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Analysis on Behaviors of Controlled Quantum Systems via Quantum Entropy

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Abstract: In this paper, we investigate the essential properties of finite dimensional measurement-based quantum feedback control systems using a kind of quantum entropy, or so-called linear entropy. We show how the terms appear in the stochastic master equation affect the purity of the conditional density matrix of the system, and clarify a limitation of control action via Hamiltonian. Moreover, applying the stochastic version of LaSalle's invariance theorem, we derive a sufficient condition under which the conditional density matrix converges in probability to the set of all pure states for any control input. The result shows a class of measurement which assures preparation of a pure state.

Keywords: Quantum systems; Stochastic differential equation; Control system analysis; Entropy

1. INTRODUCTION

Even after many years from the establishment of quantum mechanics, investigation on interesting phenomena in microscopic scale is still active. Moreover, control of quantum systems becomes one of important research topics in engineering. Recent rapid miniaturization of electronic devices motivates such research activity, since quantum mechanical effects cannot be ignored in these cases. Another motivation is the theoretical development of quantum technologies, such as quantum computation, quantum communication or precision metrology using quantum systems, which achieves high performances beyond the limits of existing technologies. In recent years, researchers in various fields such as physics, control theory, or mathematics collaborate on control of quantum systems [Mabuchi and Khaneja, 2005].

In general, feedback control is expected to attain robustness for noise or modeling error, and quantum control using continuous measurement; so-called *measurement-based quantum feedback control*, was proposed in 80's to early 90's [Belavkin, 1987, Wiseman, 1994]. Afterward, the quantum feedback control has been intensively investigated and its effectiveness has been also demonstrated by experiments [Geremia et al., 2004].

Recently, control theoretic approach to the quantum feedback control has achieved great success. For examples, control laws for a specific class of quantum systems (spin systems) attaining global asymptotic stabilization of eigenstates have been proposed by employing techniques of stochastic control theory [Mirrahimi and van Handel, 2005, Tsumura, 2006, 2007]. These results have significant importance because spin systems are expected to realize quantum technologies. On the other hand, it is also important to investigate fundamental properties of the quantum feedback control in general settings. Such investigations are naturally expected to serve for control law design in various situations. In this paper, we analyze behaviors of finite dimensional measurement-based quantum feedback control systems using the *linear entropy* (a kind of quantum entropy) as an index to characterize quantum states, and show the essential properties of the quantum feedback control. Furthermore, a condition for generating a pure state is derived using the stochastic version of LaSalle's invariance theorem.

This paper is organized as follows. In Section 2 we explain the basic idea of measurement-based quantum feedback control, and introduce a stochastic differential equation describing the systems (*stochastic master equation* (*SME*)). Section 3 is the main part of this paper. In Section 3.1 we formulate the problems and introduce the linear entropy. Section 3.2 shows the effects of each term in the SME with respect to the linear entropy. Section 3.3 is devoted to derive a condition for generating a pure state. In Section 4, we clarify a feature of the theorem derived in Section 3.3 by comparing it with related results, which is confirmed by numerical examples. We summarize the paper in Section 5.

Notation: i: imaginary unit. $\mathbb{R}$: set of all real numbers. $\mathbf{H}_n$: set of all $n \times n$ Hermitian matrices. X^*: Hermitian conjugate of a complex matrix X. $\mathrm{Tr}X$: trace of a complex matrix X. $\|X\|_2 (:= (\mathrm{Tr}\,[X^*X])^{1/2})$: Hilbert-Schmidt norm (Frobenius norm) of a complex matrix X. $[X,Y](:= XY - YX)$: commutator of complex matrices X and Y.

2. MEASUREMENT-BASED QUANTUM FEEDBACK CONTROL AND STOCHASTIC MASTER EQUATION

In this section, we explain the basic idea of measurement-based quantum feedback control and introduce the stochastic master equation (SME) which describes the control system. Among several ways to derive the SME [Bouten et al., 2004], our discussions are based on *quantum filtering* *theory* pioneered by Belavkin [1987], which is most natural from control theoretic viewpoints. Note that this paper only treats finite dimensional quantum systems.

