Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

On the source-channel coding tradeoff in networked control

Mikael Johansson *

* *School of Electrical Engineering, KTH, Stockholm, Sweden (Tel: +46-8-7907436; e-mail: mikaelj@ee.kth.se).*

Abstract: This short note considers the source-channel coding tradeoff in networked control systems. Specifically, we consider how to optimally allocate a fixed number of bits to source and channel coding in order to minimize the estimation error variance of a Kalman filter operating over a binary erasure channel. We develop an analytical model for the system performance, quantify the source-channel coding tradeoff and compare the optimal performance with what can be achieved with the optimal performance of a collocated estimator using both analytical studies and extensive monte carlo simulations.

1. INTRODUCTION

Networked control systems present a multitude of design challenges, ranging from the physical transmission schemes and networking aspects all the way to application-level algorithm design. While a lot of research has been devoted to each layer in separation (e.g. developing coding techniques for increasing link-layer reliability, routing protocols with bounded delay and jitter, etc.) the work on cross-layer design in relatively recent and is still only covering a small part of the design spectrum. A remarkable exception is the work on coding for control (see *e.g.* Nair et al. [2007], Bao [2006]). In this field, a central problem has been the one of the joint design of source coding (quantization) schemes and associated feedback control laws to ensure stability under minimal information rate. Our focus is different.

We consider the joint optimization of source and channel coding for state estimation, assuming that a total budget of B bits can be allocated to source and channel coding at each sampling instant. The transmitted data is subject to independent bit errors and our objective is to minimize the estimation error variance of a Kalman filter operating on the lossy data stream. It is clear that there is a trade-off: allocating too many bits to source coding increases the risk for packet losses while using too much channel coding demands coarse quantization and introduces significant distortion. However, quantifying this trade-off and optimally designing the associated bit allocation appears to be an unaddressed problem so far.

2. MODEL AND PROBLEM FORMULATION

We consider the networked estimation set-up in Figure 1. Observations of a linear dynamic system are encoded and transmitted over unreliable communication channel. The received data is decoded and fed into an estimator that attempts to reconstruct the process state. Specifically, we consider a linear discrete-time process

$$x_{t+1} = Ax_t + Bw_t$$
$$y_t = Cx_t + v_t$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output vector, while $w \in \mathbb{R}^m$ and $v \in \mathbb{R}^p$ are the state and output disturbances, respectively. The disturbances are supposed to be Gaussian, uncorrelated, white, with

$$\mathbf{E}\{w\} = 0, \qquad \mathbf{E}\{ww^T\} = Q$$
$$\mathbf{E}\{v\} = 0, \qquad \mathbf{E}\{vv^T\} = R$$

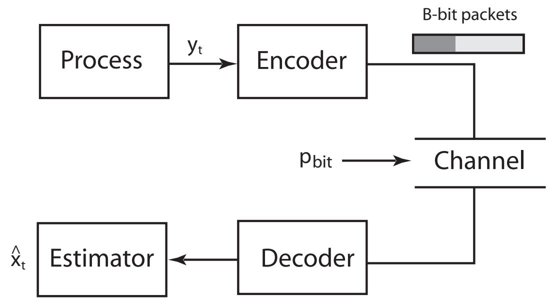

Fig. 1. Networked estimation setup.

The observations y_t are encoded and transmitted over a binary erasure channel with bit-loss probability p_{bit}. The communication rate is assumed to be B bits/sample.

The problem that we try to address in this paper is the following: given B bits per sample, how should these be allocated to source and channel coding in order to achieve a minimal estimation error variance on the receiver side.

In this first attempt, our approach is practical rather than fundamental: we separate source and channel coding and optimize over given families of source and channel coding schemes. One should bear in mind that the celebrated source-channel separation theorem (Cover and Thomas [1991]) assumes infinitely long block codes and does not hold in general for delay-constrained channels (see, *e.g.*, Bao [2006]). Moreover, quantizers engineered specifically for control-theoretic objectives are typically not uniform (Delchamps [1990], Bao [2006]). We are thus also interested in quantifying the difference between the best design within this family and the limits of achievable performance.

