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Architecture and Structure of Robust PID Controllers

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Abstract: In previous papers by the use of the architecture generating property of the optimal linear quadratic tracking theory those problems that give the PI, PD and PID controllers had been presented, thus avoiding heuristics and giving a systematic approach to explanation for their excellent performance. This approach has been used also in derivation of the family of generalized PI^mD^{n-1}controllers. In this paper by the combined use of the optimal linear quadratic tracking and system sensitivity theories a problem is formulated and solutions are shown that give a family of robust PID controllers. Examples of the control architectures and structures for first and second order systems are presented.

Keywords: PID controller, Robust Control, Control Architecture, Control Structure, System Sensitivity.

1. INTRODUCTION

The PD, PI and PID controllers are successfully applied controllers to many applications, almost from the beginning of controls applications; see D'Azzo and Houpis (1988), Franklin et al. (1994). The facts of their successful application, good performance, easiness of tuning are speaking for themselves and are sufficient rational for their use, although their structure is justified by heuristics: "These ... controls - called proportional-integral-derivative (PID) control - constitute the heuristic approach to controller design that has found wide acceptance in the process industries." Franklin et al. (1994), pp. 168.

Robustness issues are dealt with mainly by tuning algorithms of the PID controller's gains. Previous papers, Rusnak (1998, 1999), have shown that the PID controllers can be derived by the use the optimal tracking control. Namely, problems had been stated whose solutions lead to the PD, PI and PID controllers. This enables avoiding heuristics and is giving a systematic approach to explanation of the good performance of the PID controllers. This approach enabled generalization and derivation of the family of the generalized PID controllers, the PI^mD^{n-1} controllers; Rusnak (2000a, 2000b). The classical one block PID controller has been generalized, the cascade architecture – the PIV controller, has been presented for PID controller with one integrator, Rusnak (2011), and the PI2D, IPIV and PI2V controllers for controller with two integrators, Rusnak (2011), have been derived. One of the main contributions of the results in Rusnak (2011) is that they show for what problems the PID controllers are the optimal controllers and for which they are not. The generalized PI^2D controller had been implemented in real-time on Linear Motion Control System (Dvash et. all).

The use of the optimal linear quadratic tracking (LQT) theory gives the architecture as well as the structure of the controllers. By Architecture we mean, loosely, the connections between the outputs/sensors and the inputs/actuators; Structure deals with the specific realization of the controllers' blocks; and Configuration is a specific combination of architecture and structure. These issues fall within the control and feedback organization theory that have been presented in Rusnak (2006, 2008). This issue is beyond the scope of this paper. The optimal tracking theory is used here as a basis, at a system theoretic level, that enables formulation of the control-feedback loops organization problem that leads to the robust PID controllers. This paper does not deal with the numerical values of the controllers' filters coefficients/gains; rather it concentrates in organization of the control loops and structure of the filters. This is the way the optimal LQT theory is used.

The LQT theory requires a reference trajectory generator. The reference trajectory is generated by a system that reflects the nominal behaviour of the plant. The differences are the initial conditions, the input to the reference trajectory generator and the deviation of the actual plant from the nominal one. The zero steady state is imposed by integral action of the required order on the state tracking error.

The issue of sensitivity of control systems, in general, and design of robust controllers, in particular, are widely treated issues. These are for example the H_∞, QFT design techniques and more. These techniques, albeit their vitality, do not enable generalization. This is as they do not possess the architecture generating property.

The author is unaware of specific treatment of the issue of the architecture and structure of robust PID and robust generalized PID controllers.

The novelty and contribution of this paper is the simultaneous application of the LQT theory and sensitivity theory, Frank (1978), in derivation of architectures and structures of robust PID controllers. It is shown that the presented approach:

1) Can systematically rationalize the existing practices.
2) Enables, due to the architecture generating property of the LQT criterion, systematic generalization of problems for which the generalized PID controllers are optimal.
3) Enables to derive controllers that are robust with respect to specific parameters.
4) Shows the exact structure of the robust controller.

Examples of robust controllers for first and second order systems are derived.

Throughout this paper the same notation for time domain and Laplace domain is used, and the explicit Laplace variable (s) is stated to avoid confusion wherever necessary.

2. PRELIMINARIES OF PROBLEM STATEMENT

In this section preliminary definitions essential for the problem statement are introduced. Let's consider the n[th] order continuous linear time-invariant single input single output system in the observer canonical form.

$$\dot{x} = Ax + bu;\ x(t_o) = x_o, \quad y = Cx \tag{2.1}$$

$$A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 & 0 \\ -a_2 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & & 1 & \vdots \\ -a_{n-1} & 0 & 0 & 0 & 1 \\ -a_n & 0 & 0 & 0 & 0 \end{bmatrix};\ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix};\ C = c_o = [1 \ 0 \ \cdots \ 0 \ 0], \tag{2.2}$$

where x is the state; u is the input and y is the output and x_o is zero mean random vector.

