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Performance and stability analysis of discontinuous PWA systems by piecing together PWQ functions

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Abstract: An algorithm for evaluating the cost performance of discontinuous autonomous discrete-time piecewise affine systems is presented. The algorithm performs reverse reachability analysis and constructs a piecewise quadratic trajectory cost function over the entire region of attraction of the origin while explicitly taking into account the exact spatial evolution of the trajectories and the exact switching structure of the system as a whole. Available explicitly, this cost function can be integrated in order to evaluate the cost performance of the entire system. The reverse reachability algorithm is applied to the problem of constructing Lyapunov functions. The resulting Lyapunov functions are less conservative than other forms of Lyapunov function commonly used for stability analysis of autonomous discrete-time piecewise affine systems.

Keywords: Piecewise linear; Lyapunov function; performance analysis; stability analysis.

1. INTRODUCTION

Piecewise affine (PWA) systems are useful for their ability to approximate general nonlinear systems [Sontag (1981)], and to describe certain classes of hybrid system [Heemels *et al.* (2001)]. Furthermore, finite- and infinite-horizon model predictive control (MPC) laws for constrained, linear and PWA plants are given explicitly as a PWA function of the state, resulting in an autonomous PWA closed-loop system [Borrelli (2003)]. Analysis of PWA systems seems to focus on two broad areas; (i) the spatial evolution of trajectories, e.g. reachability and invariance properties [Raković *et al.* (2004); Santis *et al.* (2004)], without explicit reference to the trajectories' cost, or (ii) the energy/cost of the system and the search for Lyapunov functions for stability and performance analysis, without explicit consideration of state trajectories [Ferrari-Trecate *et al.* (2002)].

In Gondhalekar & Imura (2007) the authors introduced a novel algorithm for the performance analysis of MPC control laws for constrained linear discrete-time systems. Such systems equipped with an MPC controller result in *continuous* autonomous discrete-time PWA closed-loop systems. 'Performance' refers to both spatial and cost properties of closed-loop state trajectories. Spatial properties of interest are the (N-step) region of attraction of the origin. Cost properties of interest are the average closed-loop trajectory cost over the entire region of attraction of the origin. The algorithm iteratively performs reverse reachability analysis of the entire PWA partition, starting from a positively invariant central set. The reverse simulation is performed on entire sets, rather than considering individual trajectories. This is reminiscent of robust simulation methods [Kantner (1997)]. At each iteration the sets of states which reach the central set in the same number of steps as the current iteration are located, and the explicit piecewise quadratic (PWQ) nominal closed-loop trajectory cost function constructed on these sets. The algorithm terminates when no new states are located, i.e. when all new sets are empty. The algorithm is not guaranteed to terminate in a finite number of steps, but if it does the union of all located sets corresponds to the region of attraction of the origin. The algorithm determines four valuable parameters; (i) the region of attraction of the origin, (ii) the N-step regions of attraction of the origin, (iii) the explicit trajectory cost function over the entire region of attraction, and (iv) the explicit sets of states from which all state trajectories generate exactly the same region switching sequence.

In this paper first the algorithm introduced in Gondhalekar & Imura (2007) is generalized to *discontinuous* PWA systems. To ensure uniqueness of the system and cost function, each region of the system partition may be defined as closed, open, or a combination (clopen). This enables the analysis of discontinuous PWA systems which have no gaps between regions. Furthermore, by imposing conditions the algorithm is also generalized to systems where the origin is not strictly contained within the interior of a single region (Sections 3/4). In Section 5 two numerical examples are given. The first demonstrates how both spatial and cost performance parameters are determined by the proposed method. The second demonstrates how the method can be applied when the origin is on the boundary of multiple regions, and need not be positively invariant.

In the second part of this paper the performance analysis algorithm is applied to the problem of constructing PWQ Lyapunov functions for stability analysis of PWA systems (Section 6). Three common Lyapunov function candidates are briefly reviewed, and it is explained why the proposed method generates Lyapunov functions which are less conservative than two of them, and possibly all three of them.

2. PRELIMINARIES

The real number set is denoted by $\mathbb{R}$ ($\mathbb{R}_0$: non-negative) and the non-negative integer set by $\mathbb{N}$ ($\mathbb{N}_+ := \mathbb{N}\backslash 0$). The set of consecutive integers $\{j, \ldots, k\}$ is denoted by $\mathbb{N}_j^k$. Denote by $I_n \in \{0,1\}^{n\times n}$ the identity matrix, by $0_{\{n,m\}} \in 0^{n\times m}$ the zero matrix and by 0 without subscript the zero matrix with dimension deemed obvious by context. Element j of a vector A is denoted by $A_{[j]}$. The vector of entire row j of a matrix A is denoted by $A_{[j,:]}$. The transpose of matrix A is denoted by A^{T} and the spectral radius by $\rho(A)$. For matrices A and B of equal dimension, inequalities $A\{<,\leq,\geq,>\}B$ hold componentwise. For matrix $A \in \mathbb{R}^{n\times n}$, $A \succ 0$ if and only if $x^{\mathrm{T}}Ax > 0$ $\forall x \in \mathbb{R}^n\backslash 0$. The power set of a set $\mathbb{X}$ is denoted by $2^{\mathbb{X}}$. For a set $\mathbb{X} \in 2^{\mathbb{R}^n}$, ${}^{\mathrm{c}}\mathbb{X} := \mathbb{R}^n\backslash\mathbb{X}$ denotes the complement in $\mathbb{R}^n$, ${}^{\mathrm{cl}}\mathbb{X}$ the closure, ${}^{\circ}\mathbb{X}$ the interior and $\mathrm{Vol} : 2^{\mathbb{R}^n} \to \mathbb{R}_0$, $\mathrm{Vol}(\mathbb{X}) := \int_{\mathbb{X}} \mathrm{d}x$ the volume. A sequence of elements $x_i \in \mathbb{X}$ $\forall i \in \mathbb{N}_j^k$ is denoted by $\{x_i \in \mathbb{X}\}_{i=j}^k$. If the elements' parent set is obvious by context it is denoted by $\{x_i\}_{i=j}^k$.