Measurement for a microscopic scale system cannot be performed without probabilistic back-action. In general, the alteration of the system caused by the measurement is too drastic and instantaneous, and it prevents the implementation of feedback control. A possible way to avoid this difficulty is measuring the target system *indirectly* and *in continuous time* so that the back-action is suppressed to an allowable level and real-time (partial) information of the system is derived. This is the essential idea of continuous measurement and realized by keeping the target system interacting with another system (called *probe system*) such as laser field and measuring the probe system.

This situation is analogous to that of partially observable classical stochastic systems. Hence, as in the classical case, filtering theory for quantum systems, i.e., *quantum filtering theory* gives a basis for feedback control of quantum systems under such situations. First, we introduce the following preliminary to explain the results of quantum filtering theory.

It is necessary to use a special mathematical framework to describe probabilistic phenomena in microscopic scale. In quantum mechanics (or in *quantum probability theory*), a probability distribution (probability vector) is replaced by a *density matrix* ρ which is positive semidefinite and unital-trace. Consequently, a conditional probability distribution (conditional probability vector) is replaced by a *conditional density matrix*. We denote the set of all $n \times n$ density matrices by $\mathbf{S}_n$, i.e.,

$$\mathbf{S}_n := \{\rho \in \mathbf{H}_n \mid \rho \geq 0, \ \mathrm{Tr}\rho = 1\}. \tag{1}$$

We also call ρ a *quantum state*. A quantum state which satisfies $\rho^2 = \rho$ is called a *pure state* [1]. $\mathbf{P}_n$ denotes the set of all n-dimensional pure states, i.e.,

$$\mathbf{P}_n := \{\rho \in \mathbf{S}_n | \rho^2 = \rho\}. \tag{2}$$

Now consider a case that an n-dimensional quantum system is a control target, and it is measured by homodyne detection which is one of the methods of continuous measurement. Let $(\Omega, \mathfrak{F}, P)$ be the underlying (classical) probability space and y_t be the measurement signal at time t. Quantum filtering theory shows that the conditional density matrix $\rho_t \in \mathbf{S}_n$ of the target quantum system obeys the following equation [Belavkin, 1987, van Handel et al., 2005a,b, Bouten and van Handel, 2006, Bouten et al., 2007]:

$$\begin{aligned} \mathrm{d}\rho_t = & -\mathrm{i}u(t)[H, \rho_t]\mathrm{d}t + \mathcal{D}[C]\rho_t \mathrm{d}t \\ & + \sqrt{\eta}\mathcal{H}[C]\rho_t(\mathrm{d}y_t - \sqrt{\eta}\mathrm{Tr}\left[(C + C^*)\rho_t\right]\mathrm{d}t). \end{aligned} \tag{3}$$

This is a quantum analogue of the Wonham filter (finite dimensional version of the Kushner-Stratonovich equation). Here super-operators $\mathcal{D}[C]$ and $\mathcal{H}[C]$ are defined as follows:

$$\begin{aligned} \mathcal{D}[C]\rho &:= C\rho C^* - \frac{1}{2}C^*C\rho - \frac{1}{2}\rho C^*C, \\ \mathcal{H}[C]\rho &:= C\rho + \rho C^* - \mathrm{Tr}\left[(C + C^*)\rho\right]\rho. \end{aligned} \tag{4}$$

The $n \times n$ complex matrix C is determined by the interaction between the target system and the probe system. The $n \times n$ Hermitian matrix H denotes the control Hamiltonian and $u(t) \in \mathbb{R}$ is the control input at time t. It is assumed that the control law satisfies a regularity condition [Bouten and van Handel, 2006] ensuring the solvability of the filtering problem. We denote the set of all control laws satisfying the regularity condition by $\mathcal{U}$. The parameter η $(0 < \eta \leq 1)$ represents the measurement efficiency at detector [Jacobs and Steck, 2007] and is called *detector efficiency*. The condition $\eta = 1$ corresponds to the measurement without loss (perfect measurement).

[1] Let $\{\lambda_i\}_{i=1}^n$ be the set of all eigenvalues of $\rho \in \mathbf{S}_n$. Note that the condition $\rho^2 = \rho$ is equivalent to the condition $\lambda_{\hat{i}} = 1$ for a unique index $\hat{i}$ and $\lambda_i = 0$ for $i \neq \hat{i}$.

The conditional density matrix ρ_t is calculated by the equation (3) with measured output y_t, and used to determine control input according to a feedback control law $u(t) = u(\rho_t)$. This is the basic idea of measurement-based quantum feedback control, which can be implemented (in principle) by a computer.

