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Geometric analysis of a class of constrained mechanical control systems in the nonzero velocity setting ⋆

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Abstract: We obtain an intrinsic vector-valued symmetric bilinear form that can be associated with an underactuated constrained mechanical control system. We determine properties of the form that serve as sufficient conditions for driving a constrained mechanical system underactuated by one control to an ϵ-neighborhood of rest from an arbitrary initial configuration and velocity. We also determine properties of the form which serve as necessary conditions. These conditions are computable and coordinate invariant.

1. INTRODUCTION

1.1 Background

Mechanics and control theory are two well developed fields of study. However, their intersection still provides a rich and challenging research area commonly referred to as geometric control of mechanical systems. Applications can be found in diverse fields such as robotics, autonomous aerospace and marine vehicles, multi-body systems and constrained systems. The formalism of affine connections and distributions on a Riemannian manifold provides an elegant framework for modeling, analysis and control. This framework has given rise to new insights into nonlinear controllability in the *zero velocity setting* motivating motion planning algorithms [1].

The impetus for this work is a standing limitation in the theoretical foundation of geometric control of mechanical systems. *The modern development of geometric control of mechanical systems has been limited, for the most part, to the zero velocity setting* [2]. Controllability results that are limited to zero velocity states restrict the development of motion planning algorithms and bound the extension of controllability and motion planning results to the larger class of hybrid nonlinear mechanical systems where switching occurs at nonzero velocity states [4].

One of the practical goals of a nonlinear controllability analysis for mechanical systems is to provide a structure for the development of motion planning algorithms for underactuated systems. A control system is underactuated if it has fewer actuators than degrees of freedom. The missing actuation can often be made up for by nonholonomic constraints, however the cost for the reduction in actuation is the increase complexity of the controller design.

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Mechanical control systems with constraints can be described by a so-called constrained affine connection. Consequently, the nonlinear controllability analysis developed by Lewis and Murray [11] was adapted to simple mechanical systems with constraints [9]. The local representation was then simplified by Bullo and Zefran [3]. This simplification lead to a more efficient method for computing the Christoffel symbols of the constrained affine connection. The Christoffel symbols play an important role in computing symmetric products which are used to characterize the structure of the reachable set from zero initial velocity.

A well-known limitation of the local controllability analysis of Lewis and Murray [11] and the adaption to constrained systems is that these results are not feedback invariant. Several efforts have been made to obtain conditions in the zero velocity setting from properties of a certain intrinsic vector-valued quadratic form which does not depend upon the choice of basis for the input distribution [2], [6]. Recently, it has been observed that vector-valued quadratic forms come up in a variety of areas in control theory which has motivated a new initiative to understand the geometry of these forms [7].

1.2 Statement of Contribution

The contribution of this paper is twofold. First, we develop a novel geometric tool that can be used to characterize the set of reachable velocities in the nonzero velocity setting. Our unique approach is to use the governing equations of motion to partition or foliate the velocity-phase manifold. We develop a method to measure a constrained mechanical control system's ability to move among leaves of the foliation.

Second, we provide a general test for stopping a constrained mechanical control system. We obtain computable results which are dependent upon coordinate invariant properties of an intrinsic vector-valued symmetric bilinear form that can be associated with a constrained mechanical control system. Specifically, we provide necessary conditions for reaching rest from an arbitrary initial configuration and velocity (see Section III.C, Theorem 9). We also provide sufficient conditions for driving a class of constrained mechanical system underactuated by one control to an ϵ-neighborhood of rest from an arbitrary initial configuration and velocity (see Section III.C, Theorem 12). The constructive nature of our results naturally gives rise to a nonlinear control law for moving among the leaves of the input foliation however we do not provide an explicit algorithm. These results are applied to two well-known mechanical control systems with nonholonomic constraints: the roller racer [8] and the snakeboard [12].