The parameters of the plant are $\theta = [-a_1 \ -a_2 \ \cdots \ -a_n \ b_1 \ b_2 \ \cdots \ b_n]^T$ and we denote as well $h = [-a_1 \ -a_2 \ \ -a_n]^T$. The parameters of the plant, θ, are not known exactly or may change during the life time of the plant. It is assumed that only their estimate-nominal-average-most probable values are known. Thus the nominal system is $\left(A_m = A|_{\text{nominal}}, b_m = b|_{\text{nominal}}, c_m = c|_{\text{nominal}}\right)$. The nominal system is the "best" representation of the system and thus is used here as the model of the system.

The n[th] order reference trajectory generator is

$$\dot{x}_r = A_r x_r + b_r w_r;\ x_r(t_o) = x_{ro}, \quad y_r = c_r x_r \tag{2.3}$$

where x_r is the state; w_r is the input and y_r is the reference output. w_r is a zero mean stochastic process, and x_{ro} is zero mean random vector. The case when the orders of the plant and reference trajectory are different is beyond the scope of this paper. The parameters of the trajectory generator are $\theta_r = [-a_{r1} \ -a_{r2} \ \cdots \ -a_{rn} \ b_{r1} \ b_{r2} \ \cdots \ b_{rn}]^T$. The parameters of the trajectory generator, θ_r, are known with absolute certainty.

Further we assume that the reference system (2.3) is equal to the nominal-average system of (2.2), i.e.

$$A_r = A|_{nominal},\ c_r = c|_{nominal}.$$

In order to reject constant disturbances we introduce integral action, Kwakernaak and Sivan (1972), Anderson and Moore (1989), i.e. we consider the integral of the tracking error

$$\dot{\eta}_1 = c_1(x - x_r);\ \eta_1(t_o) = \eta_{1o}. \tag{2.4}$$

In order to achieve robustness we consider the sensitivity functions of the system performance with respect to the system's parameters, θ. That is, the gradient of the system output with respect to the plant's parameters, Frank (1978), i.e.

$$\nabla_\theta y(t) = \nabla_\theta cx(t) = C\nabla_\theta x(t) \tag{2.5}$$

and in order to drive the sensitivity to small values (ideally to zero) we consider the integral of the sensitivity function as well, i.e.

$$\dot{\eta}_2 = C\nabla_\theta x(t) = [c_3 \ \ c_4]\begin{bmatrix} \nabla_h x(t) \\ \nabla_b x(t) \end{bmatrix} = [c_3 \ \ c_4]\begin{bmatrix} x_h \\ x_b \end{bmatrix} = C\zeta(t); \qquad \eta_2(t_o) = \eta_{2o} \tag{2.6}$$

All vectors and matrices are of the appropriate dimensions.

The observer canonical form is used for convenience. For n[th] order system there are at most 2n coefficients-parameters that the state space representation can "capture". From this point of view any canonical could have fit.

3. THE SENSITIVITY SYSTEM

In section 2, eq. (2.5) the sensitivity functions, Frank (1978), have been introduced. The sensitivity functions will be minimized in the optimization process to derive robust PID controller. The sensitivity system is, Frank (1978),

$$\nabla_\theta y = \nabla_\theta Cx = C\nabla_\theta x$$
$$\frac{\partial \dot{x}}{\partial h} = A\frac{\partial x}{\partial h} + \frac{\partial A}{\partial h}x;\ \frac{\partial x}{\partial h}(t_o) = 0 \tag{3.1}$$
$$\frac{\partial \dot{x}}{\partial b} = A\frac{\partial x}{\partial b} + \frac{\partial b}{\partial b}u;\ \frac{\partial x}{\partial b}(t_o) = 0;$$

Equations (3.1) can be written

$$\frac{d}{dt}\begin{bmatrix} x \\ \frac{\partial x}{\partial h_1} \\ \vdots \\ \frac{\partial x}{\partial h_n} \\ \frac{\partial x}{\partial b_1} \\ \vdots \\ \frac{\partial x}{\partial b_n} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ \underline{\underline{E}} & bd(A) & 0 \\ \underline{\underline{0}} & 0 & bd(A) \end{bmatrix}\begin{bmatrix} x \\ \frac{\partial x}{\partial h_1} \\ \vdots \\ \frac{\partial x}{\partial h_n} \\ \frac{\partial x}{\partial b_1} \\ \vdots \\ \frac{\partial x}{\partial b_n} \end{bmatrix} + \begin{bmatrix} b \\ \underline{0} \\ \vdots \\ 0 \\ \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix} u(t) \tag{3.2}$$

