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Sum of Roots Characterization for Parametric State Feedback $\mathcal{H}_\infty$ Control

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Abstract: This paper is concerned with the solution of state feedback $\mathcal{H}_\infty$ control for a single-input-single-output plant with parameters, and an algebraic approach that utilizes the so-called 'sum of roots' is developed. A method is firstly devised that computes a polynomial which contains the optimal cost as one of its roots. Furthermore it is shown that an optimal/sub-optimal static feedback gain can be expressed in terms of plant parameters and the sum of roots. The proposed approach thus suggests that the sum of roots is useful for characterizing an achievable $\mathcal{H}_\infty$ performance level as well as some $\mathcal{H}_2$ performance limitations.

Keywords: State feedback $\mathcal{H}_\infty$ control; Characterization of achievable performance level; Sum of roots; Gröbner basis.

1. INTRODUCTION

It has been pointed out that the introduction of the notion of the *sum of roots* (SoR) reveals an intriguing relationship between polynomial spectral factorization and an algebraic method called Gröbner bases [Kanno et al., 2007b]. This observation can further be useful in that the algebraic approach yields an algorithm that can carry out polynomial spectral factorization for the parametric case. Also interesting is the fact that the SoR, initially introduced as an index of 'average stability' [Anai et al., 2005], can give direct expressions for best achievable performance levels of some $\mathcal{H}_2$ control problems [Kanno et al., 2007a].

In this paper an attempt is made for the characterization of the solution to an $\mathcal{H}_\infty$ control problem. Polynomial spectral factorization that has to be carried out in the $\mathcal{H}_\infty$ control problem exhibits a distinctive feature different from the one appearing in the $\mathcal{H}_2$ control problem. It is thus natural to expect that a different approach is required to discover the characterization of the solution in terms of the SoR. This paper suggests an approach that yields the SoR characterization for $\mathcal{H}_\infty$ control. The approach further enables one to express an optimal/sub-optimal feedback gain for the state feedback $\mathcal{H}_\infty$ control problem. The approach that makes use of the SoR has an advantage in that it is based on an algebraic method called Gröbner bases and thus compatible with the parametric case.

The paper is organized as follows. Section 2 reviews the notion of the SoR and gives the solution of polynomial spectral factorization by means of the SoR based on Gröbner basis theory. Section 3 is devoted to the state feedback $\mathcal{H}_2$ control problem, whose result is also applicable to the state feedback $\mathcal{H}_\infty$ control problem for unstable plants. Section 4 then discusses the main problem of the paper, i.e., a state feedback $\mathcal{H}_\infty$ control problem, where the difference in the optimal performance level between the stable case and the unstable case is clarified. In Section 5, the unstable case is considered, and how the sub-optimal feedback gain is written in terms of the SoR is discussed. Section 6 then deals with the stable case, and the best achievable performance level is characterized in terms of the SoR. Section 7 provides a numerical example and Section 8 gives some concluding remarks.

2. EVEN POLYNOMIAL AND THE SUM OF ROOTS

The characteristic polynomial of the Hamiltonian matrix arising from the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ control problem is an even polynomial and the solution of the associated Riccati equation is closely related to polynomial spectral factorization of the characteristic polynomial. This section reviews the SoR and how the problem of polynomial spectral factorization may be related to the SoR and solved with the aid of an algebraic method, namely, Gröbner bases. The result is exploited in the following sections.

A monic even polynomial $f(s)$ of degree $2n$ can be decomposed as

$$(-1)^n f(s) = g(s)g^{\sim}(s) \ , \tag{1}$$

where $g(s)$ is a polynomial of degree n, and $g^{\sim}(s) := g(-s)$ is the conjugate polynomial of $g(s)$. Notice that there are a number of $g(s)$ satisfying (1). Also, since $f(s)$ is monic, it is enough to consider monic $g(s)$. Polynomial spectral factorization is a special case of this decomposition. The task in that case is to find a special $g(s)$ under the assumption that $f(s)$ does not have any roots on the imaginary axis, and the stable $g(s)$ is to be calculated.

There are various numerical approaches to find $g(s)$ (stable $g(s)$, in particular), but an algebraic approach based on Gröbner basis theory is employed here. Write

$$g(s) = s^n + \sigma s^{n-1} + m_{n-2}s^{n-2} + \cdots + m_1 s + m_0 \ . \tag{2}$$

Note the coefficient σ of s^{n-1}. The quantity is called the *sum of roots* since $-\sigma$ is (literally) the sum of roots of $g(s)$. It is shown that all the polynomials $g(s)$ satisfying (1) is characterized by this quantity [Kanno et al., 2007b]. By comparing the coefficients on the both sides of (1), a set of algebraic equations in σ and m_i is obtained. The task of finding $g(s)$ thus becomes solution of this set of algebraic equations. Denote by $\mathcal{G}$ the set of the polynomials obtained from the polynomial parts of the equations. This set of polynomials has an interesting property that facilitates the use of the powerful theory of Gröbner bases.