2. GEOMETRIC MODEL

2.1 Simple Mechanical Control Systems

We consider a simple mechanical control system which is comprised of an n-dimensional configuration manifold M; a Riemannian metric $\mathbb{G}$ which represents the kinetic energy; a $\mathbb{R}$-valued function V on M which represents the potential energy; m linearly independent one forms $F^1, \ldots, F^m$ on M which represents the input forces; a distribution H on M which represents the constraint distribution; and $U = \mathbb{R}^m$ which represents the set of inputs. We do not require the set of inputs to be a subset of $\mathbb{R}^m$. This allows us to focus on the geometric properties of our system that inhibit or allow motion in the foliation as opposed to a limitation on the set of inputs. We represent the input forces as one forms and use the associated dual vector fields $Y_a = \mathbb{G}^\sharp(F^a)$, $a = 1, \ldots, m$ in our computations. Formally, we denote the control system by the tuple $\Sigma = \{M, \mathbb{G}, H, \mathcal{Y}, V, U\}$ where $\mathcal{Y} = \{Y_a \mid Y_a = \mathbb{G}^\sharp(F^a) \ \forall\, a\}$ is the *input distribution.* Note we restrict our attention to control systems where the input forces are dependent upon configuration and independent of velocity and time. DoCarmo [13] provides an excellent introduction to Riemannian geometry. A thorough description of simple mechanical control systems is provided by Bullo and Lewis [1].

If we set the Lagrangian equal to the kinetic energy minus potential energy, then the equations are given by

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = -\operatorname{grad} V(\gamma(t)) + u^a(t)Y_a(\gamma(t)) \qquad (1)$$

where ∇ is the Levi-Civita connection corresponding to $\mathbb{G}$, u is a map from $I \subset \mathbb{R} \mapsto \mathbb{R}^M$, $\gamma : I \to M$ is a curve on M. A *controlled trajectory* for Σ is taken to be the pair (γ, u) where γ and u are defined on the same interval $I \subset \mathbb{R}$. Note the usual summation notation will be assumed over repeated indices throughout this paper.

Given a constraint distribution H of rank k, we may restrict the Levi-Civita connection ∇ to H [9]. Bullo and Zefran [3] showed that given two vector fields X and Y on M then the so-called *constrained affine connection* $\tilde\nabla$ is given by

$$\tilde\nabla_X Y = P\left(\nabla_X Y\right)$$

where P is the orthogonal projection $TM \mapsto H$. The latter approach provides a computationally efficient method and is used to generate the coordinate expression for the constrained affine connection for the roller racer and the snakeboard.

The natural coordinates on TM are denoted by $((q^1, \ldots, q^n), (v^1, \ldots, v^n))$ where $(v^1, \ldots, v^n)$ are the coefficients of a tangent vector given the usual basis $\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\}$. We will denote a point in TM by v_q. We may lift the second-order differential equation defined by (1) to TM. This gives rise to the following system of first-order differential equations on TM

$$\begin{aligned} \frac{dq^k}{dt} &= v^k, \\ \frac{dv^k}{dt} &= -\Gamma^k_{ij} v^i v^j + u^a Y^k_a - \operatorname{grad} V^k, \end{aligned} \qquad (2)$$

where Γ^k_{ij} is the usual Christoffel symbol and $i, j, k = 1, \ldots, n$. Equation (2) is the local representation of the following vector field on TM

$$\begin{aligned} \dot\xi(t) = Z\left(\xi(t)\right) + u^a(t) Y_a^{\mathrm{vlft}}\left(\pi\left(\xi(t)\right)\right) \\ - \operatorname{grad} V^{\mathrm{vlft}}\left(\pi\left(\xi(t)\right)\right). \end{aligned} \qquad (3)$$

where $\xi(t)$ is the total state, Z is the geodesic spray, π is the canonical projection $TM \mapsto M$, and $Y_a^{\mathrm{vlft}}, \operatorname{grad} V^{\mathrm{vlft}}$ are the vertical lifts of the vector fields $Y_a, \operatorname{grad} V$ on M. Recall that the vertical lift of a vector field X at the point v_q is denoted by $X^{\mathrm{vlft}}_{v_q}$ and is the tangent vector at $t = 0$ to the curve $t \mapsto v + tX$.