$$\begin{bmatrix} x \\ \frac{\partial x}{\partial h_1} \\ \vdots \\ \frac{\partial x}{\partial h_n} \\ \frac{\partial x}{\partial b_1} \\ \vdots \\ \frac{\partial x}{\partial b_n} \end{bmatrix}(t_o) = \begin{bmatrix} x_o \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^{(2n^2+n)\times 1};\ \nabla_\theta y^T = \begin{bmatrix} 0 & c_o & 0 & \cdots & \cdots & 0 \\ 0 & 0 & c_o & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_o \end{bmatrix}\begin{bmatrix} x \\ \frac{\partial x}{\partial h_1} \\ \vdots \\ \frac{\partial x}{\partial h_n} \\ \frac{\partial x}{\partial b_1} \\ \vdots \\ \frac{\partial x}{\partial b_n} \end{bmatrix} \tag{3.3}$$

where

$$bd(A) = block\ diagonal(A, A, \ldots A) \in R^{n^2 \times n^2},\ n\ blocks, \tag{3.4}$$

$$\underline{\underline{E}} = \begin{bmatrix} \varepsilon_1 & \underline{0} & \cdots & 0 \\ \varepsilon_2 & 0 & & 0 \\ \vdots & & & \vdots \\ \varepsilon_n & 0 & \cdots & 0 \end{bmatrix} \in R^{n^2 \times n};\ \underline{E} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} \tag{3.5}$$

ε_i is the common unit length column.

We also denote

$$x_h=\begin{bmatrix}\frac{\partial x}{\partial h_1}\\ \vdots\\ \frac{\partial x}{\partial h_n}\end{bmatrix};\ x_b=\begin{bmatrix}\frac{\partial x}{\partial b_1}\\ \vdots\\ \frac{\partial x}{\partial b_n}\end{bmatrix};\ x_h,x_b\in R^{n^2\times 1} \tag{3.6}$$

The associated transfer functions are

$$\begin{aligned}x_h(s)&=(sI-bd(A))^{-1}\underline{\underline{E}}x(s)\\ x_b(s)&=(sI-bd(A))^{-1}\underline{E}u(s)\end{aligned} \tag{3.7}$$

Remark: For nth order system there are 2n parameters.

4. THE ROBUST PID CONTROLLER PROBLEM

The robust PID controller problem is defined. That is, the problem being dealt with here is finding the optimal control *u*(t)* that minimizes the quadratic criterion:

$$J=E\left\{\int_0^\infty\begin{bmatrix}(x(t)-x_r(t))^TQ(x(t)-x_r(t))\\ +\nabla_\theta^T y(t)Q_s\nabla_\theta y(t)\\ +\eta(t)^TQ_\eta\eta(t)+u(t)^TRu(t)\end{bmatrix}dt\right\} \tag{4.1}$$

subject to (2.1-6, 3.1-6). That is, the objective is the simultaneous minimization of the squares of the tracking error, $e=x-x_r$, the sensitivity of the output with respect to the plant's parameters, $\nabla_\theta y(t)$, their integrals, $\eta(t)=\begin{bmatrix}\eta_1^T & \eta_2^T\end{bmatrix}^T$, and the input, *u(t)*.

5. THE AUGMENTED PROBLEM

The problem of the robust PID controller in (4.1), can be written as the minimization of

$$J=E\left\{\int_0^\infty\left[\xi(t)^T\Theta\xi(t)+u(t)^TRu(t)\right]dt\right\} \tag{5.1}$$

subject to

$$\dot{\xi}=\overline{A}\,\xi+\overline{b}u+\overline{b}_rw_r';\ \ \xi(t_o)=\xi_o, \tag{5.2}$$

where the tracking error is

$$e=x-x_r \tag{5.3}$$

then

$$\xi=\begin{bmatrix}e\\x_h\\x_b\\\eta_1\\\eta_2\end{bmatrix};\ \overline{A}=\begin{bmatrix}A&0&0&0&0\\ \underline{\underline{E}}&bd(A)&0&0&0\\ 0&0&bd(A)&0&0\\ c_1&0&0&0&0\\ 0&c_3&c_4&0&0\end{bmatrix}; \tag{5.4}$$

$$\overline{b}=\begin{bmatrix}b\\0\\\varepsilon\\0\\0\end{bmatrix};\ \overline{b}_r=\begin{bmatrix}-b_r&0\\0&0\\0&\underline{E}\\0&0\\0&0\end{bmatrix};\ \ \xi(t_o)=\begin{bmatrix}x_o-x_{ro}\\0\\0\\\eta_{1o}\\\eta_{2o}\end{bmatrix} \tag{5.5}$$

$$\Theta=\begin{bmatrix}Q&0&0&0&0\\0&Q_{s1}&0&0&0\\0&0&Q_{s2}&0&0\\0&0&0&Q_{\eta1}&0\\0&0&0&0&Q_{\eta2}\end{bmatrix};w_r'=\begin{bmatrix}w_r&0\\0&L(w_r,x_{ro})\end{bmatrix} \tag{5.6}$$