Proposition 1. (Kanno et al. [2007b]). The set $\mathcal{G}$ of the polynomials forms the reduced Gröbner basis of the ideal generated by itself with respect to the graded reverse lexicographic order $\sigma \succ m_{n-2} \succ \cdots \succ m_0$. Further, generically, the ideal $\langle \mathcal{G} \rangle$ has a Gröbner basis of so-called *shape form* with respect to any elimination ordering $\{m_0, \ldots, m_{n-2}\} \succ\succ \sigma$: $\{S_f(\sigma),\, m_{n-2}-h_{n-2}(\sigma), \ldots,\, m_0-h_0(\sigma)\}$, where $S_f(\sigma)$ is a polynomial of degree exactly 2^n and $h_i(\sigma)$ are polynomials of degree strictly less than 2^n.

It is noted that $S_f(\sigma)$ and $h_i(\sigma)$ are polynomials in σ only. Also the set $\mathcal{G}$ of polynomials that is directly obtained from the problem immediately yields a Gröbner basis and thus conversion to the shape basis can be performed in an efficient manner [Faugère et al., 1993]. This intriguing and useful property is observed through Gröbner basis theory.

Remark 1. To be precise, Gröbner basis theory deals with sets of polynomials (algebraic expressions) and discusses zeros of sets of polynomials (rather than solutions of polynomial equations). However this paper supposes that a set of polynomials and a set of polynomial equations are in essence identical and considers them interchangeable.

Proposition 1 suggests that all $g(s)$ are characterized by the roots of $S_f(\sigma)$. Once roots of $S_f(\sigma)$ are found, the rest of the computation is straightforward; all the other coefficients m_i of $g(s)$ are expressed as polynomials h_i in σ and substitution of σ in h_i with a particular root of $S_f(\sigma)$ gives the corresponding $g(s)$. It can be shown that the stable $g(s)$ is obtained from the largest real root of $S_f(\sigma)$.

3. STATE FEEDBACK $\mathcal{H}_2$ CONTROL PROBLEM

3.1 Problem Formulation and Optimal Performance

The focal point of this paper is an $\mathcal{H}_\infty$ control problem, but a state feedback $\mathcal{H}_2$ control problem is considered in this section. The reason is twofold:

- To see how the solution to the $\mathcal{H}_2$ control problem is characterized in terms of the SoR, in comparison to the $\mathcal{H}_\infty$ control problem case.
- To present an approach that can in fact be utilized when dealing with the case of unstable $P(s)$ in the $\mathcal{H}_\infty$ control problem.

The $\mathcal{H}_2$ control problem is formulated as follows. In Fig. 1, $P(s)$ is the single-input-single-output (SISO) plant to be controlled and F is the constant feedback gain. The signals x, u, y, and d are the plant state, the control input, the control output, and the disturbance input, respectively. The control input u is computed by $u = Fx$. The task is to find a feedback gain F that stabilizes the closed-loop system and further minimizes the $\mathcal{H}_2$-norm of the transfer function matrix $T_{d\to\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)}$ from d to $\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$.

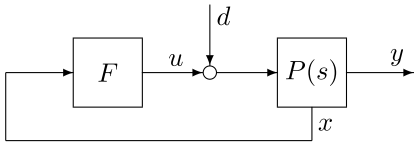

Fig. 1. State feedback system configuration

The problem can be solved as follows [Zhou et al., 1996]. Write the state-space representation of n-th order $P(s)$ as

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$, and (A, B) and (C, A) are assumed to be controllable and observable, respectively. Define the associated Hamiltonian matrix as

$$H_2 := \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix}. \tag{4}$$

The eigenvalues of H_2 are symmetric about the imaginary axis. The controllability and observability assumption guarantees that H_2 has no eigenvalues on the imaginary axis. Then there are n eigenvalues in the open left and right half planes (LHP/RHP) each. Stack the eigenvectors corresponding to the LHP eigenvalues and create the matrix

$$\begin{bmatrix} X_a^{\mathrm{T}} & X_b^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{5}$$

where $X_a, X_b \in \mathbb{C}^{n\times n}$. The now standard result shows the following [Zhou et al., 1996, Subsection 14.8.1].

