)$$

whereupon the statistical measures of performance risks are utilized locally at each distributed controllers i.

5. CONCLUSIONS

This paper shows recent advances on distributed information and decision frameworks pertaining to a linear-quadratic class of large-scale uncertain stochastic systems with state and control delays. The emphasis is on the application of a new generation of summary statistical measures associated with the generalized chi-squared performance measure. In views of performance risks, a new paradigm shift for understanding and building distributed person-by-person equilibrium decision policies is obtained, with which the self-directed but yet strongly connected decision makers, who are subject to distributed decision making, are fully capable of implementing risk-bearing actions and local best responses in the furtherance of their own goals.

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