A critical tool used to analyze distributions and mechanical control systems is the symmetric product. Given a pair of vector fields X, Y, their symmetric product is the vector field defined by

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

We denote the closure of the distribution $\mathcal{Y}$ with respect to the symmetric product by $\mathrm{Sym}^\infty(\mathcal{Y})$. Recall that a distribution $\mathcal{Y}$ is geodesically invariant if and only if it is closed with respect to the symmetric product [10]. Also, we denote the flow of a vector field X on M by $\Phi^X_t : M \to M$.

2.2 Roller Racer

In this section we introduce the geometric model of the roller racer (RR) (Fig. 1).

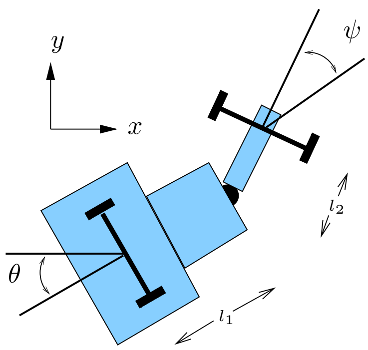

Fig. 1. Roller Racer

The configuration manifold for RR is $SE(2) \times \mathbb{S}$ with local coordinates (x, y, θ, ψ). The Riemannian metric is given by

$$\mathbb{G} = mdx \otimes dx + mdy \otimes dy + (I_1 + I_2)d\theta \otimes d\theta$$
$$+I_2 d\psi \otimes d\theta + I_2 d\theta \otimes d\psi + I_2 d\psi \otimes d\psi,$$

where $m > 0$ is the mass of the body of RR, $I_1 > 0$ is the moment of inertia of the body about its center of mass and $I_2 > 0$ is the moment of inertia of the wheel assembly about the pivot point. The constraint one-forms are given by

$$\omega_1 = \sin(\theta)\, dx - \cos(\theta)\, dy,$$
$$\omega_2 = \sin(\theta + \psi)\, dx - \cos(\theta + \psi)\, dy$$
$$-(l_2 + l_1 \cos(\psi)\, d\theta - l_2 d\psi.$$

The single control force is a pure torque $F^1 = d\psi$.

2.3 Snakeboard

In this section we introduce the geometric model of the snakeboard (SB) (Fig. 2).

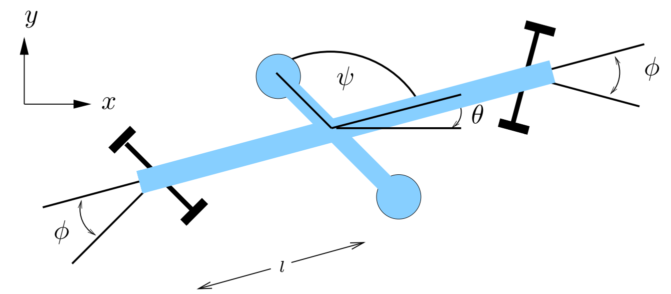

Fig. 2. Snakeboard

The configuration manifold for SB is $SE(2) \times \mathbb{S} \times \mathbb{S}$ with local coordinates $(x, y, \theta, \psi, \phi)$. The Riemannian metric is given by

$$\mathbb{G} = mdx \otimes dx + mdy \otimes dy + l^2 md\theta \otimes d\theta$$
$$+J_r d\psi \otimes d\theta + J_r d\theta \otimes d\psi + J_r d\psi \otimes d\psi + J_w d\phi \otimes d\phi,$$

where $m > 0$ is the total mass of SB, $J_r > 0$ is the moment of inertia of the rotor mounted on top of the body's center of mass, and $J_w > 0$ is the moment of inertia of the wheel axles. The constraint one-forms are given by

$$\alpha_1 = \sin(\phi - \theta)\, dx + \cos(\phi - \theta)\, dy + l\cos(\phi)\, d\theta,$$
$$\alpha_2 = -\sin(\phi + \theta)\, dx + \cos(\phi + \theta)\, dy - l\cos(\phi)\, d\theta.$$

The two control forces are pure torques $F^1 = d\psi$ and $F^2 = d\phi$.