Lemma 2. In the state feedback $\mathcal{H}_2$ control problem formulated above, X_a is invertible and $X_2 := X_b X_a^{-1} \geq 0$. Moreover, $X = X_2$ is the stabilizing solution to the algebraic Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0\ . \tag{6}$$

That is,

$$A_2 := A - BB^*X_2 \tag{7}$$

has eigenvalues in the open LHP only. Finally the optimal feedback gain is given as $F_{2,\mathrm{opt}} := -B^*X_2$, and the achieved minimum $\mathcal{H}_2$-norm is

$$\min_{F \text{ stabilizing}} \left\| T_{d\to\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)} \right\|_2 = \left(\mathrm{tr}\,\{B^*X_2B\}\right)^{\frac{1}{2}}.$$

3.2 Characterization in Terms of the Sum of Roots

The state feedback $\mathcal{H}_2$ control problem can be solved by way of the SoR and also the solution can be characterized by the SoR [Hara and Kanno, 2008]. As is mentioned the Hamiltonian matrix H_2 does not have eigenvalues on the imaginary axis and thus its characteristic polynomial $f_2(s) := \det(sI - H_2)$ does not have imaginary axis roots either, which is one of features in the $\mathcal{H}_2$ control problem. This case allows a stable monic $g_2(s)$ to be obtained such that $(-1)^n f_2(s) = g_2(s) g_2^{\sim}(s)$, and this polynomial spectral factor $g_2(s)$ in fact coincides with the characteristic polynomial of A_2. Moreover a closer relationship between $g_2(s)$ and X_2 can be revealed. Without loss of generality it can be assumed that $P(s)$ is given in controllable canonical form. That is, if $P(s)$ is written as a transfer function

$$P(s) = \frac{c_{n-1}s + \cdots + c_0}{s^n + a_{n-1}s + \cdots + a_0}\ ,$$

where the numerator and the denominator are assumed to be coprime, then its state-space representation is given as

$$P(s) = \left[\begin{array}{cccc|c} -a_{n-1} & \cdots & -a_1 & -a_0 & 1 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ \hline c_{n-1} & \cdots & c_1 & c_0 & 0 \end{array}\right] .$$

Note that the 'A' matrix is a companion matrix and that its characteristic polynomial (i.e., the denominator of $P(s)$) can immediately be determined from the first row.

Now it is shown that the stabilizing solution can be constructed from $g_2(s)$. Due to the special structure of the matrices A and B arising from the controllable canonical form assumption, A_2 shows a useful structure. Write X_2 as

$$X_2 = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{1,2} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n,n} \end{bmatrix} . \tag{8}$$

Then,

$$A_2 = -\begin{bmatrix} a_{n-1} + x_{1,1} & \cdots & a_1 + x_{1,n-1} & a_0 + x_{1,n} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} .$$

Namely, A_2 is also a companion matrix and it is immediate to relate the coefficients of $g_2(s)$ and the first row of X_2. If $g_2(s)$ is written as in the right hand side of (2), then

$$x_{1,1} = \sigma - a_{n-1} \ , \ x_{1,i} = m_{n-i} - a_{n-i} \ , \ i = 2, \ldots, n \ . \tag{9}$$

Moreover the structure of the 'B' matrix implies that the optimal cost can be expressed as $\sqrt{x_{1,1}}$. Then the above relationship reveals the following, which provides the SoR characterization for the optimal $\mathcal{H}_2$ cost.

Proposition 3. The best achievable $\mathcal{H}_2$-norm by state feedback is represented by

$$\min_{F \text{ stabilizing}} \left\| T_{d \to \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)} \right\|_2 = \sqrt{\sigma - a_{n-1}} \ .$$

Note that other elements of X_2 can also be expressed in terms of m_i, a_i and c_i. However they are not required here and hence not pursued.

Finally the characterization in terms of the SoR is discussed. Stable $g_2(s)$ that is to be found here corresponds to the largest real root of $S_f(\sigma)$ and moreover the largest real root of $S_f(\sigma)$ is always simple [Kanno et al., 2007b]. Also the optimal feedback gain is simply obtained from the first row of X_2 which has a simple relationship (9) to the SoR σ and the coefficients m_i of $g_2(s)$. Since m_i can be expressed as polynomials in σ, the optimal feedback gain is also expressed explicitly in terms of σ. Lastly, σ gives a simple expression for the optimal cost; the optimal cost is the square root of the difference between σ and a_{n-1}.

4. STATE FEEDBACK $\mathcal{H}_\infty$ CONTROL PROBLEM

4.1 Problem Formulation and Solution

Now consider the main problem of this paper, namely, the $\mathcal{H}_\infty$ control problem with state feedback. The problem formulation is identical to that of the $\mathcal{H}_2$ control problem dealt with in Section 3, except that the $\mathcal{H}_\infty$-norm is used instead of the $\mathcal{H}_2$-norm. The same feedback configuration in Fig. 1 is used and again a feedback gain F is sought that stabilizes the closed-loop system and also minimizes the $\mathcal{H}_\infty$-norm of the transfer function matrix $T_{d \to \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)}$.

