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# On the Set of Perron Exponents of Discrete Linear Systems ⋆

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**Abstract:** The Lyapunov, Bohl and Perron exponents belong to the most important numerical characteristics of dynamical systems used in control theory. Properties of the first two characteristics are well described in the literature. Properties of the Perron exponents are much less investigated. In this paper we show an example of two-dimensional discrete-time linear system with bounded coefficients for which the set of Perron exponents constitutes an interval.

*Keywords:* Perron exponent, stability, time-varying systems, Cantor-type set.

## 1. INTRODUCTION

Consider the discrete-time linear system of the form

$$x(n+1) = A(n)x(n), \; x(0) = x_0, \; n \geq 0 \qquad (1)$$

where $A(n)$ are $s$-by-$s$ real matrices. For the system (1) the transition matrix is defined as

$$\Phi(m) = A(m-1)...A(0)$$

and $\Phi(0) = I$, where $I$ is the identity matrix. For an initial condition $x_0$ the solution of (1) is denoted by $x(n, x_0)$, so

$$x(n, x_0) = \Phi(n)x_0.$$

By $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^s$ and the induced operator norm. Many properties of dynamical system (1) can be successfully characterised by certain numerical characteristics. If we are interested in exponential stability, we may use Lyapunov exponents (Barreira and Pesin (2002)) defined as follows

$$\lambda(x_0) = \limsup_{n\to\infty} \|x(n, x_0)\|^{1/n}. \qquad (2)$$

The system (1) is exponentially stable if and only if for all $x_0 \in \mathbb{R}^s$ we have $\lambda(x_0) < 1$. Moreover, Lyapunov exponent describes an upper estimation of the norm of solution. More precisely, for all $\varepsilon > 0$ and $x_0 \in \mathbb{R}^s$ there exists $N > 0$ such that

$$\|x(n, x_0)\| \leq N\left(\lambda(x_0) + \varepsilon\right)^n \qquad (3)$$

for all $n \geq 0$. Properties of Lyapunov exponents have been investigated in depth (Arnold (1991), Czornik and Nawrat (2010), Czornik (2012)). These numerical characteristics are universally applied tool in control theory. If we want to achieve a lower estimation similar to (3), we may replace the upper limit in definition (2) by lower one. In this way we obtained a definition of Perron exponent

$$\pi(x_0) = \liminf_{n\to\infty} \|x(n, x_0)\|^{1/n},$$

that was firstly introduced for continuous-time systems in (Perron (1930)).

A very important property is that the number of Lyapunov exponents of (1) is finite, not greater than size $s$ (Barreira and Pesin (2002)). Moreover, the maximal one $\lambda_{\max}$ is defined in the following way

$$\lambda_{\max} = \limsup_{n\to\infty} \|\Phi(n)\|^{1/n}.$$

In spite of Perron exponents are defined similarly to Lyapunov exponents, they have very different properties, which by contrast with Lyapunov characteristics are not so thoroughly investigated. Some properties of Perron exponents for continuous-time systems have been established in (Bylov et al. (1966)) and (Izobov (1965)-Izobov (1968)). Perron exponents for discrete-time systems have been investigated in (Czornik (2008)) (see also Czornik (2012)). In (Czornik (2008)) it has been shown that for each natural numbers $l$ there exists two-dimensional system with exactly $l$ Perron exponents. This example establishes principal distinction between the structure of the sets of Lyapunov characteristic exponents and Perron exponents. Also in (Czornik (2008)) it has been shown that in the set $\{\pi(x_0) : x_0 \in \mathbb{R}^s\}$ there is a maximal element $\pi_{\max}$, the function $\pi : \mathbb{R}^s \to \mathbb{R}$ is almost every where, in the sense of Lebesgue measure, equal to $\pi_{\max}$ and

$$\pi_{\max} = \liminf_{n\to\infty} \|\Phi(n)\|^{1/n}.$$

The two-dimensional system system constructed in (Czornik (2008)) has unbounded coefficients. The main contribution of the present paper is to give an example of a two-dimensional system with bounded coefficients for which the set of Perron exponents constitutes an interval. Such an example in continuous-time has been given in (Izobov (2006)).

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## 2. PRELIMINARY CONSTRUCTIONS. CANTOR-TYPE SET AND CANTOR-TYPE FUNCTION