The steady state solution is

$$\begin{aligned}&u(t)=-R^{-1}\overline{b}\,\overline{P}\xi(t)=-K\xi(t),\\ &0=\overline{P}\overline{A}+\overline{A}^T\overline{P}+\Theta-\overline{P}\overline{b}R^{-1}\overline{b}^T\overline{P}.\end{aligned} \tag{5.7}$$

Notice that the solution is independent of the reference trajectory driving input, w'_r. For the formal independence w'_r is required to be white. It is assumed that for all practical purposes, over the ensemble average of all reference trajectories, this is so.

6. SOLUTION OF THE ROBUST PID CONTROLLER PROBLEM

By corresponding partition of the gain matrix, *K*, in eq. (5.7) the optimal input can be written, formally, as

$$u(t)=\begin{bmatrix}K_e & K_{\zeta h} & K_{\zeta b} & K_{eI} & K_{\zeta I}\end{bmatrix}\begin{bmatrix}e\\x_h\\x_b\\\eta_1\\\eta_2\end{bmatrix} \tag{6.1}$$

Let's denote

$$K_\zeta=\begin{bmatrix}K_{\zeta h} & K_{\zeta b}\end{bmatrix};\ \zeta=\begin{bmatrix}x_h\\x_b\end{bmatrix} \tag{6.2}$$

then recalling (2.6)

$$u(s)=\left[K_e+\frac{1}{s}K_{eI}\right](x_r-x)-\left[K_\zeta+\frac{1}{s}K_{\zeta I}\right]\zeta \tag{6.3}$$

So the optimal robust PID controller is

$$u(s)=\left[K_e+\frac{1}{s}K_{eI}\right](x_r-x)-\left[K_\zeta+\frac{1}{s}K_{\zeta I}\right]\begin{bmatrix}x_h\\x_b\end{bmatrix} \tag{6.4}$$

7. ARCHITECTURES AND STRUCTURES OF THE ROBUST PID CONTROLLER PROBLEM

From section 6 several architectures emerge. Three of them are presented here.

7.1 Direct Full State Sensitivity Robust PID Architecture

This architecture and structure are direct implementation of equation (6.4), and is the same architecture as depicted in Frank (1978), chapter 9, figure 9.4-1, that is

$$\begin{aligned}u(s)&=\left[K_e+\frac{1}{s}K_{eI}\right](x_r-x)\\ &-\left[\begin{bmatrix}K_{\zeta h} & K_{\zeta b}\end{bmatrix}+\frac{1}{s}\begin{bmatrix}K_{\zeta hI} & K_{\zeta bI}\end{bmatrix}\right]\begin{bmatrix}(sI-bd(A))^{-1}\underline{\underline{E}}x(s)\\(sI-bd(A))^{-1}\underline{E}u(s)\end{bmatrix}\\ &=C_e(x_r-x)-C_{\zeta b}(sI-bd(A))^{-1}\underline{E}u(s)-C_{\zeta h}(sI-bd(A))^{-1}\underline{\underline{E}}x(s)\end{aligned} \tag{7.1}$$

This architecture requires the availability-measurement of the full state.

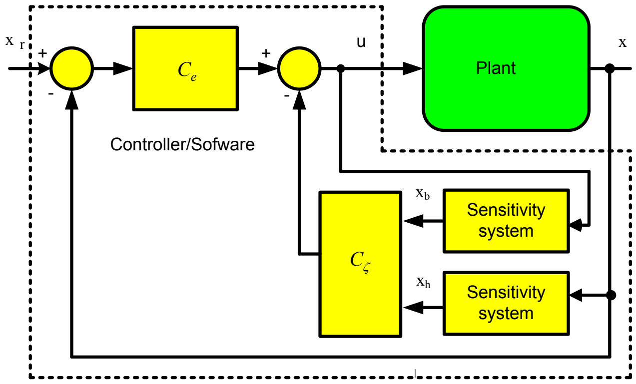

Figure 1: Block Diagram of the Direct State Sensitivity Robust PID Architecture.