Definition 1. A polyhedron $\mathbb{P} \in 2^{\mathbb{R}^n}$ is defined by a triple $(G, W, D) \in \mathbb{R}^{g\times n} \times \mathbb{R}^g \times \{0,1\}^g$, $g \in \mathbb{N}_+$: $\mathbb{P}(G, W, D)$

$$= \left\{ x \in \mathbb{R}^n \mid \forall l \in \mathbb{N}_1^g \ , \ \begin{cases} G_{[l,:]}x \leq W_{[l]} & \text{if} \quad D_{[l]} = 1 \\ G_{[l,:]}x < W_{[l]} & \text{if} \quad D_{[l]} = 0 \end{cases} \right\}.$$

The polyhedron $\mathbb{P}$ is defined by g linear inequalities. Binary vector D indicates whether the inequalities are strict or not. Polyhedron $\mathbb{P}$ may thus be closed, open or clopen: ${}^{\mathrm{cl}}\mathbb{P} = \{x \in \mathbb{R}^n | G \leq W\}$, ${}^{\circ}\mathbb{P} = \{x \in \mathbb{R}^n | G < W\}$.

Definition 2. A PWQ function is defined by a collection of sextets $Y = (\mathcal{G}, \mathcal{W}, \mathcal{D}, \mathcal{H}, \mathcal{L}, \mathcal{C})_k$ $\forall k \in \mathbb{Q}$, with set of region indices $\mathbb{Q} := \mathbb{N}_1^{\mathrm{Q}}$, where $\mathrm{Q} \in \mathbb{N}_+$ denotes the total number of regions of the PWQ partition. Each region is defined by: $\mathbb{Y}_k := \mathbb{P}(\mathcal{G}_k, \mathcal{W}_k, \mathcal{D}_k)$, $\mathcal{G}_k \in \mathbb{R}^{\sigma_k\times n}$, $\sigma_k \in \mathbb{N}_+$, $\mathcal{W}_k \in \mathbb{R}^{\sigma_k}$, $\mathcal{D}_k \in \{0,1\}^{\sigma_k}$. In each region $\mathbb{Y}_k$, elements $\mathcal{H}_k \in \mathbb{R}^{n\times n}$, $\mathcal{L}_k \in \mathbb{R}^{1\times n}$, $\mathcal{C}_k \in \mathbb{R}$ define a unique quadratic function $f_k : \mathbb{R}^n \to \mathbb{R}$, $f_k(x) := x^{\mathrm{T}}\mathcal{H}_k x + \mathcal{L}_k x + \mathcal{C}_k$. The set $\mathbb{Y} := \bigcup_{k=1}^{Q} \mathbb{Y}_k$ is termed the *domain* of the PWQ function.

Definition 3. A family of PWQ functions is denoted by $Y^{[p]} = (\mathcal{G}, \mathcal{W}, \mathcal{D}, \mathcal{H}, \mathcal{L}, \mathcal{C})_k^{[p]}$ $\forall k \in \mathbb{Q}^{[p]}$, $\forall p \in \mathcal{P}$, $\mathcal{P} = \mathbb{N}_1^P$, where $P \in \mathbb{N}_+$ is the number of PWQ functions in the family, and $\mathbb{Q}^{[p]}$ is the set of regions in the partition of PWQ function with index p. The set $\bar{\mathbb{Y}} := \bigcup_{p=1}^{P} \mathbb{Y}^{[p]}$ is termed the *domain* of the family of PWQ functions.

Definition 4. An autonomous discrete-time PWA system is defined by a collection of octets $(G, W, D, A, a, H, L, C)_j$ $\forall j \in \mathbb{S}$, with set $\mathbb{S} := \mathbb{N}_1^{\mathrm{S}}$ of region indices, where $\mathrm{S} \in \mathbb{N}_+$ denotes the total number of regions of the PWA partition. Each region is defined by: $\mathbb{X}_j := \mathbb{P}(G_j, W_j, D_j)$, $G_j \in \mathbb{R}^{g_j\times n}$, $g_j \in \mathbb{N}_+$, $W_j \in \mathbb{R}^{g_j}$, $D_j \in \{0,1\}^{g_j}$. In each region $\mathbb{X}_j$ the dynamics are given by $x(i+1) = A_j x(i) + a_j$, with time step index $i \in \mathbb{N}$, state transition matrix $A_j \in \mathbb{R}^{n\times n}$ and affine term $a_j \in \mathbb{R}^n$. Elements $H_j \in \mathbb{R}^{n\times n}$, $L_j \in \mathbb{R}^{1\times n}$ and $C_j \in \mathbb{R}$ define a quadratic single-step cost function $\mathscr{L}_j : \mathbb{R}^n \to \mathbb{R}$, $\mathscr{L}_j(x) := x^{\mathrm{T}}H_j x + L_j x + C_j$. Any two distinct regions have disjoint interiors: ${}^{\circ}\mathbb{X}_j \cap {}^{\circ}\mathbb{X}_k = \emptyset$ if $j \neq k$ $\forall (j,k) \in \mathbb{S}\times\mathbb{S}$. The dynamics and cost function are continuous over common closed boundaries: $A_j x + a_j = A_k x + a_k \wedge \mathscr{L}_j(x) = \mathscr{L}_k(x)$ $\forall (j,k) \in \mathbb{S}\times\mathbb{S}$ s.t. $x \in \mathbb{X}_j \wedge x \in \mathbb{X}_k$. The set $\mathbb{X} := \bigcup_{j=1}^{\mathrm{S}} \mathbb{X}_j$ is termed the system's *domain.* A state trajectory is denoted by $\{x(i)\}_{i=j}^k$, with $\{x(i)\}_{i=j}^{\infty}$ termed *well-defined* if $x(i) \in \mathbb{X}$ $\forall i \in \mathbb{N}$.