Let

$$\varepsilon_n = \exp\left(-\exp\left(\sum_{i=1}^{n} 2^i\right)\right).$$

For this sequence we define a Cantor-type set and denote it by $P_0$. Take closed interval $\Delta = [0, 1]$. Let $C_1$ be the set consisting of two disjoint closed subintervals of $\Delta$ of length $\varepsilon_1$, the left one $\Delta_1^{(1)}$ (for which the left endpoint coincides with the left endpoint of $\Delta$) and the right one $\Delta_1^{(2)}$(for which the right endpoint coincides with the right endpoint of $\Delta$). Now continue recursively, if $J \in C_n$, then include in the set $C_{n+1}$ its left-and-right-closed subintervals of length $\varepsilon_{n+1}$. Denote $\Delta_n^{(m)}$, $m = 1, ..., 2^n$ the elements of $C_n$ and by $\alpha_n^{(m)}$ their middle points. Moreover $\alpha_n^{(0)} := \alpha_n^{(2^n-1)}$ for $n > 1$ and $\alpha_1^{(0)} = 0$. We define the set $P_0$ as follows

$$P_0 = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} \Delta_n^{(i)}.$$

On the closed interval $\Delta = [0, 1]$ we have shown the sets $C_1 = \left\{\Delta_1^{(1)}, \Delta_1^{(2)}\right\}$, $C_2 = \left\{\Delta_2^{(1)}, \Delta_2^{(2)}, \Delta_2^{(3)}, \Delta_2^{(4)}\right\}$ with middle points of their elements in the Figure (1).

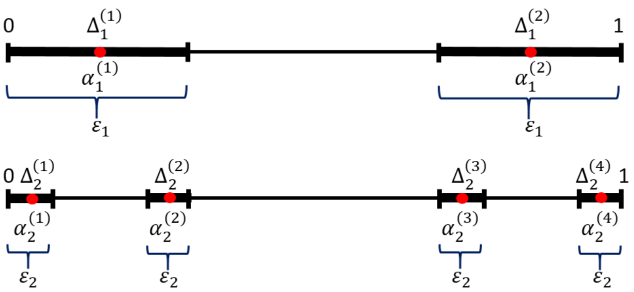


Fig. 1. Sets $C_1$, $C_2$ with middle points of their elements.

Now we construct a Cantor-type function corresponding to the Cantor-type set defined above. Define the function $f_1 : \Delta \to \Delta$ so that it has values:

- 0 at the left endpoint of the interval $\Delta$,
- 1 at the right endpoint of the interval $\Delta$,
- 1/2 on $\Delta \backslash C_1$ and interpolate linearly on the intervals in $C_1$.

Recursively, construct the function $f_{n+1}$ so that:

(1) for every interval $J = [s, t] \in C_n$, the function $f_{n+1}$ agrees with $f_n$ at $s$ and $t$,
(2) it has value equal to $[f_n(s) + f_n(t)]/2$ on $J \backslash C_{n+1}$ and interpolate linearly on the intervals in $C_{n+1}$.

The functions $f_1$ and $f_2$ are shown in the Figure (2).

The sequence of continuous functions $f_n$ converges uniformly on $\Delta$. We define Cantor-type function $\Phi$ as the limit. For any $n$ and all $n \leq m$ the functions $\Phi$ and $f_n$ agree at the endpoints of intervals $J \in C_m$.

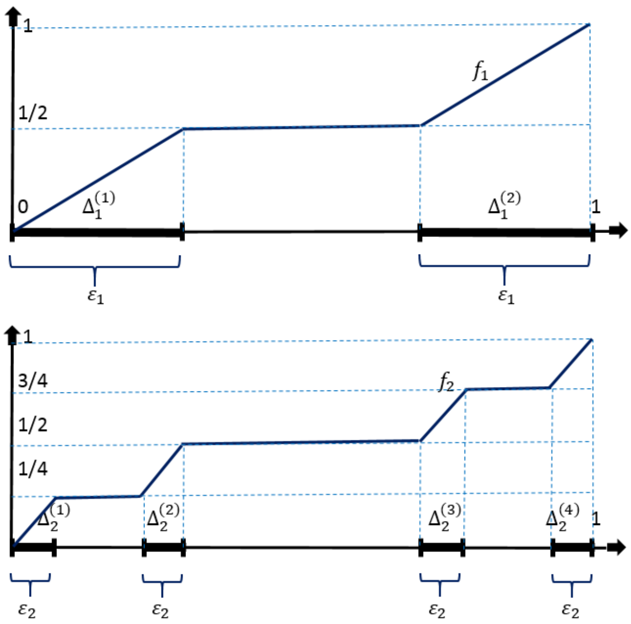


Fig. 2. Functions $f_1$ and $f_2$.

## 3. MAIN RESULT

In this section we will construct an example of two-dimensional system with bounded coefficient for which the set $\{\pi(x_0) : x_0 \in \mathbb{R}^s\}$ is an interval.

Let start the construction with definition of two auxiliary sequences $f(n)$ and $F(n)$. Let divide the halfline $[0, \infty)$ by the points $T_k = e^k$, $k \geq 0$ into left-closed intervals $s_n^{(m)}$ in the following way (presented also in the Figure (3)):

- for any natural $n \geq 1$ values of $m$ varies from 1 to $2^n$,
- the right endpoint $s_n^{(m)}$ coincides with left endpoint of $s_n^{(m+1)}$ (for $m = 2^n$ the right endpoint $s_n^{(2^n)}$ coincides with left endpoint of the next interval $s_{n+1}^{(1)}$).