7.2 Indirect State Sensitivity Robust PID Architecture

This architecture and structure do not require the availability-measurement of the full state and the measurement of the input and output is sufficient. The outputs are not used for the sensitivity function computation. This is called the indirect state sensitivity robust PID architecture. The state used for derivation of the sensitivity functions are derive from a plant emulator that is using (2.1)

$$x(s)=(sI-A_m)^{-1}\underline{b_m}u(s)$$

and we get

$$\begin{aligned}u(s)&=\left[K_e+\frac{1}{s}K_{eI}\right](x_r-x)\\&-\left[\left[K_{\zeta h}\quad K_{\zeta b}\right]+\frac{1}{s}\left[K_{\zeta hI}\quad K_{\zeta bI}\right]\right]\begin{bmatrix}(sI-bd(A))^{-1}\underline{\underline{E}}(sI-A_m)^{-1}\underline{b_m}u(s)\\(sI-bd(A))^{-1}\underline{E}u(s)\end{bmatrix}\\&=C_ee-C_{\zeta b}(sI-bd(A))^{-1}\underline{E}u(s)-C_{\zeta h}(sI-bd(A))^{-1}(sI-A_m)^{-1}\underline{b_m}u(s)\end{aligned} \tag{7.2}$$

Figure 2 presents the block diagram of the indirect state sensitivity robust PID architecture.

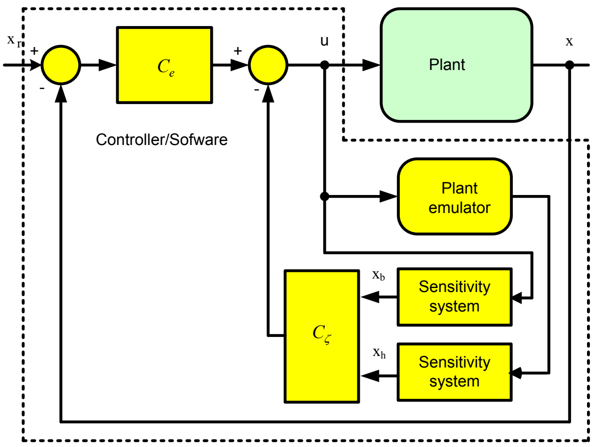

Figure 2: Block Diagram of the Indirect State Sensitivity Robust PID Architecture.

For single input systems we get

$$u(s)=C_{PID}(s)\;C_{Robust}(s)\;e(s)$$

$$C_{PID}(s)=\left[K_e+\frac{1}{s}K_{eI}\right] \tag{7.3}$$

$C_{Robust}(s)=$

$$\frac{1}{1+\left[\left[K_{\zeta h}\quad K_{\zeta b}\right]+\frac{1}{s}\left[K_{\zeta hI}\quad K_{\zeta bI}\right]\right]\begin{bmatrix}(sI-bd(A))^{-1}\underline{\underline{E}}(sI-A_m)^{-1}\underline{b_m}\\(sI-bd(A))^{-1}\underline{E}\end{bmatrix}}$$

The above means that the robust PID controller applies the PID controller, $C_{PID}(s)$ and then in cascade the robustifying controller $C_{Robust}(s)$. This is a one block controller architecture whose structure is given by

$$C(s)=\frac{\left[K_e+\frac{1}{s}K_{eI}\right]}{1+(sI-bd(A))^{-1}\left[C_{\zeta h}\underline{\underline{E}}(sI-A_m)^{-1}\underline{b_m}+C_{\zeta b}\underline{E}\right]} \tag{7.4}$$

7.3 Mixed State Sensitivity Robust PID Architecture

In section 7.1 availability of the full state has been assumed. In this section partial state availability is assumed. The rest of the state is derived by model of the system as in section 7.2 (in stochastic system it will an estimator, issue that is not treated in this paper). For deterministic system for those states that are measured it is the direct state sensitivity and for those states that are not measured it is the indirect state sensitivity. Block diagram of this approach is presented in figure 3.

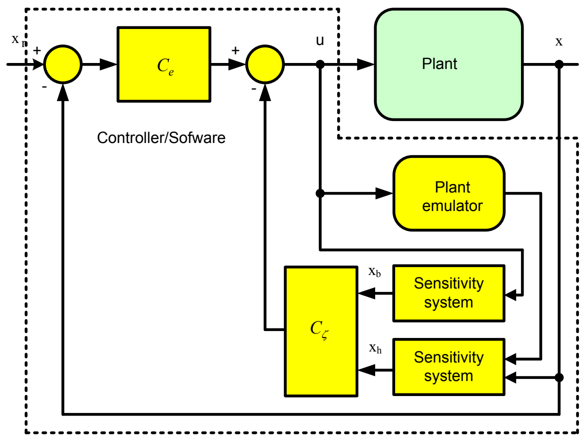

Figure 3: Block diagram of the Mixed State Sensitivity Robust PID Architecture.

7.4 Discussion

<u>The Direct Full State Sensitivity Robust PID Architecture</u>, Figure 1, section 7.1, is the optimal architecture. However in practice not all states are measurable.