Using clopen regions allows to model uniquely defined, discontinuous PWA systems with no gaps between regions. Using only closed regions this is not possible without using switching rules for states in multiple regions. Such rules cannot be incorporated into the proposed algorithm. However, the continuity condition of Definition 4 allows to model continuous portions of discontinuous PWA systems, or continuous PWA systems, using only closed regions. Not every uniquely defined PWA system according to other definitions can be modeled according to Definition 4.

3. PERFORMANCE ANALYSIS PROBLEM

Define the region of attraction of the origin of a PWA system of Definition 4 as: $\mathbb{A} := \{x \in \mathbb{X} | \lim_{i\to\infty} x(i) = 0,$ $x(0) = x\}$, and the running cost $J : \mathbb{X} \to \{\mathbb{R}, \emptyset\}$ of a trajectory starting from state x: $J(x) := \sum_{i=0}^{\infty} \mathscr{L}(x(i))$, $x(0) = x$, $\mathscr{L}(x(i)) = \mathscr{L}_j(x(i))$ if $x(i) \in \mathbb{X}_j$. For a state $x \in \mathbb{X}$ which does not result in a well-defined state trajectory $\{x(i) \in \mathbb{X}\}_{i=0}^{\infty}$ we write $J(x) = \emptyset$. The performance analysis problem is then stated as follows:

Problem 5. Determine:

a) Region of attraction of origin: $\mathbb{A}$.

b) Average running cost: $\bar{J} := \dfrac{1}{\mathrm{Vol}(\mathbb{A})} \displaystyle\int_{\mathbb{A}} J(x)\mathrm{d}x.$

In order to be able to apply the proposed method to Problem 5 we make a number of assumptions.

Assumption 6. The PWA partition has a finite number of regions: $S < \infty$.

Assumption 7. $\forall \{x(i) \in \mathbb{X}\}_{i=0}^{\infty}$ s.t. $\lim_{i\to\infty} x(i) = 0$ $\exists (l, j) \in \mathbb{N}\times\mathbb{S}$, $l < \infty$ s.t. $x(i) \in \mathbb{X}_j$ $\forall i \geq l$.

Remark 8. Assumption 7 states that all state trajectories which converge to the origin require a finite number of steps to reach a positively invariant set which either contains or borders the origin [1], and that this positively invariant set must be a subset of or equal to a region of the PWA partition. This implies that a state trajectory which converges to the origin makes a finite number of region switches. Assumption 7 is critical for the proposed algorithm to work, and is the major limitation to its application. Entering a positively invariant reach set is required for initialization of the proposed algorithm. The finiteness condition $l < \infty$ could be relaxed. In that case the proposed algorithm is not guaranteed to terminate within a finite number of iterations.

Define the set of indices of regions into any one of which all state trajectories which converge to the origin must pass into and remain: $\mathbb{S}_0 := \{j \in \mathbb{S} | \exists (x(0), l) \in \mathbb{X}\times\mathbb{N}, l < \infty$ s.t. $x(i) \in \mathbb{X}_j \forall i \geq l\}$.

Assumption 9. $\left[x(i) \in \mathbb{X}_j \ \forall i \in \mathbb{N} \quad \wedge \quad \lim_{i\to\infty} x(i) = 0\right]$ $\forall x(0) \in \mathbb{X}_j$ $\forall j \in \mathbb{S}_0$.

Remark 10. Assumption 9 states that regions $\mathbb{X}_j$ $\forall j \in \mathbb{S}_0$ are positively invariant with stable dynamics. If Assumption 7 but not 9 is satisfied, then regions $\mathbb{X}_j$ $\forall j \in \mathbb{S}_0$ can

[1] Note that it is not strictly necessary for the region to contain the origin. Rather, the closure must contain the origin (see Section 5.2).

be re-partitioned so that Assumption 9 is satisfied. The resulting system is more complex, i.e. has more regions.

Remark 11. Suppose the origin is contained in the interior of a single region with stable dynamics: $0 \in {}^{\circ}\mathbb{X}_j$, $\rho(A_j) < 1$, $a_j = 0$. If region $\mathbb{X}_j$ is not positively invariant then it can be re-partitioned such that the origin is contained within the interior of a positively invariant region. Thus, the proposed approach can always be employed when the origin is contained in the interior on one region, although finite termination is not automatically guaranteed.

Assumption 12.
(i) $\mathscr{L}_j(x) \geq 0\ \forall x \in \mathbb{X}_j\ \forall j \in \mathbb{S}$ (ii) $C_j = 0\ \forall j \in \mathbb{S}_0$

Remark 13. Assumption 12 implies that the one-step cost is positive semi-definite, and that $\mathscr{L}_j(0) = 0\ \forall j \in \mathbb{S}_0$. The latter condition ensures that the running cost is finite for state trajectories which converge to the origin. Note that this does not imply a zero one step cost at the origin. Possibly $\mathscr{L}_j(0) \neq 0$ if $0 \in \mathbb{X}_j$. It is possible that $j \notin \mathbb{S}_0$ for some $\mathbb{X}_j \ni 0$ (see the example in Section 5.2).

Problem 5 is hard to tackle directly. To simplify it, define the N-step regions of attraction of the origin $\mathbb{A}_N$: $\mathbb{A}_0 := \bigcup_{j\in\mathbb{S}_0} \mathbb{X}_j$, $\mathbb{A}_N := \{x \in \mathbb{X} | x(N) \in \mathbb{A}_0, x(i) \notin \mathbb{A}_0\ \forall i \in \mathbb{N}_0^{N-1}, x(0) = x\}\ \forall N \in \mathbb{N}_+$. The positively invariant region $\mathbb{A}_0$ is called the *central set*. The state trajectory from any initial state in $\mathbb{A}_N$ requires exactly N steps to reach $\mathbb{A}_0$. Clearly $\bigcup_{k=0}^{N+1} \mathbb{A}_k \supseteq \bigcup_{k=0}^{N} \mathbb{A}_k$, as a trajectory which reaches the central set in $N+1$ steps must pass through the set of states which reach the central set in N steps. By Assumption 7 there exists an $\hat{N} < \infty$ s.t. $\mathbb{A}_{\hat{N}+1} = \emptyset$. Thus $\mathbb{A} = \bigcup_{k=0}^{\hat{N}} \mathbb{A}_k$. The state trajectory from any initial state within the region $\mathbb{A}$ of attraction of the origin requires at most $\hat{N}$ steps to reach the central set $\mathbb{A}_0$.