3. GEOMETRIC ANALYSIS

3.1 Construction

In this section we expand upon and adapt the definition of an affine subbundle found in Hirschorn and Lewis [6]. We restrict our attention to configuration manifolds that admit a well defined global set of basis vector fields however our results generalize under appropriate conditions. *The basic geometry of our construction can be captured by assuming $H = TM$ however we can always relax this assumption by properly accounting for the orthogonal projection P.*

Recall that an input distribution $\mathcal{Y}$ on M is a subset $\mathcal{Y} \subset TM$ having the property that for each $q \in M$ there exists a family of vector fields $\{Y_1, \ldots, Y_m\}$ on M so that for each $q \in M$ we have

$$\mathcal{Y}_q \equiv \mathcal{Y} \cap T_q M = \mathrm{span}_{\mathbb{R}}\{Y_1(q), \ldots, Y_m(q)\}.$$

We refer to the vector fields $\{Y_1, \ldots, Y_m\}$ as generators for $\mathcal{Y}$. Let $\mathcal{Y}^\perp$ denote an orthonormal frame $\{Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ that generates the $\mathbb{G}$-orthogonal complement of the input distribution $\mathcal{Y}$. Note that even though $\mathcal{Y}^\perp$ is canonically define, we must choose an orthonormal basis. It is clear that $\{\mathcal{Y}_q, \mathcal{Y}_q^\perp\}$ form a basis for $T_q M$ at each $q \in M$. Note that $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ is a set of m *linearly independent* vector fields while $\mathcal{Y}^\perp = \{Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ is a set of $n - m$ *orthonormal* vector fields. This basis will be used to define an affine subbundle and construct an affine foliation of the tangent bundle.

An *affine subbundle* on M is a subset $A \subset TM$ having the property that for each $q \in M$ there exists a family of vector fields $\{Y_0, \ldots, Y_m\}$ so that for each $q \in U$ we have

$$A_q \equiv A \cap T_q M$$
$$= \{Y_0(q) = Y_1^\perp(q) + \cdots + Y_{n-m}^\perp(q)\}$$
$$+ \mathrm{span}_{\mathbb{R}}\{Y_1(q), \ldots, Y_m(q)\}.$$

An *affine foliation*, $\mathcal{A}$, on TM is a collection of disjoint immersed affine subbundles of TM whose disjoint union equals TM. Each connected affine subbundle A is called an *affine leaf* of the affine foliation. Now let us apply this framework to a simple mechanical control system.

Definition 1. (Input Foliation). Let $(M, \mathbb{G}, V, \mathcal{Y}, U)$ be a simple mechanical control system with the input distribution $\mathcal{Y}$ generated by $\{Y_1, \ldots, Y_m\}$ and the corresponding $\mathbb{G}$-orthogonal distribution $\mathcal{Y}^\perp$ generated by $\{Y_1^\perp, \ldots, Y_{n-m}^\perp\}$. An *input foliation* $\mathcal{A}_{\mathcal{Y}}$ is an affine foliation whose affine leaves are affine subbundles given by

$$A_s(q) = \{v_q \in TM \mid \langle\langle Y^\perp, v_q \rangle\rangle = s, s \in \mathbb{R}^{n-m}\}.$$

Remark 2. The input foliation is parametrized by $s \in \mathbb{R}^{n-m}$. Note that when $s = 0$, $A_0 = \mathcal{Y}$ and $A_0(q) = \mathcal{Y}_q$ where $\mathcal{Y}$ is an immersed submanifold of TM and $\mathcal{Y}_q$ is a linear subspace of $T_q M$. Thus, the input distribution $\mathcal{Y}$ is a single leaf of the affine foliation.

Given a basis of vector fields $\{X_1, \ldots, X_n\}$ on M, we define the *generalized Christoffel symbols* of ∇ to be

$$\nabla_{X_i} X_j = \hat{\Gamma}_{ij}^k X_k.$$

Note that when $X_i = \frac{\partial}{\partial q^i}$ we recover the usual Christoffel symbols of ∇. We introduce the *symmetrization* of the generalized Christoffel symbols.