It is assumed in the remainder of the paper that we can appropriately set the initial value ρ_0. Then, stochastic properties of the innovations process

$$y_t - \sqrt{\eta} \int_0^t \mathrm{Tr}\left[(C + C^*)\rho_s\right]\mathrm{d}s \tag{5}$$

are those of the standard Wiener process [van Handel et al., 2005a, Bouten et al., 2007]. Thus, the equation (3) can be represented as the following matrix-valued Ito type nonlinear stochastic differential equation:

$$\mathrm{d}\rho_t = -\mathrm{i}u(t)[H, \rho_t]\mathrm{d}t + \mathcal{D}[C]\rho_t\mathrm{d}t + \sqrt{\eta}\mathcal{H}[C]\rho_t\mathrm{d}w_t, \tag{6}$$

where $\mathrm{d}w_t$ denotes the standard Wiener increment. This is known as *stochastic master equation (SME)* in physics.

A typical experimental setting described by the SME is a spin system in optical cavity measured by laser and actuated by magnetic field depicted in Fig. 1. A quantum dot system measured by quantum point contact is also described by the SME in [Goan et al., 2001].

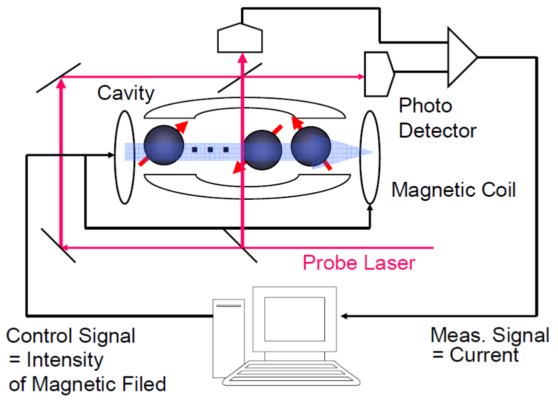

Fig. 1. A typical measurement-based quantum feedback control system.

Basic theoretical issues on the quantum feedback control are 1) analysis on the behaviors of the solution ρ_t of the SME, and 2) design of a feedback control law $u(t) = u(\rho_t)$ which makes ρ_t behave to be desired. This paper focuses on the analysis problem from a new perspective (refer [Yamamoto et al., 2005, Sasaki et al., 2006] for some discussion on the reachability of the SME).

3. ANALYSIS USING QUANTUM ENTROPY

In this section, we analyze the measurement-based feedback control systems using a kind of entropy as an index to evaluate its behavior.

3.1 Problem Setting

We here formulate the problems to be investigated and introduce a kind of quantum entropy as an evaluation index for the problems.

One of the objectives of quantum feedback control is preparation of particular quantum states which play important roles in applications, such as quantum information processing or precision metrology [van Handel et al., 2005b]. Although the target quantum states to be prepared are different in each application, they have a common feature in general. That is, *they should be pure states* [2]. From this viewpoint, a quantum feedback control system has to guarantee the preparation of a pure state at least. Hence, important questions are the following:

i) Do the terms of the right hand side of the SME (6) make ρ_t approach to or go away from $\mathbf{P}_n$ (the set of all pure states)?
ii) Does ρ_t converge to $\mathbf{P}_n$?

This paper treats these two questions in general settings, i.e., without imposing preconditions to n, C, H, $u(\in \mathcal{U})$ and η. Essential properties of measurement-based quantum feedback control are clarified through answering these questions.

We introduce a quantity defined by

$$S_L(\rho) := 1 - \mathrm{Tr}\left[\rho^2\right] = 1 - \|\rho\|_2^2, \tag{7}$$

which represents how pure a quantum state is to investigate the problems stated above. $S_L(\rho)$ is called *linear entropy* [Breuer and Petruccione, 2002] or impurity, and is equivalent to the Tsallis entropy of order 2 [Plastino and Plastino, 1993]. It is easy to see that the linear entropy satisfies $0 \leq S_L(\rho) \leq 1$, and $S_L(\rho) = 0$ holds if and only if ρ is a pure state. In addition, S_L is obviously continuous with respect to the Hilbert-Schmidt norm.