The following conceptual algorithm solves Problem 5:

Algorithm 14.

1. Set $\mathbb{A}_0 = \bigcup_{j\in\mathbb{S}_0} \mathbb{X}_j$ and compute the explicit PWQ cost function $Y^{[1]}$ for $\mathbb{A}_0$.
2. For $N = 1, 2, 3, \ldots$ determine $\mathbb{A}_N$ and compute the explicit PWQ cost function $Y^{[N+1]}$ for each $\mathbb{A}_N$.
3. Stop when $\mathbb{A}_N = \mathbb{Y}^{[N+1]} = \emptyset$. Set $\hat{N} = N - 1$. Then $\mathbb{A} = \bar{\mathbb{Y}} = \bigcup_{k=0}^{\hat{N}} \mathbb{A}_k$.
4. Integrate the explicit PWQ cost function family $Y^{[p]}$ to find $\bar{J}$.

Step 1 is solved by $Y^{[1]} = (\mathcal{G}_j, \mathcal{W}_j, \mathcal{D}_j, \mathcal{H}_j, \mathcal{L}_j, \mathcal{C}_j)_{j\in\mathbb{S}_0}$:

$$\left.\begin{array}{l} \mathcal{G}_j = G_j\ ,\ \mathcal{W}_j = W_j\ ,\ \mathcal{D}_j = D_j\ , \\ \mathcal{H}_j \quad \text{solves} \quad A_j^{\mathrm{T}}\mathcal{H}_j A_j - \mathcal{H}_j + H_j = 0\ , \\ \mathcal{L}_j = L_j(I_n - A_j)^{-1}\ ,\ \mathcal{C}_j = 0 \end{array}\right\}. \tag{1}$$

Step 2 is solved by the reverse reachability algorithm of Section 4. For step 3, existence of an $\hat{N} < \infty$ s.t. $\mathbb{A}_N = \emptyset$ is ensured by Assumption 7. The integration of Step 4 can be performed numerically, or analytically in simple cases. Performing step 4 is not discussed in this paper.

4. REVERSE REACHABILITY ALGORITHM

The reverse reachability algorithm proposed here solves step 2 of Algorithm 14 by repeated use of the following geometric fact for a PWA system region $\mathbb{X}_j$ and PWQ function region $\mathbb{Y}_k$: $\{x(0) \in \mathbb{X}_j | x(1) \in \mathbb{Y}_k\} = \mathbb{P}(\bar{\mathcal{G}}, \bar{\mathcal{W}}, \bar{\mathcal{D}})$,

$$\bar{\mathcal{G}} := \begin{bmatrix} G_j \\ \mathcal{G}_k A_j \end{bmatrix},\ \bar{\mathcal{W}} := \begin{bmatrix} W_j \\ \mathcal{W}_k - \mathcal{G}_k a_j \end{bmatrix},\ \bar{\mathcal{D}} := \begin{bmatrix} D_j \\ \mathcal{D}_k \end{bmatrix}.$$

The running cost $J(x)\ \forall x \in \mathbb{P}(\bar{\mathcal{G}}, \bar{\mathcal{W}}, \bar{\mathcal{D}})$ is given by:

$$\begin{aligned} J(x) &= \mathscr{L}_j(x) + f_k(A_j x + a_j) \\ &= x^{\mathrm{T}} H_j x + L_j x + C_j + (A_j x + a_j)^{\mathrm{T}} \mathcal{H}_k (A_j x + a_j) \\ &\quad + \mathcal{L}_k (A_j x + a_j) + \mathcal{C}_k \\ &= x^{\mathrm{T}} \left[H_j + A_j^{\mathrm{T}} \mathcal{H}_k A_j\right] x + \left[L_j + 2a_j^{\mathrm{T}} \mathcal{H}_k A_j + \mathcal{L}_k A_j\right] x \\ &\quad + \left[C_j + a_j^{\mathrm{T}} \mathcal{H}_k a_j + \mathcal{L}_k a_j + \mathcal{C}_k\right]\ . \end{aligned}$$

These two results are applied iteratively to construct a family of PWQ functions $Y^{[p]}\ \forall p \in \mathbb{N}_+$. For each p the domain of $Y^{[p]}$ corresponds to the $(p-1)$-step region of attraction of the origin: $\mathbb{A}_{p-1}$. Each PWQ function $Y^{[p]}$ in turn is constructed by using the above two tools for each combination of PWA system region $j \in \mathbb{S}\backslash\mathbb{S}_0$, and every region $\mathbb{Y}_k^{[p-1]}\ \forall k \in \mathbb{N}_1^{Q^{[p-1]}}$, where $Q^{[p-1]}$ is the number of regions of the PWQ function generated the iteration previously. The algorithm is formalized in Algorithm 15.