Definition 3. We define the *generalized symmetric Christoffel symbols* for ∇ with respect to the basis of vector fields $\{X_1, \ldots, X_n\}$ on M as the n^3 functions $\tilde{\Gamma}^k_{ij} : M \to \mathbb{R}$ defined by

$$\begin{aligned}\tilde{\Gamma}^k_{ij} X_k &= \frac{1}{2}\left(\hat{\Gamma}^k_{ij} + \hat{\Gamma}^k_{ji}\right) X_k \\ &= \frac{1}{2}\langle X_i : X_j \rangle.\end{aligned}$$

We may define the velocity vector $\dot{\gamma}(t) = \dot{\gamma}^i(t)\frac{\partial}{\partial q^i}$ of the curve $\gamma(t)$ in terms of the family of vector fields $\{\mathcal{Y}, \mathcal{Y}^\perp\}$. The new expression for $\dot{\gamma}(t)$ is in the form

$$\dot{\gamma}(t) = w^a(t) Y_a(\gamma(t)) + s^b(t) Y_b^\perp(\gamma(t)) \tag{4}$$

where $s^b(t) = \langle\langle \dot{\gamma}(t), Y_b^\perp \rangle\rangle_{\gamma(t)}$. We now provide a local expression for a measure of a simple mechanical control system's ability to move among the leaves of the input foliation $\mathcal{A}_\mathcal{Y}$.

Lemma 4. Let $(M, \mathbb{G}, V, \mathcal{Y}, U)$ be a simple mechanical control system with an input foliation $\mathcal{A}_\mathcal{Y}$ defined above. The following holds along the curve $\gamma(t)$ satisfying (1):

$$\begin{aligned}\frac{d}{dt}s^b(t) = &-\frac{1}{2}w^a(t)w^p(t)\langle\langle\langle Y_a : Y_p\rangle, Y_b^\perp\rangle\rangle \\ &-\frac{1}{2}w^a(t)s^r(t)\langle\langle\langle Y_a : Y_r^\perp\rangle, Y_b^\perp\rangle\rangle \\ &-\frac{1}{2}s^r(t)w^p(t)\langle\langle\langle Y_r^\perp : Y_p\rangle, Y_b^\perp\rangle\rangle \\ &-\frac{1}{2}s^r(t)s^k(t)\langle\langle\langle Y_r^\perp : Y_k^\perp\rangle, Y_b^\perp\rangle\rangle \\ &-\langle\langle \operatorname{grad} V, Y_b^\perp\rangle\rangle\end{aligned} \tag{5}$$

where $a, p \in \{1, \ldots, m\}$, $b, k, r \in \{1, \ldots, n-m\}$.

Proof. Recall from the definition of an input foliation that

$$s^b(t) = \langle\langle Y_b^\perp, \dot{\gamma}(t)\rangle\rangle. \tag{6}$$

We could proceed by substituting (4) into (6) and differentiating taking advantage of the compatibility associated with the Levi-Civita connection. Alternatively, we use the notion of a generalized symmetric Christoffel symbol. It follows from the construction of $\mathcal{Y}^\perp$ that the bth component of $\tilde{\Gamma}^b_{ij}$ along the the orthonormal vector field $Y_b^\perp$ can be expressed as a projection using $\mathbb{G}$.

We observe that (5) is quadratic in the parameter $w(t)$. Now we relate an intrinsic vector-valued symmetric bilinear form to the measure derived in Lemma 4.

Definition 5. Let $(M, \mathbb{G}, V, \mathcal{Y}, U)$ be a simple mechanical control system with the input distribution $\mathcal{Y}$ generated by $\{Y_1, \ldots, Y_m\}$ and the corresponding $\mathbb{G}$-orthogonal distribution $\mathcal{Y}^\perp$ generated by $\{Y_{m+1}^\perp, \ldots, Y_n^\perp\}$. We define the *intrinsic vector-valued symmetric bilinear form* to be $B : \mathcal{Y}_q \times \mathcal{Y}_q \to \mathcal{D}_q^\perp$ given in coordinates by

$$B^b_{ap} w^a w^p = \frac{1}{2}\langle\langle\langle Y_a : Y_p\rangle, Y_b^\perp\rangle\rangle w^a w^p,$$

where $a, p \in \{1, \ldots, m\}, b \in \{1, \ldots, n-m\}$..

Remark 6. If Σ is underactuated by one control then $b = 1$ and B is a real-valued symmetric bilinear form.