According to (6) and the Ito rule ($\mathrm{d}t^2 = 0, \mathrm{d}t\mathrm{d}w_t = 0, \mathrm{d}w_t^2 = \mathrm{d}t$), we can calculate the increment of $S_L(\rho_t)$ as follows:

$$\begin{aligned} \mathrm{d}S_L(\rho_t) =& - \mathrm{Tr}\left[\mathrm{d}\rho_t\rho_t\right] - \mathrm{Tr}\left[\rho_t\mathrm{d}\rho_t\right] - \mathrm{Tr}\left[\mathrm{d}\rho_t\mathrm{d}\rho_t\right] \\ =& - 2\mathrm{Tr}\left[(\mathcal{D}[C]\rho_t)\rho_t\right]\mathrm{d}t - \eta\mathrm{Tr}\left[(\mathcal{H}[C]\rho_t)^2\right]\mathrm{d}t \\ & - 2\sqrt{\eta}\mathrm{Tr}\left[(\mathcal{H}[C]\rho_t)\rho_t\right]\mathrm{d}w_t. \end{aligned} \tag{8}$$

We consider the questions i) and ii) based on this equation.

Remark . The von Neumann entropy expressed as

$$S(\rho) := -\mathrm{Tr}\left[\rho\log\rho\right] \tag{9}$$

is a quantum analogue of the Shannon entropy and is frequently used in quantum information theory. However, it is much easier to handle the increment of the linear entropy than to handle that of the von Neumann entropy. Thus we use the linear entropy in this paper.

3.2 Effects of Terms in SME

In this subsection, we show the effects of each term in the SME (6) by analyzing the equation (8) and answer the question i).

[2] This paper focus on the purity of target quantum states rather than another important feature; *entanglement*.

Since neither $u(t)$ nor H appear in the equation (8), the effect of the control input to the purity of the conditional density matrix is indirect, i.e.,

- Any control input cannot change the purity of the conditional density matrix directly.

According to the definition of the linear entropy, it can be transformed into the following geometrical description: the effect of control input at a point $\rho \in \mathbf{S}_n$ is restricted to the tangent hyperplane at the point ρ of the hypersphere $\{X \in \mathbf{H}_n \mid \|X\|_2 = \|\rho\|_2\}$. This result is quite natural or almost obvious because control input affects the target quantum system via Hamiltonian which causes unitary evolution. However, it is important to notice this limitation in the design process for a feedback control law.

The first term in (8) expresses the effect of $\mathcal{D}[C]\rho_t\mathrm{d}t$ to the purity of the conditional density matrix. We can apply the result of Lidar et al. [2006] to our case. That is, if C is a normal matrix, or C satisfies $CC^* = C^*C$, the following inequality holds:

$$-2\mathrm{Tr}\left[(\mathcal{D}[C]\rho)\rho\right] \geq 0, \text{ for all } \rho \in \mathbf{S}_n. \tag{10}$$

We can interpret this inequality as follows:

- If C is a normal matrix, $\mathcal{D}[C]\rho_t\mathrm{d}t$ always causes undesirable effects for the preparation of a pure state.

In many cases of measurement-based quantum feedback control, C is a Hermitian matrix and thus a normal matrix. Consequently, the statement above tells us the negative effect of $\mathcal{D}[C]\rho_t\mathrm{d}t$ in practical situations.

The second term in (8) represents the averaged effect of $\mathcal{H}[C]\rho_t\mathrm{d}w_t$ to the purity of the conditional density matrix. Since the super-operator $\mathcal{H}[C]$ maps a Hermitian matrix to a Hermitian matrix, $(\mathcal{H}[C]\rho)^2 = (\mathcal{H}[C]\rho)^*(\mathcal{H}[C]\rho)$ holds. Thus, we have the following relation:

$$-\eta\mathrm{Tr}\left[(\mathcal{H}[C]\rho)^2\right] = -\eta\|\mathcal{H}[C]\rho\|_2^2 \leq 0, \text{ for all } \rho \in \mathbf{S}_n. \tag{11}$$

This inequality implies the following:

- $\mathcal{H}[C]\rho_t\mathrm{d}w_t$ always yields desirable effects for the preparation of a pure state on average.

This is a fairly reasonable result because w_t is originally the innovation process (5) and thus $\mathcal{H}[C]\rho_t\mathrm{d}w_t$ is a term updating or correcting the conditional density matrix associated with the measured output.