Algorithm 15. (Reverse reachability algorithm)

– Initialize : $p \leftarrow 1$, $Y_1^{[p]}$ s.t. Eq.(1)
– While $\mathbb{Y}^{[p]} \neq \emptyset$ Do
– $l \leftarrow 0$
– For : $k = 1, \ldots, Q^{[p]}$
– For : $j \in \mathbb{S}\backslash\mathbb{S}_0$
– $l \leftarrow l + 1$
– $\mathcal{G}_l^{[p+1]} \leftarrow \begin{bmatrix} G_j \\ \mathcal{G}_k^{[p]} A_j \end{bmatrix}$
– $\mathcal{W}_l^{[p+1]} \leftarrow \begin{bmatrix} W_j \\ \mathcal{W}_k^{[p]} - \mathcal{G}_k^{[p]} a_j \end{bmatrix}$
– $\mathcal{D}_l^{[p+1]} \leftarrow \begin{bmatrix} D_j \\ \mathcal{D}_k^{[p]} \end{bmatrix}$
– $\mathcal{H}_l^{[p+1]} \leftarrow H_j + A_j^{\mathrm{T}} \mathcal{H}_k^{[p]} A_j$
– $\mathcal{L}_l^{[p+1]} \leftarrow L_j + 2a_j^{\mathrm{T}} \mathcal{H}_k^{[p]} A_j + \mathcal{L}_k^{[p]} A_j$
– $\mathcal{C}_l^{[p+1]} \leftarrow C_j + a_j^{\mathrm{T}} \mathcal{H}_k^{[p]} a_j + \mathcal{L}_k^{[p]} a_j + \mathcal{C}_k^{[p]}$
– End For
– End For
– $p \leftarrow p + 1$
– End While

5. NUMERICAL EXAMPLE

5.1 Continuous PWA System: Linear-Quadratic Regulator

The double integrator with sample-period $\tau = 0.2$s,

$$x(i+1) = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} x(i) + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix} u(i),$$

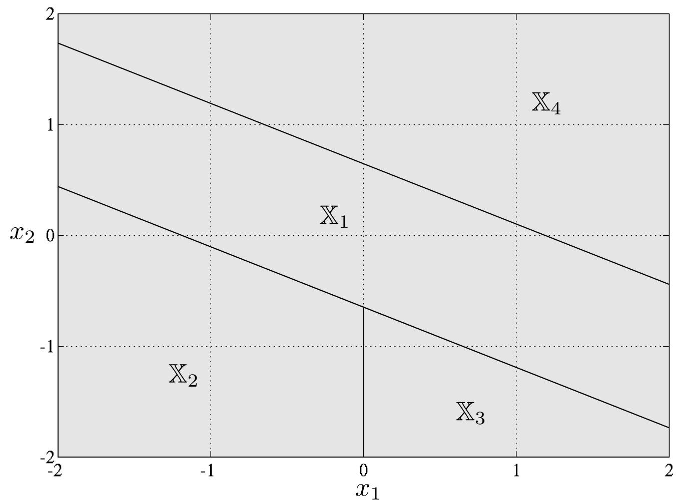

Fig. 1. PWA system partition. 4 regions.

control input $u(i)$, constraints $u(i) \in \{u \in \mathbb{R} | \|u\|_\infty \leq 1\} \wedge x(i) \in \{x \in \mathbb{R}^2 | \|x\|_\infty \leq 2\}$ $\forall i \in \mathbb{N}$ and running cost function $J(x) = \sum_{i=0}^{\infty} \left[x^{\mathrm{T}}(i)x(i) + u^{\mathrm{T}}(i)u(i)\right]$, $x(0) = x$ is considered. The constrained linear-quadratic regulator is given by the PWA control law of Eq. (2) for the partition plotted in Fig. 1 with $K = -[0.841, 1.546]$. Regions $\mathbb{X}_2$ and $\mathbb{X}_3$ could be combined into one. The split has been retained in order to visualize the region switching behavior.

$$u^*(i) = \begin{cases} Kx(i) & \text{if } x(i) \in \mathbb{X}_1 \\ +1 & \text{if } x(i) \in (\mathbb{X}_2 \cup \mathbb{X}_3) \\ -1 & \text{if } x(i) \in \mathbb{X}_4 \end{cases} \tag{2}$$

The cost functions are defined by: $H_1 = I_2 + K^{\mathrm{T}}K$, $L_1 = 0$, $C_1 = 0$, $H_j = I_2$, $L_j = 0$, $C_j = 1$, $\forall j \in \{2, 3, 4\}$.

Plotted in Fig. 3 is the PWQ partition of the explicit running cost function over the entire region of attraction of the origin: $\mathbb{A}$. In this case $\mathbb{A}$ also corresponds to the maximal positively invariant set. The state trajectories from two initial states close to the lower bound of the plot are indicated by black dots. These give an indication of how all states from one particular region of the PWQ partition switch into the same destination region.

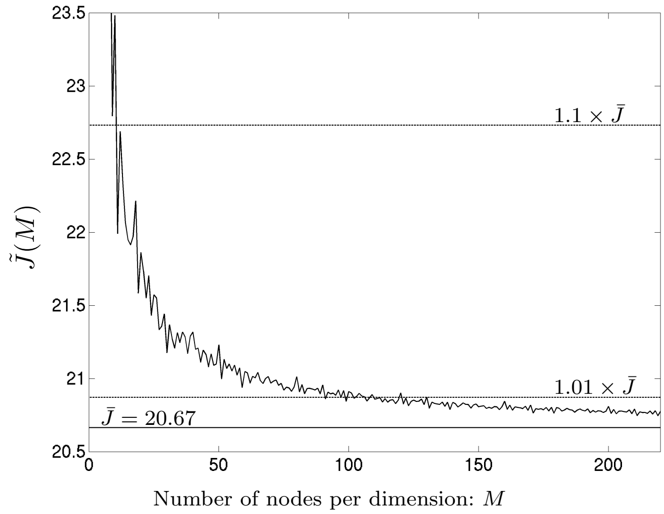

Fig. 2. Numerical approximation of average cost $\tilde{J}(M)$ vs. number of nodes per dimension M. Exact: $\bar{J} = 20.67$.