<u>The Indirect State Sensitivity Robust PID Architecture</u>, Figure 2, section 7.2, assumes that no state is available. It is suboptimal but it is the one that is usually implemented. This architecture is the one practiced in design of robust systems.

<u>The Mixed State Sensitivity Robust PID Architecture</u>, Figure 3, section 7.3, a newly proposed architecture, uses the measured states for building the respective sensitivity function and the remaining sensitivity function are derived from a plant's emulator.

8. ARCHITECTURE AND STRUCTURE OF ROBUST PI CONTROLLERS FOR FIRST ORDER SYSTEM

The case of desensitizing of the performance with respect to the time constant of a first order system is considered. For

known systems this approach leads to the P and PI controllers, Rusnak (2011).
The system and the respective transfer function are

$$\dot{y} = -\frac{1}{\tau} y + \frac{1}{\tau} u; \quad H(s) = \frac{1}{s\tau + 1} \tag{8.1}$$

The objective is (4.1). The sensitivity system is

$$\frac{d}{dt} \frac{\partial y}{\partial(1/\tau)} = \frac{\partial \dot{y}}{\partial(1/\tau)} = -\frac{1}{\tau} \frac{\partial y}{\partial(1/\tau)} - y + u \tag{8.2}$$

Let's denote $\zeta = \partial y / \partial(1/\tau)$, then augmentation of the state and sensitivity function gives

$$\frac{d}{dt}\begin{bmatrix} y \\ \zeta \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} & 0 \\ -1 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} y \\ \zeta \end{bmatrix} + \begin{bmatrix} \frac{1}{\tau} \\ 1 \end{bmatrix} u \tag{8.3}$$

whose solution is

$$\frac{\zeta}{u} = \frac{s}{(s+1/\tau)(s+1/\tau)}; \quad \frac{\zeta}{y} = \frac{s\tau}{(s+1/\tau)} \tag{8.4}$$

In realization of the controller one must remember that the actual time constant, τ, of the system is unknown. Therefore, the controller is realized with an estimate (the nominal value) of this time constant. The computation of the sensitivity function is realized in the controller.

8.1 Direct Full State Sensitivity Robust PID Architecture

$$u = \left(C_{eP} + \frac{C_{eI}}{s} \right) e - C_\zeta \zeta = C_e e - C_\zeta \left(\frac{\zeta}{y} \right) y$$
$$= C_e e - C_\zeta \frac{s\tau_m^2}{(s\tau_m + 1)} y = C_{PI} e - C_{Robust} y \tag{8.5}$$

Figure 4 presents a block diagram of this architecture.

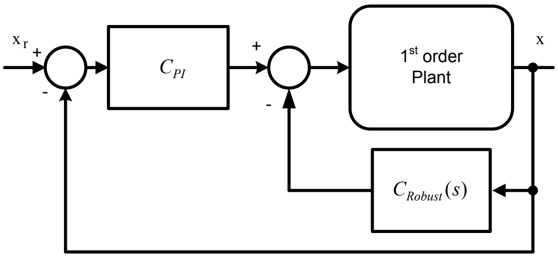

Figure 4: Two block controller architecture generated by the direct full state sensitivity robust PID problem for 1st order system (8.5).

8.2 Indirect Full State Sensitivity Robust PID Architecture

8.2.1 P controller

For P controller in the ζ loop we have

$$C_\zeta(s) = C_{\zeta P} \tag{8.6}$$

$$u = \left(C_{eP} + \frac{C_{eI}}{s} \right) e - C_\zeta \zeta = C_{PI} e - C_\zeta \frac{\zeta}{u} u \tag{8.7}$$

$$u = C_{PI} \frac{1}{1 + \frac{C_\zeta s}{(s+1/\tau)^2}} e = C_{PI} \frac{(s+1/\tau)^2}{(s+1/\tau)^2 + C_{\zeta P} s} y$$
$$= C_{PI} \frac{s^2 + 2s/\tau + (1/\tau)^2}{s^2 + (2/\tau + C_{\zeta P})s + (1/\tau)^2} y \tag{8.8}$$

8.2.2 PI controller

For PI controller in the ζ loop we have

$$C_\zeta = C_{\zeta P} + \frac{C_{\zeta I}}{s} \tag{8.9}$$

$$u = C_{PI} e - C_\zeta \zeta = C_{PI} e - \left(C_{\zeta P} + \frac{C_{\zeta I}}{s} \right) \frac{\xi}{u} u \tag{8.10}$$

$$u = C_{PI} \frac{1}{1 + \frac{(C_{\zeta P} + C_{\xi I}/s)s}{(s+1/\tau)^2}} e = C_{PI} \frac{(s+1/\tau)^2}{(s+1/\tau)^2 + (C_{\zeta P} + C_{\xi I}/s)s} e \tag{8.11}$$

$$u = C_{PI} \frac{s^2 + 2/\tau\, s + (1/\tau)^2}{s^2 + (2/\tau\, C_{\zeta P})s + (1/\tau)^2 + C_{\zeta I}} e \tag{8.12}$$

These are the robust P and PI controllers. The main observation is that the robust P/PI controller is a cascade of classical P/PI controller and a second order Lead-Lag. Figure 5 presents a block diagram of this architecture.