The above investigations are summarized as follows.

Proposition 1. Any control input cannot change the purity of the conditional density matrix directly, and $\mathcal{H}[C]\rho_t\mathrm{d}w_t$ always yields desirable effects for the preparation of a pure state on average. Moreover, $\mathcal{D}[C]\rho_t\mathrm{d}t$ always causes undesirable effects for the preparation of a pure state if C is a normal matrix.

Note that these results were derived without imposing preconditions to n, C, H, u and η (except the regularity of u). Therefore, we can regard them as fundamental properties of measurement-based quantum feedback control systems.

3.3 Asymptotic Behavior of Conditional Density Matrix

This subsection focuses on the question ii), i.e., the asymptotic behavior of the conditional density matrix. First, we introduce the following definition of convergence.

Definition 1. Let $\mathbf{M}_n$ be a subset of $\mathbf{H}_n$. An $\mathbf{H}_n$-valued stochastic process $\{X_t\}_{t\in[0,\infty)}$ is said to converge in probability to $\mathbf{M}_n$ if

$$\lim_{t\to\infty} P\left(\{\omega\in\Omega \mid \inf_{Y\in\mathbf{M}_n} \|X_t(\omega)-Y\|_2\} \geq \epsilon\right) = 0 \quad (12)$$

holds for arbitrary small $\epsilon > 0$.

We can show the following result using the stochastic version of LaSalle's invariance theorem by Kushner [1967, 1968, 1972] (see also [Mirrahimi and van Handel, 2005]).

Theorem 2. Suppose η (detector efficiency) is equal to 1 and $C + C^*$ has distinct eigenvalues. Then the solution ρ_t of stochastic master equation (6) converges in probability to $\mathbf{P}_n$ (the set of all pure states) for any control law $u \in \mathcal{U}$.

We need the following lemma to prove the theorem. In what follows, $\mathscr{A}$ denotes the weak infinitesimal operator of ρ_t [Mirrahimi and van Handel, 2005].

Lemma 3. If η is equal to 1, the following relation holds for any n, C, H, and $u \in \mathcal{U}$:

$$\mathscr{A}S_L(\rho) \leq 0, \text{ for all } \rho \in \mathbf{S}_n. \quad (13)$$

Proof . When η is equal to 1, we can see from (8) that

$$\mathscr{A}S_L(\rho) = -2\mathrm{Tr}\left[(\mathcal{D}[C]\rho)\rho\right] - \mathrm{Tr}\left[(\mathcal{H}[C]\rho)^2\right] \quad (14)$$

holds. By substituting definitions of $\mathcal{D}[C]\rho$ and $\mathcal{H}[C]\rho$ into (14), we have

$$\mathscr{A}S_L(\rho) = -K(\rho), \quad (15)$$

where

$$\begin{aligned} K(\rho) :=& \mathrm{Tr}\left[(C+C^*)\rho(C+C^*)\rho\right] \\ &- 2\mathrm{Tr}\left[(C+C^*)\rho\right]\mathrm{Tr}\left[(C+C^*)\rho^2\right] \\ &+ (\mathrm{Tr}\left[(C+C^*)\rho\right])^2\mathrm{Tr}\left[\rho^2\right]. \end{aligned} \quad (16)$$

Setting $A := C + C^*$, $K(\rho)$ can be described as follows:

$$\begin{aligned} K(\rho) =& \mathrm{Tr}\left[\rho^2\right]\left[\mathrm{Tr}\left[A\rho\right] - \frac{\mathrm{Tr}\left[A\rho^2\right]}{\mathrm{Tr}\left[\rho^2\right]}\right]^2 \\ &+ \frac{1}{\mathrm{Tr}\left[\rho^2\right]}\left\{\mathrm{Tr}\left[A\rho A\rho\right]\mathrm{Tr}\left[\rho^2\right] - \left(\mathrm{Tr}\left[A\rho^2\right]\right)^2\right\}. \end{aligned} \quad (17)$$