Given the state-space representation of $P(s)$ in (3), where (A, B) and (C, A) are assumed controllable and observable, respectively, the Hamiltonian matrix associated to the sub-optimal problem with the $\mathcal{H}_\infty$-norm level $\gamma > 0$, i.e.,

$$\left\| T_{d \to \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)} \right\|_\infty < \gamma \tag{10}$$

is written as

$$H_\infty := \begin{bmatrix} A & (\gamma^{-2} - 1)BB^* \\ -C^*C & -A^* \end{bmatrix} . \tag{11}$$

Superficially the only difference from H_2 in (4) is the $(1, 2)$-block element. The eigenvalues of H_∞ are symmetric about the imaginary axis as well. However there are a number of differences between H_∞ and H_2:

- The matrix H_∞ contains a real parameter γ that is related to the performance level.
- The $(1, 2)$-block element of H_∞ can be both positive and negative semi-definite, depending on the value of γ, while that of H_2 is always negative semi-definite.
- The matrix H_∞ may have imaginary axis eigenvalues, depending on the value of γ, while H_2 is always free from eigenvalues on the imaginary axis.

When H_∞ does not have eigenvalues on the imaginary axis, the matrix (5) can be constructed based on the eigenvectors corresponding to the LHP eigenvalues, as in the $\mathcal{H}_2$ control problem case. The following is immediate from the standard result [Zhou et al., 1996, Theorem 16.9].

Lemma 4. There exists a stabilizing feedback gain F satisfying (10) if and only if

a) H_∞ has no eigenvalues on the imaginary axis;
b) X_a is invertible; and
c) $X_\infty := X_b X_a^{-1} \geq 0$.

Furthermore, when the above conditions are satisfied, a (sub-optimal) stabilizing feedback gain achieving the norm requirement is given by

$$F_{\infty,\mathrm{sub}} := -B^* X_\infty \ . \tag{12}$$

Also, $X = X_\infty$ is the stabilizing solution to the following algebraic Riccati equation:

$$A^*X + XA + (\gamma^{-2} - 1)XBB^*X + C^*C = 0 \ . \tag{13}$$

4.2 Optimal Performance

Given γ, Lemma 4 yields a method that determines whether the condition (10) can be achieved and, if achievable, finds a feedback gain satisfying it. Thus, by way of a bi-section approach, one may find the best possible γ, i.e.,

$$\gamma_{\mathrm{opt}} := \inf_{F \text{ stabilizing}} \left\| T_{d \to \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)} \right\|_\infty .$$

Nevertheless this approach of finding γ_{opt} is not suited in case of parametric $P(s)$ since the eigenvalues/eigenvectors cannot in general be computed explicitly. Indeed it has not been clear whether there is a tractable way to link plant parameters and the optimal performance γ_{opt}.

This section provides a result that serves as a basis when devising a method that can express γ_{opt} in the parametric case. The following result shows that characterizing γ_{opt} is relatively easy in the state feedback $\mathcal{H}_\infty$ control problem and also indicates which condition to investigate for finding $\gamma_{\rm opt}$. The proof is omitted due to space limitation.

Lemma 5. In the $\mathcal{H}_\infty$ state feedback problem formulated in Subsection 4.1, the best achievable performance level $\gamma_{\rm opt}$ is strictly less than 1 when $P(s)$ is stable, and is equal to 1 when $P(s)$ is unstable:

$$\gamma_{\rm opt} \begin{cases} < 1 & P(s) \text{ stable },\\ = 1 & P(s) \text{ unstable }. \end{cases}$$

Moreover, for stable $P(s)$, when $\gamma = \gamma_{\rm opt}$, Condition a) is violated, i.e., H_∞ has eigenvalues on the imaginary axis.

Lemma 5 first suggests that the stable and unstable cases should be treated differently. For unstable $P(s)$, one can deduce that $\gamma_{\rm opt} = 1$ without any calculation, and it may be proven that $\gamma_{\rm opt} = 1$ cannot in general be accomplished. Thus the main task in this case will be to derive an expression of the sub-optimal gain $F_{\infty,\rm sub}$ for $\gamma > 1$.

On the contrary Lemma 5 implies that finding $\gamma_{\rm opt}$ for stable $P(s)$ requires some computation. Lemma 5 also suggests an idea on how to compute $\gamma_{\rm opt}$. More specifically candidates for $\gamma_{\rm opt}$ can be found by investigating the condition under which H_∞ has imaginary axis eigenvalues. This point is exploited in Section 6.