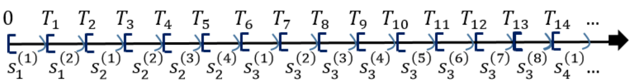


Fig. 3. Halfline $[0, \infty)$ divided by the points $T_k = e^k$, $k \geq 0$ into left-closed intervals $s_n^{(m)}$.

Set

$$F(1) = 0,\ F(k) = \alpha_n^{(m)} \text{ if } k \in s_n^{(m)},\ k \geq 2$$

and

$$f(k) = \begin{cases} 0 \text{ if } k \in s_n^{(m)} \backslash \widetilde{s}_n^{(m)} \\ -\Phi\left(\alpha_n^{(m)}\right) \text{ if } k \in \widetilde{s}_n^{(m)} \end{cases},$$

where $\widetilde{s}_n^{(m)}$ is the open righthalf of the interval $s_n^{(m)}$. Define the matrices $A(n)$ for system (1) in the following way

$$A(n) = \begin{bmatrix} a(n) & 0 \\ b(n) & 1 \end{bmatrix},\ n \geq 0$$

where

$$a(0) = 1,\ a(n) = \exp\left((n+1) f(n+1) - n f(n)\right), n \geq 1$$
$$b(0) = 0,\ b(n) = \left(F(n+1) - F(n)\right) \exp\left(-n f(n)\right),\ n \geq 1.$$

From the definition of $f(n)$ and $F(n)$ it is clear that $A(n)$ is a bounded sequence. It is easy to check, that solution

$x(n,x_0) = [\,x_1(n)\ x_2(n)\,]^T$ of (1) with initial condition $x_0 = [\,x_{01}\ x_{02}\,]^T$ is given by

$$x_1(n) = \exp(nf(n))\,x_{01},$$
$$x_2(n) = F(n)x_{01} + x_{02}. \qquad (4)$$

We will calculate Perron exponent $p(\alpha)$ for $x_0 = [\,1\ -\alpha\,]^T$ for $\alpha \in P_0, \alpha \neq 0$. We have

$$p(\alpha) = \liminf_{n\to\infty}\left[\exp(2nf(n)) + (F(n)-\alpha)^2\right]^{\frac{1}{2n}} \qquad (5)$$

According to the definition of $P_0$ for all $\alpha \in P_0, \alpha \neq 0$ and $n \geq 1$ there exists $m_n(\alpha) \leq 2^n$ such that

$$\left|\alpha_n^{(m_n(\alpha))} - \alpha\right| \leq \frac{\varepsilon_n}{2}. \qquad (6)$$

Denote by $\tau_n$ the integer part of the number in the middle of $\widetilde{s}_n^{(m_n(\alpha))}$. With this notation we have $\tau_n \to \infty$ and

$$\exp(-\Phi(\gamma)\,\tau_n) > \varepsilon_n,\ \gamma \in \Delta. \qquad (7)$$

From the last two inequalities we obtain

$$2\ln p(\alpha) \leq \liminf_{n\to\infty}\frac{1}{\tau_n}\ln\left(2\exp\left(-2\Phi\left(\alpha_n^{(m_n(\alpha))}\right)\tau_n\right)\right).$$

The last limit is equal to $-2\Phi(\alpha)$ since $\Phi$ is continuous. Therefore

$$\ln p(\alpha) \leq -\Phi(\alpha) < 0.$$

Now we show the opposite inequality. Let $n_k$ be the sequence for which

$$p(\alpha) = \lim_{k\to\infty}\left[\exp(2n_kf(n_k)) + (F(n_k)-\alpha)^2\right]^{\frac{1}{2n_k}}.$$

Because $\ln p(\alpha) < 0$, then

$$\limsup_{k\to\infty}\frac{1}{n_k}\ln|F(n_k)-\alpha| < 0 \qquad (8)$$

and

$$\limsup_{k\to\infty} f(n_k) < 0. \qquad (9)$$

From the inequality (8) we have

$$\lim_{k\to\infty} F(n_k) = \alpha,$$

which together with (9) implies that $F(n_k) = \alpha_{n_k}^{(m_k)}$ for large enough $k$ and therefore

$$\lim_{k\to\infty}\alpha_{n_k}^{(m_k)} = \alpha.$$

Finally

$$\ln p(\alpha) \geq \liminf_{k\to\infty} f(n_k) \geq \liminf_{k\to\infty}\left(-\Phi\left(\alpha_{n_k}^{(m_k)}\right)\right) = -\Phi(\alpha)$$

and we have proved, that

$$\left\{\pi(x_0) : x_0 = [\,1\ -\alpha\,]^T, \alpha \in P_0, \alpha \neq 0\right\} = \left(e^{-1}, 1\right).$$

## 4. CONCLUSION

In this paper we have shown that the set of Perron exponents of two-dimensional discrete linear systems with bounded coefficients may be an interval. The constructed example strongly motivates investigation of the range structure for the Perron exponents of the system (1).

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