978-1-1234-7890-2/08/$20.00 © 2008 IFAC

10.3182/20080706-5-KR-1001.3943

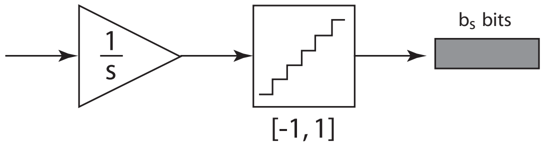

Fig. 2. Source coding: scaling and uniform unit quantizer.

3. SOURCE CODING

Within the control community, source-coding research has focused on lossy coding and quantization, see Delchamps [1990], Nair and Evans [1998], Tatikonda [2000], Picasso et al. [2002]. We follow the developments in Xiao et al. [2003] and assume uniform quantization. Although this is not optimal, it is conventional, easily implementable and leads to a simple model of how the system performance depends on the number of source coding bits.

The unit range uniform b-bit quantizer partitions the range $[-1, 1]$ into 2^b intervals of uniform width 2^{1-b}. Each quantization interval is assigned a codeword of b bits. Given one of these codewords, the numerical signal is reconstructed by taking the midpoint of the corresponding quantization interval. As long as the quantizer does not overflow, the error between the original and reconstructed signal lies in the interval $[-2^b, 2^b]$.

The behavior of the quantizer for inputs z with $|z| > 1$ is not specified. To avoid overflow, the signal is scaled by a factor s^{-1} prior to encoding, and re-scaled by a factor s after decoding. Under this scaling, the error between original and reconstructed signal lies in the interval $[-s2^{-b}, s2^{-b}]$. To minimize the quantization error while ensuring no overflow (or that overflow is rare), the scale factors s should be chosen as the maximum possible value of the original signal, or as a value that with very high probability is higher than the magnitude of the signal. As in Xiao et al. [2003], we will use the 3σ-rule and let $s_i = 3\mathbf{rms}(y)$. If y_i is Gaussian, this scaling ensures that overflow occurs only about 0.3% of the time. The structure of our solution is illustrated in Figure 2.

Assuming that overflow is rare, we model the quantization errors as independent random variables, uniformly distributed on the interval $s[-2^b, 2^b]$ (cf. Franklin et al. [1990]). In other words, we model the effect of quantizing y as an additive white noise source q such that

$$y_t^{(r)} = y_t + q_t$$

where $\mathbf{E}\{q\} = 0$ and $\mathbf{Var}\{q\} = (1/3)s^2 2^{-2b}$. See Widrow et al. [1996] for further details about the statistical properties of the quantization error.

4. CHANNEL CODING

When the quantized sensor data is sent over the communication channel, there will always be a risk for transmission errors. The information can be protected by adding carefully selected redundancy to the packets. This process is knows as channel coding, see Figure 3.

In our model, we assume that sensor samples are sent over an time-invariant independent binary erasure channel with bit-loss probability p_{bit}. If all B message bits are used for source coding, the corresponding packet loss probability

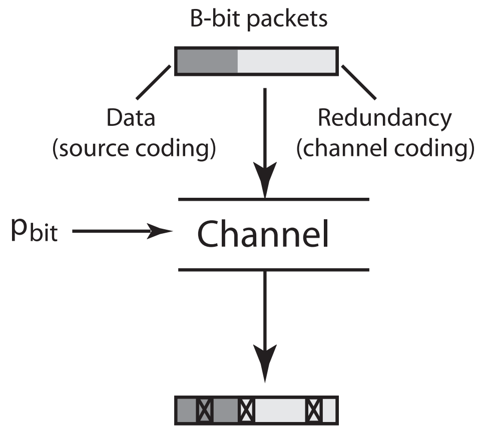

Fig. 3. Channel coding: The packet contains $B - b_s$ channel coding bits. If the received packet has fewer than $B - b_s$ bit errors, the original source code word can be recovered.

is $1 - (1 - p_{\text{bit}})^B$ (which, for small p_{bit} is roughly equal to Bp_{bit}). This packet loss probability can be efficiently reduced using channel coding.