Before closing this section, new variables are introduced to simplify expressions appearing in the sequel:

$$\rho := 1 - \gamma^{-2}\ , \quad \rho_{\rm opt} := 1 - \gamma_{\rm opt}^{-2}\ .$$

The following relationship between γ and ρ is immediate:

$$\gamma \geq 1 \iff 0 \leq \rho \leq 1\ , \quad 0 < \gamma < 1 \iff \rho < 0\ .$$

The sign of ρ directly indicates which case, stable or unstable $P(s)$, is treated, and thus this notation is also expected to facilitate to clarify the stable/unstable situation.

5. SUM OF ROOTS CHARACTERIZATION FOR THE UNSTABLE CASE

This section discusses the unstable case. It is immediate that $\rho_{\rm opt} = 0$ from Lemma 5 and the optimum cannot in general be achieved. Therefore the only interest may be how the feedback gain (12) is expressed in terms of $\rho > 0$.

With the 'ρ' notation, H_∞ in (11) is expressed as

$$H_\infty := \begin{bmatrix} A & -\rho BB^* \\ -C^*C & -A^* \end{bmatrix}.$$

Since $\rho > 0$, this Hamiltonian matrix is considered identical in essence to the Hamiltonian matrix H_2 appearing in the $\mathcal{H}_2$ control problem by considering $\sqrt{\rho}B$ as B in (4). Hence the approach stated for the $\mathcal{H}_2$ control problem in Section 2 is directly applicable for finding the feedback gain (12) for any $\rho > 0$, or, equivalently, $\gamma > 1$.

A simple numerical example is provided to demonstrated how the performance level ρ and the 'average stability' σ may be related by using the techniques presented in Sections 2 and 3. Consider the plant $P(s) = \frac{p+s}{s(p-s)}$, $p > 0$. The associated Hamiltonian matrix is

$$H_\infty := \begin{bmatrix} p & 0 & -\rho & 0 \\ 1 & 0 & 0 & 0 \\ -1 & -p & -p & -1 \\ -p & -p^2 & 0 & 0 \end{bmatrix}.$$

This is an unstable plant and thus ρ should be taken to be $\rho > 0$. The characteristic polynomial of H_∞ is $f_\infty(s) = s^4 - (p^2+\rho)s^2 + p^2\rho$. The approach in Section 2 gives a polynomial equation relating σ and ρ: $\sigma^4 - 2(p^2 + 2\rho)\sigma^2 + p^4 - 2p^2\rho + \rho^2 = 0$. This equation can in fact be solved for σ and the largest real root is $\sigma = p + \sqrt{\rho}$. This suggests that the 'average stability' σ monotonically increases as ρ increases from 0 to 1. In this particular example, one of the close-loop poles is $-p$ irrespective of the value of ρ, and thus the other pole moves from 0 to 1 as ρ increases from 0 to 1.

6. SUM OF ROOTS CHARACTERIZATION FOR THE STABLE CASE

This section focuses on the case of stable $P(s)$ where, unlike the unstable case, $\rho_{\rm opt}$ depends on $P(s)$. An attempt is made to characterize ρ in terms of the SoR and it is shown that such characterization is possible for $P(s)$ with parameters. Subsection 6.1 considers the non-parametric case and discusses how σ and ρ can be related. Subsection 6.2 then deals with the parametric case and shows that the main task is to divide the parameter region into several sub-regions of identical property. A method of computing the optimal feedback gain is discussed in Subsection 6.3.

6.1 Characterization of $\gamma_{\rm opt}$: Non-parametric Case

This subsection states a condition derived from Lemma 5 that shows how the existence of imaginary axis eigenvalues of the Hamiltonian matrix H_∞ can be characterized in terms of the SoR σ. The result can then yield an expression linking $\rho_{\rm opt}$ and σ. Denote by $f_\infty(s)$ the characteristic polynomial of H_∞ in (11), i.e.,

$$f_\infty(s) := \det(sI - H_\infty)\ , \tag{14}$$

and consider its polynomial spectral factorization $(-1)^n f_\infty(s) = g_\infty(s)g_\infty^\sim(s)$. Write $g_\infty(s)$ as in the right hand side of (2). A set of polynomial equations in terms of σ and m_i can be obtained by comparing the coefficients of the both sides of the equation. The shape basis of the set of equations is then computed. Denote by $S_f(\sigma)$ the polynomial in σ only in the shape basis, as in Proposition 1.

Theorem 6. For stable $P(s)$, the following holds.