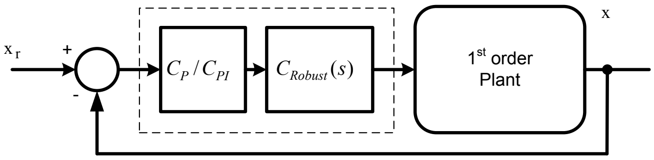

Figure 5: One block controller architecture generated by the indirect full state sensitivity robust PID problem for 1st order system (8.12).

9. ARCHITECTURE AND STRUCTURE OF ROBUST PID CONTROLLERS FOR SECOND ORDER SYSTEM

The case of desensitizing a second order system with respect to the time constant is dealt with. For known systems this approach leads to the PD and PID controllers, Rusnak (2011). (Here, for convenience the system is represented in the companion canonical form.)
The system and the respective transfer function are

$$\ddot{y} = -\frac{1}{\tau} \dot{y} + \frac{k}{\tau} u; \quad H(s) = \frac{k}{s(s\tau + 1)} \tag{9.1}$$

So that we have

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tau} k \end{bmatrix} u \tag{9.3}$$

and

$$\frac{d}{dt} \frac{\partial \dot{y}}{\partial(1/\tau)} = \frac{\partial \ddot{y}}{\partial(1/\tau)} = -\frac{1}{\tau} \frac{\partial \dot{y}}{\partial(1/\tau)} - \dot{y} + ku \tag{9.4}$$

The objective is (4.1). Denote $\zeta = \partial y / \partial(1/\tau), \dot{\zeta} = \partial \dot{y} / \partial(1/\tau)$, then augmentation gives

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \\ \zeta \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{\tau} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \zeta \\ \dot{\zeta} \end{bmatrix} + \begin{bmatrix} 0 \\ k \\ 0 \\ k \end{bmatrix} u \tag{9.5}$$

$$\frac{\zeta}{u} = \frac{k}{(s+1/\tau)^2} = \frac{k\tau^2}{(s\tau+1)^2}; \quad \frac{\zeta}{y} = \frac{s}{(s+1/\tau)} = \frac{s\tau^2}{(s\tau+1)} \tag{9.6}$$

9.1 Direct Full State Sensitivity Robust PID Architecture

$$u = C_e e + C_{\dot e}\dot e - C_\zeta \zeta - C_{\dot\zeta}\dot\zeta = C_e e + C_{\dot e}\dot e - \left(C_\zeta + sC_{\dot\zeta}\right)\frac{\zeta}{y} y \qquad (9.7)$$

$$= C_e e + C_{\dot e}\dot e - \left(C_\zeta + sC_{\dot\zeta}\right)\frac{s\tau}{(s\tau+1)} y \qquad (9.8)$$

$$= C_e e + C_{\dot e}\dot e - \left(C_\zeta + sC_{\dot\zeta}\right)\frac{\tau}{(s\tau+1)}\dot y;\ parallel\ robust\ PID \qquad (9.9)$$

$$= C_{\dot e}\left[\dot e + \frac{C_e}{C_{\dot e}} e\right] - \left(C_\zeta + sC_{\dot\zeta}\right)\frac{\tau}{(s\tau+1)}\dot y;\ cascade\ robust\ PID \qquad (9.10)$$

$$= C_{PID}\, e - C_{Robust}\, y;\ Direct\ Robust\ PID$$

9.2 Indirect State Sensitivity Robust PID Architecture

The controller is

$$u = C_y e + C_{\dot y}\dot e - C_\zeta\zeta - C_{\dot\zeta}\dot\zeta = C_y e + C_{\dot y}\dot y - \left(C_\zeta + sC_{\dot\zeta}\right)\frac{\zeta}{u} u \qquad (9.11)$$

Then

$$u = \frac{1}{1+\frac{C_\zeta + sC_{\dot\zeta}}{(s\tau+1)^2}k'}\left(C_y y + C_{\dot y}\dot y\right) = \frac{(s+a_1)^2}{(s\tau+1)^2 + C_{\dot\zeta}k's + C_\zeta k'}\left(C_y y + C_{\dot y}\dot y\right);\ parallel \qquad (9.12)$$

$$u = \frac{(s\tau+1)^2}{(s\tau+1)^2 + C_{\dot\zeta}k's + C_\zeta k'} C_{\dot y}\left[\dot y + \frac{C_y}{C_{\dot y}} y\right];\ cascade\ PIV\ setup \qquad (9.13)$$