One class of powerful block codes are the Reed-Solomon codes. If b_c bits are used for channel coding, the packet can be recovered if any $B - b_c$ bits are received correctly. The packet loss probability is thus

$$p_{\text{pkt}}(b_c) = 1 - \sum_{i=0}^{b_c} \binom{B}{i} p_{\text{bit}}^i (1 - p_{\text{bit}})^{B-i} \qquad (1)$$

5. OPTIMAL ESTIMATION UNDER PACKET LOSS

Optimal estimation under packet loss has been investigated recently in Sinopoli et al. [2004]. In our setting, the model of the system evolution and the observations available at the receiver side takes the form

$$x_{t+1} = Ax_t + w_t$$
$$y_t^{(r)} = \gamma_t(Cx_t + v_t + q_t)$$

The stochastic variable γ_t models the packet loss process: when a sensor packet is received correctly, $\gamma_t = 1$, while $\gamma_t = 0$ corresponds to packet loss. According to our channel model γ_t is an i.i.d. Bernoulli random variable with $\mathbf{Prob}\{\gamma_t = 0\} = p_{\text{pkt}}$. The optimal estimator can be derived using similar techniques as the standard Kalman filter, and performs an innovation step

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t}$$
$$P_{t+1|t} = AP_{t|t}A^T + Q$$

at each sample, while the correction step

$$K_{t+1} = P_{t+1|t}C^T(CP_{t+1|t}C^T + \tilde{R})^{-1}$$
$$\hat{x}_{t|t} = \hat{x}_{t+1|t} + \gamma_{t+1}K_{k+1}(y_{t+1} - C\hat{x}_{t+1|t})$$
$$P_{t+1|t+1} = P_{t+1|t} - \gamma_{t+1}K_{t+1}CP_{t+1|t}$$

is only executed when new data is received ($\gamma_{t+1} = 1$).

Although this scheme for dealing with packet losses has a long history, the first comprehensive analysis of its properties appears to be the work by Sinopoli et al. [2004]. The authors establish the existence of a critical packet loss probability $p_{\text{pkt}}^{\text{crit}}$: the estimator covariance matrix converges in mean if $p_{\text{pkt}} < p_{\text{pkt}}^{\text{crit}}$ and diverges otherwise.

Further, they prove that when the loss probability is below the critical value, the expected value of P_t can be bounded,

$$0 < S_t \leq \mathbf{E}\{P_t\} \leq V_t$$

where $\lim_{t\to\infty} S_t = \overline{S}$ solves the algebraic Riccati equation

$$\overline{S} = p_{\text{pkt}} A\overline{S}A^T + Q$$

while $\lim_{t\to\infty} V_t = \overline{V}$ can be found as the solution to the modified algebraic Riccati equation

$$\overline{V} = A\overline{V}A^T + Q - (1 - p_{\text{pkt}})A\overline{V}C^T(C\overline{V}C^T + R)^{-1}C\overline{V}A^T$$

6. THE SOURCE-CHANNEL CODING TRADEOFF

In our problem there is a clear trade-off between source and channel coding. Allocating too few bits to source coding induces a large quantization noise, while allocating too many (and hence, too few to channel coding) results in a high packet loss rate. We will now combine the results from §3–§5 to quantify the source-channel coding tradeoff. Specifically, we consider a stable scalar process,

$$x_{t+1} = ax_t + w_t$$
$$y_t = x_t + v_t$$

with $\mathbf{E}\{w^2\} = 1$ and $\mathbf{E}\{v^2\} = r$. For this system, it holds that $\mathbf{E}\{y^2\} = r + (1 - a^2)^{-1}$, so using the 3σ-rule for scalings and b_s bits for channel coding induces an additive quantization noise q with variance

$$\mathbf{E}\{q^2\} = 3(r + (1 - a^2)^{-1})2^{-2b_s} \quad (2)$$

and the associated packet loss is given by $p_{\text{pkt}}(B - b_s)$. The numerical examples in this section use $a = 0.95, r = 0.01$ and $B = 16$.

The first investigation we perform is to study how the optimal allocation of bits to source and channel coding depends on the loss probability p_{bit}. The optimal allocation is done by exhaustive search: for each value of $b_s \in [0, B]$, we determine the effective variance of the quantization noise q via (2) and the packet loss probability $p_{\text{pkt}}(B - b_s)$ using (1). We then solve the modified algebraic Riccati equation for $\overline{V}$ and call the bit allocation that achieves the smallest value of $\overline{V}$ the optimal one. The results are shown in Figure 4. The results are intuitive: when there are no packet losses, all bits should be allocated to source coding. As the loss probability increases, more and more bits should be allocated to channel coding.