1) When ρ is fixed such that $\rho > \rho_{\rm opt}$, or, equivalently, when H_∞ does not have eigenvalues on the imaginary axis, there exists at least two real roots in $S_f(\sigma)$ and the largest real root is always simple.
2) When ρ is $\rho_{\rm opt}$ (i.e., when H_∞ has multiple eigenvalues on the imaginary axis), $S_f(\sigma)$ has at least one real root and the largest real root of $S_f(\sigma)$ is repeated.
3) For a fixed $\rho < \rho_{\rm opt}$, either
   a) *[Generic Case]* $S_f(\sigma)$ does not have any real roots; or
   b) *[Degenerate Case]* $S_f(\sigma)$ has real roots and the largest real root is repeated.

The above theorem is stated to elucidate the mode transition as ρ decreases from 0. However the case $\rho = \rho_{\rm opt}$ can in fact be considered as a special case of 3-b). In the generic situation, when ρ is reduced from 0, the largest real root of $S_f(\sigma)$ which is initially a simple root becomes repeated at $\rho = \rho_{\rm opt}$ and then the repeated roots leave the real axis to become a pair of complex conjugate roots. Also, at that time, $S_f(\sigma)$ does not have real roots.

Theorem 6 gives an algorithm deriving a polynomial one of whose roots is $\rho_{\rm opt}$. When $\rho = \rho_{\rm opt}$, $S_f(\sigma)$ has a multiple root, so a condition that $\rho_{\rm opt}$ has to satisfy can be obtained from the discriminant $h(\rho)$ of $S_f(\sigma)$ with respect to σ. One can further choose the correct $\rho_{\rm opt}$ from the roots of $h(\rho)$ based on the property of the SoR stated in Case 1).

Algorithm O (Preparation)

Step O-1: Construct the Hamiltonian matrix H_∞ in (11) from the state-space representation of $P(s)$. Compute the characteristic polynomial $f_\infty(s)$ of H_∞ in (14).

Step O-2: Write $g_\infty(s)$ as in the right hand side of (2). Compare the both sides of (1) to get $\mathcal{G}$. Compute the shape basis for $\langle \mathcal{G} \rangle$. Compute the discriminant $h(\rho)$ of $S_f(\sigma)$ with respect to σ.

Algorithm I

Step I-1: Compute the negative roots of $h(\rho)$. They are candidates for $\rho_{\rm opt}$.

Step I-2: Compute the middle points of these candidates. From the largest middle point, substitute the value of each middle point in ρ of $S_f(\sigma)$ and check whether there exist real roots in $S_f(\sigma)$. If yes, then pick up the next largest middle point and repeat this step. Otherwise, set to $\rho_{\rm opt}$ the candidate which is just above the chosen middle point and terminate.

6.2 Characterization of $\gamma_{\rm opt}$: Parametric Case

This subsection discusses the case of parametric $P(s)$ and indicates that the crucial point in this case is to identify the regions of parameters exhibiting identical properties for $\rho_{\rm opt}$ in each region. This is in fact cylindrical algebraic decomposition (CAD) [Collins, 1975], an established method of computer algebra. Once such parameter regions are obtained, one has to select a sample point from each region and then to examine the property for $h(\rho)$ constructed from those sample points. In that sense the parametric case requires an outer loop for Algorithm I.

The discussion in the sequel is confined to the one parameter case for simplicity, but it is emphasized that a more general multiple parameter case can be dealt with by the CAD approach in a systematic manner. Denote the only parameter as α. Note first that Algorithm O works for the parametric case without any change. The discriminant $h(\rho)$ of $S_f(\sigma)$ is a polynomial in ρ and α in this case. Compute further the discriminant of $h(\rho)$ with respect to ρ, which is a polynomial in α. The real roots α of this polynomial indicate at which values of α the discriminant $h(\rho)$ has multiple roots when seen as a polynomial in ρ, and further those roots divide parameter regions. Taking (any) one value from each region, one can examine which root of $h(\rho)$ corresponds to $\rho_{\rm opt}$ throughout the region. The examination can be carried out by means of Algorithm I. To sum up the outer loop required for the parametric case is written as follows.

Algorithm II

Step II-1: Compute the discriminant of $h(\rho)$ with respect to ρ. Compute the real roots α_i of the discriminant.

Step II-2: Choose arbitrarily a value between two consecutive α_i and substitute α in $h(\rho)$ by that value. Execute Algorithm I to find out which root of $h(\rho)$ corresponds to $\rho_{\rm opt}$.

6.3 Characterization of the Feedback Gain

Now consider the computation of the feedback gain F. To this end the solution X_∞ to the Riccati equation (13) is constructed from $g_\infty(s)$, as in the $\mathcal{H}_2$ problem stated in Section 3, and it is shown that X_∞ is also characterized in terms of σ. So as to ensure the existence of the positive semi-definite X_∞, it is assumed that $\rho \geq \rho_{\rm opt}$ throughout.