Let $B = (B_{ij}) := U^*AU$, where U is a unitary matrix which diagonalizes ρ. In addition, let $\{\lambda_i\}_{i=1}^n$ be the set of all eigenvalues of ρ. Note that λ_i is nonnegative for all i, since ρ is positive semidefinite. We get the following:

$$\begin{aligned} &\mathrm{Tr}\left[A\rho A\rho\right]\mathrm{Tr}\left[\rho^2\right] - \left(\mathrm{Tr}\left[A\rho^2\right]\right)^2 \\ &= \mathrm{Tr}\left[U^*AUU^*\rho UU^*AUU^*\rho U\right]\mathrm{Tr}\left[\rho^2\right] \\ &- \left(\mathrm{Tr}\left[U^*AUU^*\rho^2 U\right]\right)^2 \\ &= \sum_{i,j}\lambda_i\lambda_j B_{ij}B_{ji}\sum_k {\lambda_k}^2 - \sum_{i,j}{\lambda_i}^2{\lambda_j}^2 B_{ii}B_{jj} \\ &\geq \sum_{i,j}{\lambda_i}^2{\lambda_j}^2 {B_{ii}}^2 - \sum_{i,j}{\lambda_i}^2{\lambda_j}^2 B_{ii}B_{jj} \\ &= \sum_{i<j}{\lambda_i}^2{\lambda_j}^2(B_{ii}-B_{jj})^2 \geq 0. \end{aligned} \quad (18)$$

Note that B is a Hermitian matrix, and hence $B_{ij} = \bar{B}_{ji}$ holds ($\bar{B}_{ji}$ is the complex conjugate of B_{ji}). The first inequality is obvious, because we just drop nonnegative terms $\lambda_i\lambda_j B_{ij}B_{ji} = \lambda_i\lambda_j|B_{ij}|^2$ $(i \neq j)$.

The first term of (17) is obviously nonnegative, and thus $K(\rho) \geq 0$ and $\mathscr{A}S_L(\rho) = -K(\rho) \leq 0$ holds for any $\rho \in \mathbf{S}_n$. It should be noted that the above discussion does not depend on n, C, H, nor u. □

This lemma shows that if the measurement is perfect, or $\eta = 1$, the entropy of ρ_t does not increase on average for any n, C, H and $u \in \mathcal{U}$.

We are now ready to prove the theorem.

Proof of Theorem 2 . Let $\mathbf{U}_n$ be the set of all $\rho \in \mathbf{S}_n$ which satisfies $\mathscr{A}S_L(\rho) = 0$. As addressed in Lemma 3, $\mathscr{A}S_L(\rho) \leq 0$ holds in $\mathbf{S}_n$ when η is equal to 1. Other conditions of the stochastic version of LaSalle's invariance theorem are also satisfied [Mirrahimi and van Handel, 2005]. As the conclusion of the stochastic version of LaSalle's invariance theorem, ρ_t converges in probability to the largest invariant set contained in $\mathbf{U}_n$.

We can see from (18) that $B_{ij} = 0$ $(i \neq j)$ is a necessary condition for $\mathscr{A}S_L(\rho) = 0$. Under this condition, the diagonal elements of B are the eigenvalues of $C + C^*$. If $C + C^*$ has distinct eigenvalues, $\mathscr{A}S_L(\rho) = 0$ is satisfied if and only if $\lambda_i = 1$ holds for some i, or ρ is a pure state. This implies that $\mathbf{U}_n$ is identical to $\mathbf{P}_n$.

Furthermore, according to (8), $\mathrm{d}S_L(\rho_t)|_{\rho_t=\rho} = 0$ holds if ρ is a pure state. This means that $\mathbf{P}_n$ is an invariant set. Hence, the largest invariant set contained in $\mathbf{P}_n$ ($= \mathbf{U}_n$) is $\mathbf{P}_n$ itself. This completes the proof. □

Theorem 2 characterizes a class of measurement which guarantees the preparation of a pure state. Under the condition of the theorem, feedback control is expected to attain additional purposes such as

(1) speeding up the convergence of ρ_t to $\mathbf{P}_n$ [3],
(2) making ρ_t converge to a *particular* pure state.

A feedback control law assuring the convergence to the target state with an pure initial state, is a possible candidate for the second purpose, because the conditional density matrix is expected to become pure after a long time. This can be a guideline to design feedback control laws.

4. DISCUSSIONS WITH NUMERICAL EXAMPLES

4.1 Comparison with Related Results

We here clarify a feature of the Theorem 2 by comparing it with related results.