Fixing the bit loss probability to (the arguably high value of) $p_{\text{bit}} = 0.1$, Figure 5 shows the estimator performance as a function of the number of source coding bits. Although there is indeed a trade-off, it does not appear to be very important: as long as some bits are allocated to source and channel coding, the performance curve is very flat and changes in bit allocation does not change the actual performance very much. The results remain qualitatively the same when the bit loss probability is varied.

Although our coding scheme (the separation into source and channel codes, as well as the specific codes) in suboptimal, it is interesting to note that the performance comes very close to the optimal performance of a sensor collated with the process using full precision measurements; see Figure 5 as soon as a handful of bits are allocated to both source and channel code. Thus, in this specific example there appears to be little room for improvement by using more advanced coding schemes (including jointly designed, non-uniform and dynamic source-channel codes).

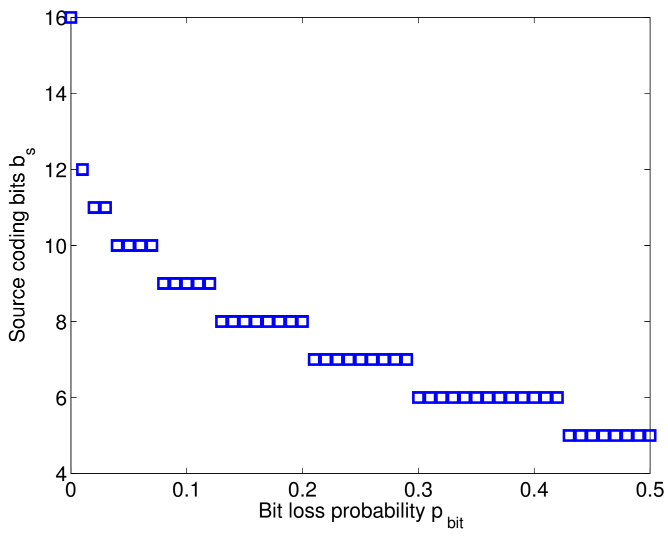

Fig. 4. Optimal allocation of bits to source coding as function of bit loss probability.

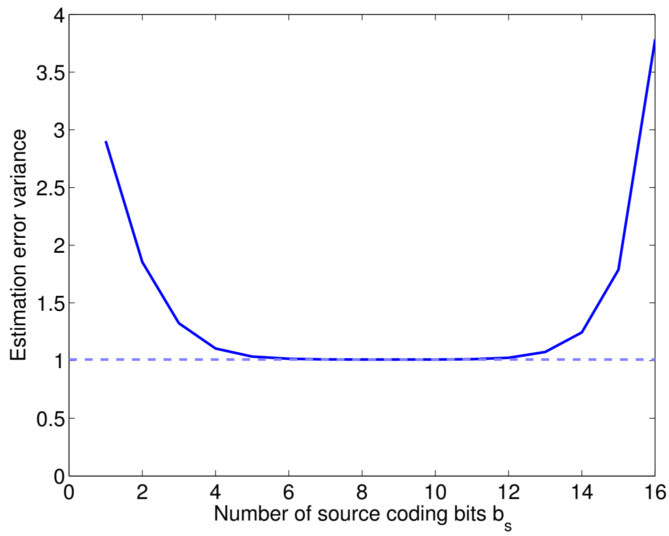

Fig. 5. Estimation error vs. number of source coding bits for $p_{\text{bit}} = 0.1$. The dashed line shows the optimal performance obtained by a Kalman filter collocated with the process that uses full-precision observations.

The observation is further strengthened by the results of a Monte Carlo simulation of the complete encoder-decoder-estimator scheme shown in Figure 6 (i.e. simulations with actual quantization and reconstruction rather than the white noise approximation, Bernoulli bit errors and the actual time-varying Kalman filter working on the lossy data stream). The 90% confidence intervals shown are the results of 100 simulations with random initial state on the process, and different noise realizations. One can clearly see that the approximate model used for the theoretical analysis is highly accurate, as soon as a couple of bits are allocated to the source coding.