Again it is assumed that $P(s)$ is given in controllable canonical form. Also write X_∞ as in the right hand side of (8). First note that the characteristic polynomial of

$$A - \rho B B^* X_\infty \tag{15}$$

is identical to $g_\infty(s)$. Again, due to the special structure of the matrices A and B arising from the controllable canonical form assumption, it is easy to see that

$$A - \rho BB^*X_\infty = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} - \rho \begin{bmatrix} x_{1,1} & \cdots & x_{1,n-1} & x_{1,n} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} .$$

Notice thus that (15) is a companion matrix as well. Therefore the following relationship can be observed:

$$\begin{cases} \rho x_{1,1} = \sigma - a_{n-1} , \\ \rho x_{1,i} = m_{n-i} - a_{n-i} , \quad i = 2, \ldots, n . \end{cases} \tag{16}$$

The optimal feedback gain is the first row of X_∞ and hence it can be seen that the elements of the feedback gain matrix can be characterized in terms of the SoR.

7. ILLUSTRATIVE EXAMPLE

This section demonstrates the developed approach by way of the following numerical example:

$$P(s) = \frac{1}{s^2+s+\alpha} , \quad \alpha > 0 ,$$

where α is a parameter. For any value of $\alpha > 0$, $P(s)$ is stable and hence $\rho_{\rm opt} < 0$. Thus an investigation is made on how the value of $\rho_{\rm opt}$ changes as α changes by means of Algorithms O, I, and II. Algorithms O is executed first.

Step O-1: The state-space representation of $P(s)$ in controllable canonical form is given as

$$P(s) = \left[\begin{array}{cc|c} -1 & -\alpha & 1 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right] .$$

The associated Hamiltonian matrix is then

$$H_\infty := \begin{bmatrix} -1 & -\alpha & -\rho & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & \alpha & 0 \end{bmatrix} .$$

Its characteristic polynomial is

$$f_\infty(s) := \det(sI - H_\infty) = s^4 + (2\alpha - 1)s^2 + \rho + \alpha^2 .$$

Step O-2: Write $g_\infty(s)$ as $g_\infty(s) = s^2 + \sigma s + m_0$, and compare the coefficients of the both sides of (1) to get

$$\sigma^2 - 2m_0 + 2\alpha - 1 = 0 , \quad m_0^2 - \rho - \alpha^2 = 0 .$$

Then, $S_f(\sigma)$ is obtained as

$$S_f(\sigma) = \sigma^4 + (4\alpha - 2)\sigma^2 - 4\rho - 4\alpha + 1 .$$

Take the discriminant of $S_f(\sigma)$ with respect to σ and compute its square-free part to get

$$h(\rho) = (4\rho + 4\alpha - 1)(\rho + \alpha^2) \ . \tag{17}$$

Since this is a parametric case, Algorithm II is invoked.
Step II-1: Compute the square-free part of the discriminant of $h(\rho)$ with respect to ρ:

$$2\alpha - 1 \ . \tag{18}$$

This suggests that, at $\alpha = \frac{1}{2}$, the polynomial (17) has multiple roots when seen as a polynomial in ρ. The two roots of (17) are plotted against α in Fig. 2, which confirms that (17) has multiple roots when $\alpha = \frac{1}{2} (= 0.5)$. For other values of α, there are two distinctive roots. One of the two roots is the true ρ_{opt}, which can be selected based on Theorem 6. For each region, $(0, \frac{1}{2})$ and $(\frac{1}{2}, +\infty)$, it is invariant which root (the larger or the smaller) corresponds to the true ρ_{opt}. Thus one has to choose a sample point from each region and examine the two roots of (17) in order to identify the true ρ_{opt}. Technically, (18) helps to find out delineable regions. As is mentioned before the approach is CAD [Collins, 1975] manually carried out.
Step II-2: Now that two parameter regions to be examined are identified, a sample point is selected from each region and Algorithm I is then invoked. Here two values of α, $\alpha = \frac{1}{5} (= 0.2)$, $\frac{4}{5} (= 0.8)$, are investigated.