First, we review Theorem 2 by focusing on two aspects, namely 1) properties of convergence and 2) class of systems. More specific, 1-a) type of convergence, 1-b) region of convergence, 2-a) constraints on C, 2-b) constraints on H and $u \in \mathcal{U}$, and 2-c) constraints on η are considered. In these viewpoints, Theorem 2 is summarized as follows:

- Theorem 2:
  - 1-a) convergence in probability, 1-b) $\mathbf{P}_n$ (the set of all pure states),
  - 2-a) $C + C^*$ has distinct eigenvalues,
    2-b) <u>no constraint</u>, 2-c) $\eta = 1$.

[3] As seen in Section 3.2, control via Hamiltonian does not change the purity directly. However, the speed-up of convergence might be possible in a certain (indirect) way. Combes et al. [2007] discusses such speed-up using feedback control.

It should be emphasized that Theorem 2 holds for any choices of H and $u \in \mathcal{U}$. Comparisons with related results may highlight this feature.

Van Handel et al. investigated basic properties of continuous measurement for a spin (angular momentum) system in [van Handel et al., 2005a]. In the paper, they showed that if we only measure a spin system, the conditional density matrix converges almost surely to one of the eigenstates [4]. For the same system, Mirrahimi and van Handel [2005] and Tsumura [2007] considered a control problem and proposed feedback control laws (a switching type in the former and a continuous type in the latter) which attain global stabilization of an arbitrary target eigenstate.

These studies can be reinterpreted as the results stating convergence properties of ρ_t in each situation. In order to compare these studies with Theorem 2, we here describe them in the same way as follows:

- Van Handel et al. [2005a]:
  - 1-a) almost sure convergence, 1-b) the set of all eigenstates,
  - 2-a) C is an angular momentum operator [5], 2-b) $u(t) \equiv 0$, 2-c) no constraint.
- Mirrahimi and van Handel [2005] and Tsumura [2007]:
  - 1-a) almost sure convergence, 1-b) an arbitrary target eigenstate,
  - 2-a) C is an angular momentum operator, 2-b) H is another angular momentum operator and u is a specific feedback control law, 2-c) no constraint.

Note that H and u are specified in these studies, while Theorem 2 holds for general H and $u \in \mathcal{U}$.

As addressed in Section 3.2, the entropy is not directly affected by the control input. The above-mentioned feature of Theorem 2 is due to this property of the entropy.

4.2 Numerical Examples

We will confirm Theorem 2 by numerical examples, focusing on the main point of discussions in the previous subsection, or the irrelevance of the type of control law.

Here we consider a 2-dimensional quantum system. In this case, it is easy to see that $\rho \in \mathbf{S}_2$ can be parameterized as

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-\mathrm{i}y \\ x+\mathrm{i}y & 1-z \end{bmatrix}, \tag{19}$$

where x, y, z are real scalars. Furthermore, positive semidefiniteness of ρ leads to the condition $x^2+y^2+z^2 \le 1$. Thus, we can identify ρ with a point included in a solid sphere with unit radius in $\mathbb{R}^3$, which is called *Bloch sphere*. The surface of the Bloch sphere corresponds to $\mathbf{P}_2$ (the set of all 2-dimensional pure states).

We set C and H as

$$C = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ H = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{20}$$

[4] Eigenstates are specific pure states determined by the measurement.

[5] If we assume $u(t) \equiv 0$, this condition is easily generalized to the condition " $C^* = C$".

This C obviously satisfies the condition of Theorem 2. This setting corresponds to the situation that we measure a single particle with spin 1/2 by laser in z-direction and apply a magnetic field in y-direction. Furthermore, we assume $\eta = 1$.

Figure 2 shows the simulated trajectory of ρ_t in the Bloch sphere with $u(t) \equiv 1$, where the initial point of the simulation is $(x, y, z) = (-1/2, 1/2, 1/2)$. The trajectory of ρ_t converges to the surface of the Bloch sphere, i.e., $\mathbf{P}_2$. The convergence is also verified by Fig. 3, which illustrates the time evolution of $S_L(\rho_t)$. We can see that the entropy converges to zero, which means that ρ_t converges to $\mathbf{P}_2$.