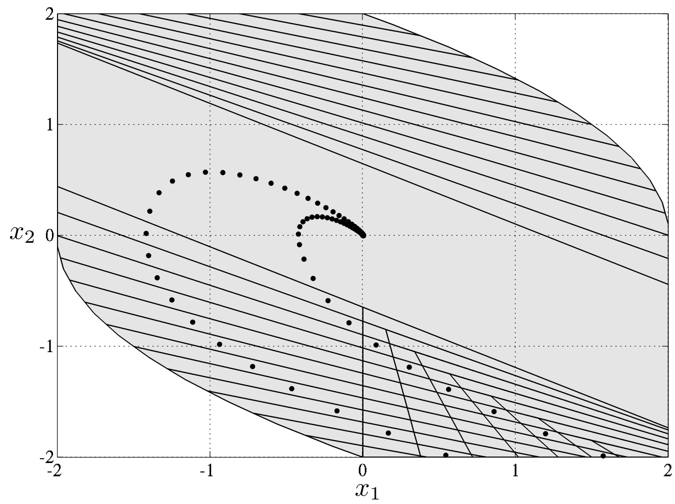

Fig. 3. PWQ running cost function partition. 76 regions.

The average running cost over the entire region of attraction was computed: $\bar{J} = 20.67$. Completion of the reachability algorithm and integration of the explicit PWQ cost function required about 0.53s and 0.51s, respectively. Suppose one attempted to approximate the average running cost by performing simulations from a finite number of initial states, given by an even grid with M nodes per dimension. Plotted in Fig. 2 is this numerical approximation $\tilde{J}(M)$ against the number of nodes M per dimension. The solid, horizontal line denotes the exact average $\bar{J}$. The dashed, horizontal lines indicate an over-approximation by 1% and 10%. The following result is obtained: $\bar{J} \leq \tilde{J}(M) \leq 1.01\bar{J}$ $\forall M \geq 131$. A grid with $M = 131$ corresponds to $131^2 = 17,161$ simulations. Computing $\tilde{J}(131)$ required[2] roughly 20.4s.

5.2 Discontinuous PWA System

Consider the following system:

$$\mathbb{X}_1: \quad x(i+1) = x(i) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \quad \text{if } \begin{cases} -1 \leq x_1 \leq 0 \\ 0 \leq x_2 \leq 1 \end{cases}$$

$$\mathbb{X}_2: \quad x(i+1) = x(i) + \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} \quad \text{if } \begin{cases} 0 < x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \end{cases}$$

$$\mathbb{X}_3: \quad x(i+1) = x(i) + \begin{bmatrix} -0.5 \\ 0 \end{bmatrix} \quad \text{if } \begin{cases} 0 \leq x_1 \leq 1 \\ -1 \leq x_2 < 0 \end{cases}$$

$$\mathbb{X}_4: \quad x(i+1) = 0.8 I_2 x(i) \quad \text{if } \begin{cases} -1 \leq x_1 < 0 \\ -1 \leq x_2 < 0 \end{cases}$$

The system partition is plotted in Figure 4. Region boundaries are slightly separated in order to emphasize which boundaries are closed and which are open. Closed boundaries are denoted by solid lines, open ones by dashed lines. Note that this PWA system is discontinuous, uniquely defined, has no gaps between regions and that $0 \in \mathbb{X}_1$. Region $\mathbb{X}_4$ is positively invariant, and it is easy to verify that all state trajectories $\{x(i)\}_{i=0}^{\infty}$ $\forall x(0) \in \mathbb{X} \backslash \mathbb{X}_4$ enter region $\mathbb{X}_4$. This system thus satisfies Assumptions 7 and 9. The one-step cost function is not relevant for the discussion below, so for simplicity the details are omitted.

[2] Computed using MATLAB. No rigorous attempt was made to optimize programs used for obtaining run-time results.

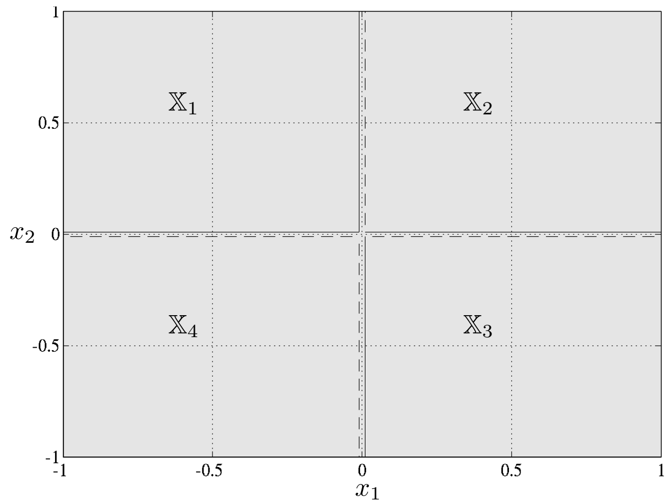

Fig. 4. PWA system partition. 4 regions. Solid line: Closed boundary. Dashed line: Open boundary.

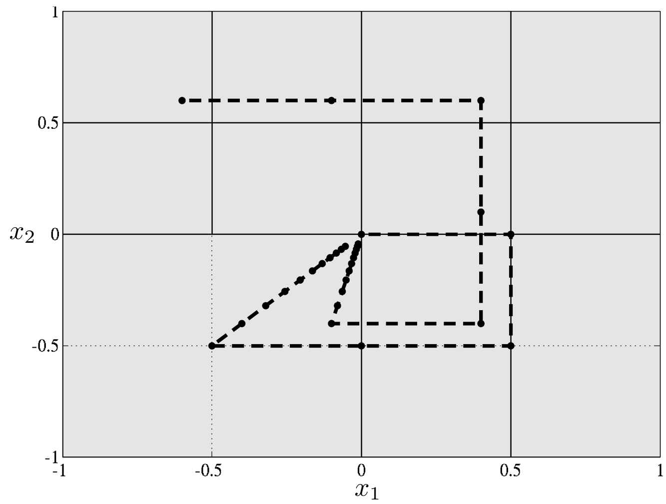

Fig. 5. PWQ running cost function partition. 11 regions.