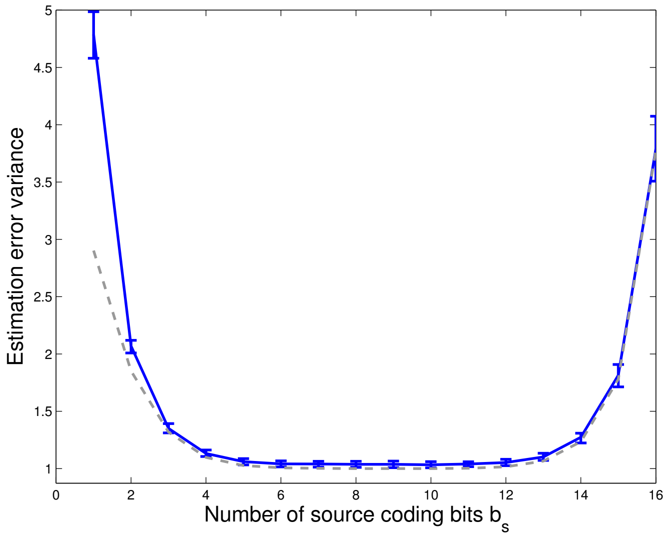

Fig. 6. Estimation error vs. number of source coding bits for $p_{\text{bit}} = 0.1$. The dashed line shows the theoretically derived source-channel coding tradeoff curve, while the full line shows 90% confidence bounds for the actual performance estimated from 100 Monte Carlo simulations.

7. CONCLUSIONS

In this short note, we have considered the joint optimization of source and channel coding for state estimation. We have demonstrated that there is indeed a trade-off: allocating too many bits to source coding increases the risk for packet losses while using too much channel coding demands coarse quantization and introduces significant distortion. However, somewhat disappointingly, we noticed that the trade-off curve is very flat once a handful bits is allocated to each code, indicating that the trade-off is perhaps of little relevance. Comparing our restricted design, which optimizes over fixed families of source and channel codes, we further noted that the performance can come very close to the optimal that can be achieved by a collocated estimator using full precision measurements. Thus, there appears to be little room for improvements using more advanced source and channel coding schemes.

This paper has only considered the case of stable linear systems. Although the estimation problem becomes more interesting if the process is unstable, the source coding problem becomes non-trivial as the observed variable grows unboundedly. Moreover, a more realistic channel model would incorporate correlated losses. It is well-known that forward error correcting codes becomes less efficient in this case, indicating that our trade-off curves would be shifted. We leave these issues to our future work.

Acknowledgements This work is supported in part by The Swedish Agency for Innovation Systems through the Nordite WiSA project, the Swedish Research Council and the European Commission. The paper is based on an oral presentation given at the Hycon-Newcom Workshop on Embedded Systems and Infrastructureless Networks.

REFERENCES

L. Bao. *Source-Channel Coding for Closed-Loop Control*. Licentiate thesis, School of Electrical Engineering, Royal Institute of Technology, Stockholm, Sweden, 2006.

T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley & Sons, Inc., 1991.

D. F. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Transactions on Automatic Control*, 51(2):916–924, February 1990.

G. F. Franklin, J. D. Powell, and M. L. Workman. *Digital Control of Dynamic Systems*. Addision-Wesley, 3rd edition, 1990.

G. N. Nair and R. J. Evans. State estimation under bit-rate constraints. In *Proceedings of the 37th IEEE Conference on Decision and Control*, Tampa, FL, December 1998.

G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans. Feedback control under rate constraints: an overview. *Proceedings of the IEEE*, 95(1):108–137, 2007.

B. Picasso, F. Gouaisbaut, and A. Bicchi. Construction of invariant and attractive sets for quantized-input linear systems. In *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, NV, December 2002.

B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry. Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Conrtol*, 49(9):1453–1464, September 2004.

S. Tatikonda. *Control under communication constraints*. Phd thesis, Massachusetts Institute of Technology, Cambridge, MA, 2000.

B. Widrow, I. Kóllar, and M.-L. Liu. Statistical theory of quantization. *IEEE Transactions on Instrumentation and Measurements*, 45:353–361, April 1996.

L. Xiao, M. Johansson, H. Hindi, S. Boyd, and G. Goldsmith. Joint optimization of communication rates and linear systems. *IEEE Transactions on Automatic Control*, 48(1):148–153, January 2003.