$\underline{\alpha = \frac{1}{5}}$:

Step I-1: In this case, (17) has two roots: $\rho = \frac{1}{20} (= 0.05)$, $-\frac{1}{25} (= -0.04)$.
Step I-2: There is only one root below 0 and it is clear that the negative root corresponds to the true ρ_{opt}. However here it is confirmed based on Theorem 6. Let $\rho = \frac{1}{25} (= 0.04)$. Then, $S_f(\sigma) = \sigma^4 - \frac{6}{5}\sigma^2 + \frac{1}{25}$, and this has 4 simple real roots, which concludes that $\rho_{\mathrm{opt}} < \frac{1}{25}$. Next let $\rho = -\frac{1}{20} (= -0.05)$. Then, $S_f(\sigma) = \sigma^4 - \frac{6}{5}\sigma^2 + \frac{2}{5}$. This polynomial has no real root, i.e., $\rho_{\mathrm{opt}} > -\frac{1}{20}$. These facts conclude that $\rho_{\mathrm{opt}} = -\frac{1}{25}$.

$\underline{\alpha = \frac{4}{5}}$:

Now consider the other region. This time, (17) has two negative roots: $\rho = -\frac{11}{20} (= -0.55)$, $-\frac{16}{25} (= -0.64)$, and thus the both roots are candidates for ρ_{opt}. The following sample values of ρ are examined in turn: $\rho = -\frac{3}{5} (= -0.6)$, $-\frac{4}{5} (= -0.8)$.

- $\rho = -\frac{3}{5}$: This case yields $S_f(\sigma) = \sigma^4 + \frac{6}{5}\sigma^2 + \frac{1}{5}$. This one has purely imaginary roots only and it can be conclude that $\rho_{\mathrm{opt}} > -\frac{3}{5}$. That is the larger root of (17) corresponds to the true ρ_{opt} in the region $(\frac{1}{2}, +\infty)$ for α.
- $\rho = -\frac{4}{5}$: Just to make sure, this case is examined and $S_f(\sigma) = \sigma^4 + \frac{6}{5}\sigma^2 + 1$. Again this one has no real roots, confirming that $\rho_{\mathrm{opt}} > -\frac{4}{5}$.

The above examination allows one to draw the solid curve in Fig. 2 for the true ρ_{opt} against α.

8. CONCLUDING REMARKS

This paper has considered the characterization of the solution to the parametric state feedback $\mathcal{H}_\infty$ control problem in terms of the sum of roots. An approach is developed that can link the optimal performance level and also the optimal/sub-optimal feedback gain with plant parameters by means of the sum of roots and the Gröbner basis technique. In this way the sum of roots characterization is shown possible to some $\mathcal{H}_\infty$ control problem.

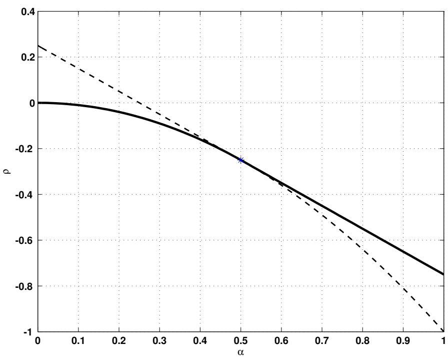

Fig. 2. Roots ρ of (17) against α

It is expected that optimization of the best performance level over parameters can be formulated in an algebraic manner, just as in the $\mathcal{H}_2$ control problem [Kanno et al., 2007b]. Then the suggested approach would work as a preprocessor to algebraic optimization methods such as the one based on quantifier elimination [Weispfenning, 1997].

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REFERENCES

H. Anai, S. Hara, and K. Yokoyama. Sum of roots with positive real parts. In *Proceedings of the ACM SIGSAM International Symposium on Symbolic and Algebraic Computation, ISSAC2005*, pages 21–28, 2005.

G. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In *Proceedings Second GI Conference on Automata Theory and Formal Languages*, volume 33 of *Lecture Notes in Computer Science*, pages 134–183, Berlin, 1975. Springer-Verlag.

J. C. Faugère, P. Gianni, D. Lazard, and T. Mora. Efficient computation of zero-dimensional Gröbner bases by change of ordering. *Journal of Symbolic Computation*, 16(4):329–344, October 1993.

S. Hara and M. Kanno. Sum of roots characterization for $\mathcal{H}_2$ control performance limitations. *SICE Journal of Control, Measurement, and System Integration*, 1(1): 58–65, January 2008.

M. Kanno, S. Hara, H. Anai, and K. Yokoyama. Sum of roots, polynomial spectral factorization, and control performance limitations. In *Proceedings of the 46th IEEE Conference on Decision and Control*, pages 2968–2973, New Orleans, Louisiana USA, December 2007a.

M. Kanno, K. Yokoyama, H. Anai, and S. Hara. Parametric optimization in control using the sum of roots for parametric polynomial spectral factorization. In *Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISSAC 2007*, pages 211–218, Waterloo, Ontario, Canada, July-August 2007b.

V. Weispfenning. Simulation and optimization by quantifier elimination. *Journal of Symbolic Computation*, 24 (2):189–208, August 1997.

K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, NJ, 1996.