The PWQ running cost function partition is plotted in Figure 5. No distinction is made between closed and open boundaries, although the algorithm does properly take this into account. The cost function is uniquely defined and has no gaps between regions. Also plotted in Figure 5 are two state trajectories, from initial states $x(0) = 0$ and $x(0) = [-0.6, 0.6]^{\mathrm{T}}$. Of particular interest is the trajectory starting at the origin. Clearly this system is not stable, because the origin is not invariant. It is therefore futile to attempt performance analysis by Lyapunov methods [Ferrari-Trecate *et al.* (2002)], because no Lyapunov function can exists. In a practical sense however, the system is stable, as all trajectories converge to the origin. Another point of interest is that the running cost function is discontinuous. Consider the origin. As the origin is not invariant $J(0) > 0$. However for $x \in \mathbb{X}_4$, $J(x) \to 0$ as $x \to 0$.

6. STABILITY ANALYSIS PROBLEM

Definition 16. Consider a PWA system of Definition 4 and functions $V : \mathbb{D} \to \mathbb{R}_0$, $\Delta V : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$, $\mathbb{D} \in 2^{\mathbb{X}}$, $\mathbb{D} \ni 0$:

$$V(x) > 0 \quad \forall x \in \mathbb{D}\backslash 0 \ ,$$
$$V(0) = 0 \ ,$$
$$\Delta V(x(0), x(1)) := V(x(1)) - V(x(0)) < 0 \ .$$

Then $V(x)$ is called a Lyapunov function, $\mathbb{D}$ the domain of V, and the origin of the system is termed Lyapunov stable.

Problem 17. For a PWA system of Definition 4, find a Lyapunov function $V(x)$ with largest possible domain $\mathbb{D}$.

6.1 Current Approaches for Solving Problem 17

In Ferrari-Trecate *et al.* (2002) three common Lyapunov function candidates for autonomous discrete-time PWA systems are presented and their relative merits discussed:

$$V_1(x) = x^{\mathrm{T}} P x \qquad \forall x \in \mathbb{X} \ ,$$
$$V_2(x) = x^{\mathrm{T}} P_j x \qquad \forall x \in \mathbb{X}_j \qquad \forall j \in \mathbb{S} \ ,$$
$$V_3(x) = x^{\mathrm{T}} P_j(x) x \qquad \forall x \in \mathbb{X}_j \qquad \forall j \in \mathbb{S} \ .$$

Candidate V_1 results in so-called quadratic Lyapunov stability. Note that $P \succ 0$. Candidate V_2 results in so-called PWQ stability. Note that it is not necessary that $P_j \succ 0 \ \forall j \in \mathbb{S}$, because each local function is only required to be positive-definite within its region: $x^{\mathrm{T}} P_j x > 0 \ \forall x \in \mathbb{X}_j \backslash 0 \ \forall j \in \mathbb{S}$. For V_3 the state dependent cost matrices are given by $P_j(x) = \sum_{l=1}^{N} P_{j(l)} \rho_{j(l)}$, with parameter matrices $P_{j(l)} \in \mathbb{R}^{n \times x} \ \forall l \in \mathbb{N}_1^N \ \forall j \in \mathbb{S}$, and bounded basis functions $\rho_{j(l)} : \mathbb{X}_j \to \mathbb{R} \ \forall l \in \mathbb{N}_1^N \ \forall j \in \mathbb{S}$. The Lyapunov function candidates are numbered in increasing order of complexity, but decreasing order of conservativeness.

Assuming the origin of some PWA system is in fact Lyapunov stable, there are two main reasons why it may not be possible to determine a Lyapunov function by using the above Lyapunov function candidates. Firstly, the domain of the Lyapunov function is assumed to be the domain of the PWA system. However, a Lyapunov function can only be located on the domain of attraction of the origin. Secondly, the conservativeness introduced by the form of the Lyapunov function candidate may not permit an actual Lyapunov function to accept that particular form. There are two key aspects of the candidates' form. First, the candidate functions are in some sense purely quadratic. Candidate V_1 is purely quadratic, candidate V_2 is piecewise purely quadratic, while candidate V_3 is piecewise weighted purely quadratic. This sets bounds on the amount of 'energy' which the system can dissipate. Second, the partitions of candidates V_2 and V_3 are the partition of the PWA system itself. This prevents the Lyapunov function candidate from rigorously taking into account that for discrete-time PWA systems the state trajectory from a state within some region may remain within that region, can transition into any other region, or may even exit the domain of the system altogether.

6.2 PWQ Lyapunov Functions via Reverse Reachability

The problems of computing cost functions and Lyapunov functions are closely related, because the 'running cost' of a system can be interpreted as an energy function. The running cost $J : \mathbb{A} \to \mathbb{R}_0$ of Section 5.1 is in fact [3] a Lyapunov function for the region of attraction of the origin, with domain $\mathbb{D} = \mathbb{A}$. Conversely, locating Lyapunov functions has been used for evaluating the performance of

[3] However, note that in the example of Section 5.2 the running cost $J : \mathbb{X} \to \mathbb{R}_0$ is not a Lyapunov function, because $J(0) \neq 0$.

PWA systems [Ferrari-Trecate *et al.* (2002)]. The following discussion focuses on Lyapunov function candidate V_2, which is a generalization of V_1. Candidate V_3 is excluded, as it cannot be incorporated into the proposed algorithm.

For a PWA system of Definition 4, suppose it is not possible (or one is just not able) to establish a Lyapunov function by employing candidate V_2. Suppose, however, the following Lyapunov function has been determined:

$$\hat{V}(x) = x^{\mathrm{T}} P_j x \quad \forall x \in \mathbb{X}_j \quad \forall j \in \mathbb{S}_0 \subset \mathbb{S}.$$

Reminiscent of Section 3 we call the domain of $\hat{V}$ the *central set*: $\mathbb{A}_0 = \bigcup_{j \in \mathbb{S}_0} \mathbb{X}_j$. Reminiscent of Assumptions 7 and 9 the following assumption is made:

Assumption 18.