One block Robust PID (see Figure 5)

$$C_{robustPID}(s) = C_{PID}(s)\, C_{Robust}(s)$$

$$C_{PID}(s) = \left(C_y y + C_{\dot y}\dot y\right) = \left(k_P + \frac{k_I}{s} + k_D s\right) y \qquad (9.14)$$

$$C_{Robust}(s) = \frac{(s\tau+1)^2}{(s\tau+1)^2 + C_{\dot\zeta}k's + C_\zeta k'} \qquad (9.15)$$

As an example figures 6 and 7 present the architecture of parallel and cascade indirect state sensitivity robust PID controller architectures.

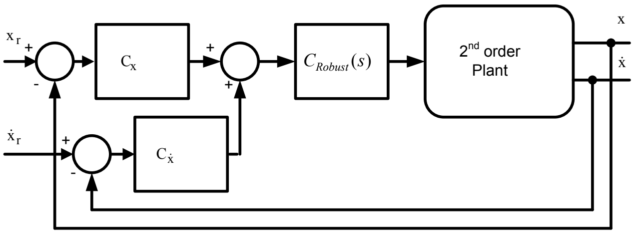

Figure 6: Architecture of parallel indirect state sensitivity robust PID controller for 2nd order system (9.12).

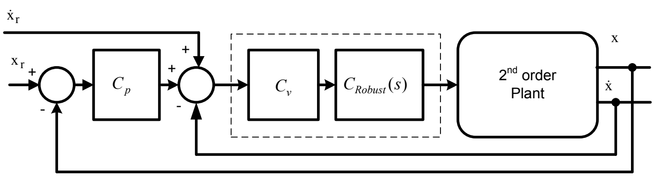

Figure 7: Architecture of cascade indirect state sensitivity robust PID controller for 2nd order system (9.13).

10. CONCLUSIONS

The optimal linear quadratic tracking theory and system sensitivity theory have been used to formulate a problem and show solutions that give a family of robust PID controllers. The architecture generating property of the LQT criterion has been exploited to derive architectures and structure of the family of optimal robust PID controllers. This way heuristics were avoided and systematic approach has been presented in derivation of robust PID controllers. Examples of architectures and structures for first and second order systems were presented.

REFERENCES

Anderson, B.D.O. and Moore, J.B. (1989). Optimal Control: Linear Quadratic Methods, Prentice-Hall.

D'Azzo, J.J. and Houpis, C.H. (1988). Linear Control Systems Analysis And Design: Conventional and Modern, McGraw-Hill Book Company, third edition.

Dvash, D., Ashkinezer, B. and Rusnak, I. (2013). Implementation of Generalized PI^2D Controller on Linear Motion Control System. Control, Robotics and Machine Learning Laboratory (CRML), Technion, Israel. (in Hebrew)

Frank, P.M. (1978). Introduction to Sensitivity Theory, Academic Press, Inc.

Franklin, G.F., Powell, J.D. and Emani-Naeini, A. (1994). Feedback Control of Dynamic Systems, third edition, Addison-Wesley Publishing Company, Inc., 1994.

Kwakernaak, and Sivan, R. (1972). Linear Optimal Control, John Wiley & Sons, Inc.

Rusnak, I. (1998). The Optimality of PID Controllers, 9th Mediterranean Electrotechnical Conference, MELECON '98, May 18-20, Tel-Aviv, Israel.

Rusnak, I. (1999). Generalized PID Controllers, the 7th IEEE Mediterranean Conference on Control & Automation, MED 99, 28-29 June 1999, Haifa, Israel.

Rusnak, I. (2000a). The Generalized PID Controller and its Application to Control of Ultrasonic and Electric Motors, PID 2000, IFAC Workshop on Digital Control, Past, Present and Future of PID Control, Terrassa, Spain, 5-7 April 2000.

Rusnak, I. (2000b): Generalized PID Controller for Stochastic Systems, The 21st Convention of IEEE in Israel, April 11-12, Tel-Aviv, Israel.

Rusnak, I. (2006). Control and Feedback Organization - Review and Terminology, The 24nd IEEE Conference of Electrical and Electronics Engineers in Israel in Israel, November 1, Eilat, Israel.

Rusnak, I. (2008). Control Organization – Survey and Application, The 9th Biennial ASME Conference on Engineering Systems Design and Analysis, ESDA 2008, July 7-9, Haifa, Israel.

Rusnak, I. (2011). Family of the PID Controllers, chapter in "Introduction to PID Controllers – Theory, Tuning and Application to Frontier Areas", InTech - Open Access Publisher, www.intechopen.com, ISBN: 978-953-307-927-1.