We have made more simulations with different types of control law u to confirm the property in Theorem 2. Figures 4 and 5 respectively illustrate the time evolutions of $S_L(\rho_t)$ with random piecewise constant input and that with a feedback control law $u(t) = 1 - \mathrm{Tr}\,[\rho_t \rho_\mathrm{f}]$, where $\rho_\mathrm{f} = \mathrm{diag}[1, 0]$. The initial points and the simulated Wiener processes are same as the first case. We can see that the values of entropy converge to zero, i.e., the conditional density matrices also converges to $\mathbf{P}_2$ in these cases as stated in Theorem 2.

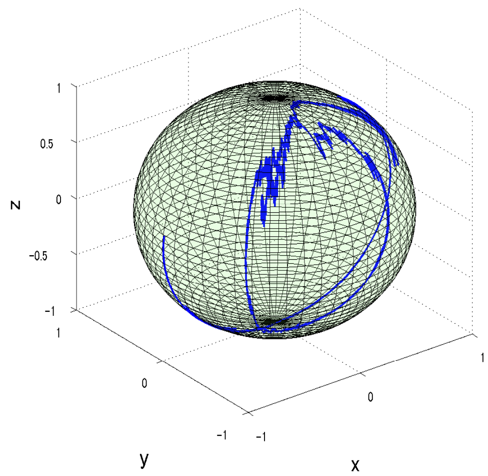

Fig. 2. Trajectory of ρ_t in the Bloch sphere with $u(t) \equiv 1$.

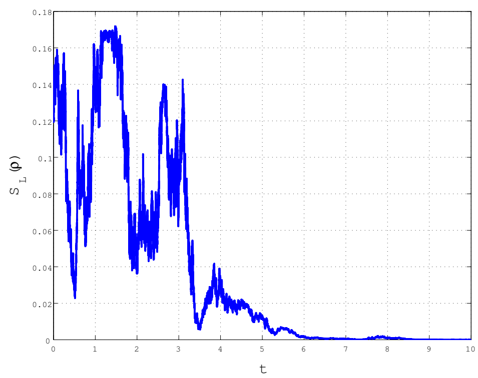

Fig. 3. Time evolution of $S_L(\rho_t)$ with $u(t) \equiv 1$.

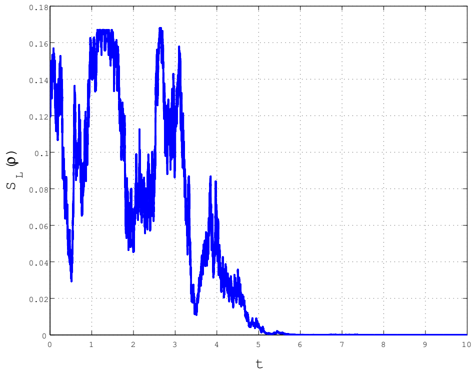

Fig. 4. Time evolution of $S_L(\rho_t)$ with random piecewise constant input.

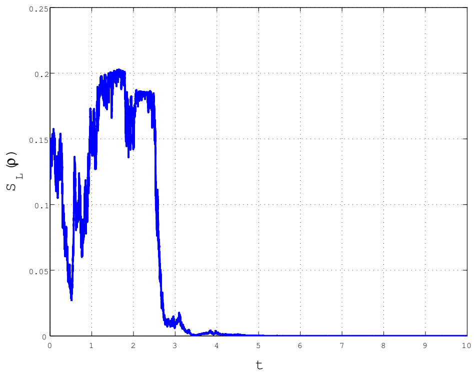

Fig. 5. Time evolution of $S_L(\rho_t)$ with a feedback control law $u(t) = 1 - \mathrm{Tr}\,[\rho_t \rho_{\mathrm{f}}]$.

5. CONCLUSION

In this paper, we have analyzed behaviors of measurement-based quantum feedback control systems using the linear entropy as an index. We first made the effects of terms appearing in the stochastic master equation (6) clear and clarified a limitation of control action via Hamiltonian. We have also shown that $\mathcal{H}[C]\rho_t \mathrm{d}t$ always yields desirable effects for preparation of a pure state and $\mathcal{D}[C]\rho_t \mathrm{d}t$ causes undesirable effects if C is a normal matrix. These results are fundamental properties of measurement-based quantum feedback control systems. Furthermore, we have derived a condition which assures the conditional density matrix converges in probability to the set of all pure states for any control input.

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