(i) $\forall \{x(i) \in \mathbb{X}\}_{i=0}^{\infty}$ s.t. $\lim_{i\to\infty} x(i) = 0 \ \exists l \in \mathbb{N},\ l < \infty$ s.t. $x(i) \in \mathbb{A}_0 \ \forall i \geq l$

(ii) $x(i) \in \mathbb{A}_0 \ \forall i \in \mathbb{N} \ \forall x(0) \in \mathbb{A}_0$

Remark 19. Assumption 18 states that all state trajectories which converge to the origin enter the positively invariant central set $\mathbb{A}_0$ (domain of $\hat{V}$). However, Assumption 18 is crucially different from Assumptions 7 and 9. The number of region switches after a trajectory has entered the positively invariant central set $\mathbb{A}_0$ is not assumed finite.

Remark 20. To satisfy Assumption 18 it may be necessary to re-partition the PWA system, similarly to Remark 10.

Lyapunov function $\hat{V}$ solves step 1 of Algorithm 14, adapted for Problem 17. Step 2 is now performed analogously to solving Problem 5 in Section 3. Assumption 18 (i) ensures that Step 3 of Algorithm 14 is possible, i.e. that the reverse reachability algorithm terminates in a finite number of steps. Step 4 of Algorithm 14 is irrelevant for Problem 17. The resulting Lyapunov function is as follows:

$$V(x) = x^{\mathrm{T}} \mathcal{H}_k x + \mathcal{L}_k x + \mathcal{C}_k \quad \forall x \in \mathcal{X}_k \quad \forall k \in \mathbb{V}\ , \quad (3)$$

with region index set $\mathbb{V}$ and regions $\mathcal{X}_k$ (rather than $\mathbb{X}_k$). The change of region symbol emphasizes that the PWQ Lyapunov function partition is different from the PWA system partition. This point is crucial. The reverse reachability algorithm constructs the regions $\mathcal{X}_k$ such that trajectories from all states $x \in \mathcal{X}_k$ enter the same region at the next step. Furthermore, in regions outside the central set $\mathbb{A}_0$ the Lyapunov function is not purely quadratic, but may contain linear and constant terms also. This indicates that the Lyapunov function is both arithmetically less conservative, because the class of piecewise quadratic functions is a super-class of piecewise purely quadratic functions, and also spatially less conservative, because the PWQ Lyapunov function partition properly takes into account the region switching behavior of the entire system.

6.3 Controlling Energy Dissipation

In the performance analysis problem of Section 3 the single-step cost function of the system, $\mathscr{L}_j(x)\ \forall j \in \mathbb{S}$, was given, and would have some physical significance (see Section 5.1). The running cost function $J(x)$ would therefore inherit this physical significance. For the problem of constructing Lyapunov functions, the single-step cost corresponds to the energy dissipation of the system, and is a design parameter for tuning the Lyapunov function.

Without loss of generality Lyapunov function $\hat{V}(x)\ \forall x \in \mathbb{A}_0$ covering the central set may be assumed to be a piecewise purely quadratic function. This is because a Lyapunov function candidate requires that $V(0) = 0$, and that a purely quadratic function can approximate any other smooth function if the domain is chosen small enough.

However, a piecewise purely quadratic Lyapunov function implies that the single-step cost function has the following form: $\mathscr{L}_j(x) = x^{\mathrm{T}} \left[H_j - A_j^{\mathrm{T}} \mathcal{H}_k A_j\right] x - 2a_j^{\mathrm{T}} \mathcal{H}_k A_j x - a_j^{\mathrm{T}} \mathcal{H}_k a_j$ $\forall x \in \mathbb{X}_j$. This is on the one hand impossible to achieve, in general, because from the same PWA partition region (same j) the state trajectory can switch into any other target region (different k). Furthermore, the restriction seems arbitrary. Being able to choose the single-step cost function $\mathscr{L}_j(x) = x^{\mathrm{T}} H_j x + L_j x + C_j\ \forall x \in \mathbb{X}_j$ freely gives much more flexibility and control to determine the system's energy dissipation on the entire state-space.

7. CONCLUSION

An algorithm for determining the exact cost performance of autonomous discrete-time PWA systems was presented. The algorithm constructs the explicit PWQ running cost function over the entire region of attraction of the origin. Available explicitly, this cost function can be integrated in order to determine the performance of the system as a whole. This alleviates the need to perform a large number of simulations. The algorithm was further applied to the problem of constructing PWQ Lyapunov functions. The resulting Lyapunov functions are less conservative than some commonly used Lyapunov function candidates.

The partitions of PWQ functions generated by the proposed algorithm can be very complex. For the performance analysis problem there is no way to circumnavigate this. However, it is useful to generate Lyapunov functions with simple partitions. In future work, determining Lyapunov functions of the form of Eq. (3) with low-complexity or minimal partitions will be investigated.

REFERENCES

F. Borrelli, *Constrained Optimal Control of Linear and Hybrid Systems*, Springer, ISBN 3-540-00257-X, 2003.

G. Ferrari-Trecate, F.A. Cuzzola, D. Mignone, M. Morari, *Analysis of Discrete-Time Piecewise Affine and Hybrid Systems*, Automatica 38(2002)2139-2146.

R. Gondhalekar, J. Imura, *Performance Measures in Model Predictive Control with Non-linear Prediction Horizon Time-Discretization*, Proc. ECC, Greece, 2007.

W. P. M. H. Heemels, B. De Schutter, A. Bemporad, *Equivalence of Hybrid Dynamical Models*, Automatica 37(2001)1085-1091.

M. Kantner, *Robust Stability of Piecewise Linear Discrete Time Systems*, Proc. ACC, USA, 1997.

S. V. Raković, P. Grieder, M. Kvasnika, D. Q. Mayne, M. Morari, *Computation of Invariant Sets for Piecewise Affine Discrete Time Systems subject to Bounded Disturbances*, Proc. 43rd IEEE CDC, Bahamas, 2004.

E. De Santis, M. D. Di Benedetto, L. Berardi, *Computation of Maximal Safe Sets for Switching Systems*, IEEE TAC, Vol. 49, No. 2, 2004.

E. D. Sontag, *Nonlinear Regulation: The Piecewise Linear Approach*, IEEE TAC, Vol. 26, No. 2, 1981.