The intrinsic vector-valued symmetric bilinear form defined above is an important measure of how the velocity components w parallel to the input forces influence the velocity components s orthogonal to the input forces. The remainder of the paper will focus on characterizing computable, coordinate invariant properties of B.

3.2 Control Definitions

The following section contains several control definitions which are used in the statement of our main results.

Definition 7. We say that Σ is **ϵ-stabilizable to rest** (ϵ-STR) if for any $\epsilon > 0$ there exists a piecewise continuous function $\tilde{u} : TM \to \mathbb{R}^m$ such that the solution to the initial value problem

$$\dot{\xi}(t) = Z(\xi(t)) + \tilde{u}^a(\xi(t)) Y_a^{vlft}(\xi(t)), \quad \xi(0) = (q_0, v_0),$$

satisfies $|v(T)| < \epsilon$ for some $q \in M$ and finite T.

Definition 8. Let B be a real-valued symmetric bilinear form on M and $w \in \mathcal{Y}_q$.

(ii) The *positive set* M^+ is the set of $q \in M$ such that $w^T B w > 0$ holds for $w \neq 0$.
(ii) The *negative set* M^- is the set of $q \in M$ such that $w^T B w < 0$ holds for $w \neq 0$.
(iii) The *indefinite set* $M^{+/-}$ is the set of $q \in M$ such that $w^T B w$ may take positive, negative and zero value for $w \neq 0$.
(iv) The *degenerate set* $M^\emptyset$ is the set of $q \in M$ such that $w^T B w = 0$ holds for all w.

3.3 Control Results

The following section contains the main results of this paper. Our goal is to determine conditions that can be expressed in terms of coordinate invariant properties of a real-valued symmetric bilinear form. Our first result is independent of the control set.

Theorem 9. Let $\Sigma = \{M, \mathbb{G}, V = 0, \mathcal{Y}\}$ be a simple mechanical system underactuated by one control. If $M = M^+$, $M = M^-$ or $M = M^\emptyset$ then Σ cannot be driven to rest from an arbitrary initial configuration and velocity.

Proof. It follows from our construction that if Σ is underactuated by one control then the input foliation $\mathcal{A}_\mathcal{Y}$ is a one-parameter family of affine leaves A_s. The affine leaf A_0 contains the zero-velocity vector. The first two assumptions above imply that the system can either reach all leaves above the initial leaf or reach all leaves below the initial leaf, respectively. Clearly, if the initial leaf lies above or below A_0, respectively, the system cannot be driven to A_0. The third assumption implies that the input foliation is invariant and that the system must remain on the initial leaf thus cannot move toward A_0 from an arbitrary initial configuration and velocity.

It follows from (1) that if we choose u^a sufficiently large, w^a can achieve any value for each $a = 1, \ldots, m$. We need to show that it is possible to do this with an arbitrarily small influence on the configuration q and orthogonal velocity component s. This idea is captured by the following lemma.

Lemma 10. Let $q^i(t_1)$, $w^a(t_1)$ and $s^b(t_1)$ be initial conditions for (1). For any constants $D, \delta, \epsilon > 0$ and d^a with $-D < d^a < D$ there exists a constant $M > 0$ and $u^a(t) \in C(t_1, t_2)$ such that $|t_1 - t_2| < \delta$ and $|u^a(t)| < M$ for all $t \in [t_1, t_2]$ and the following conditions are satisfied:

(i) $|w(t) - w(t_1)| < D$ for all $t \in [t_1, t_2]$,
(ii) $w^a(t_2) = d^a$,
(iii) $|q(t) - q(t_1)| < \epsilon$, $|s(t) - s(t_1)| < \epsilon$ for all $t \in [t_1, t_2]$,

where $|\cdot|$ is the appropriate Euclidean norm.

Proof. The proof is a simple consequence of the Mean Value Theorem for vector-valued functions.

Lemma 11. Let $q^i(t_1)$, $w^a(t_1)$ and $s(t_1) < 0$ be initial conditions for (1) and let $q \in M^+$. For any $C, \beta, \sigma > 0$ there exists a constant $N > 0$ and $u^a(t) \in C(t_1, t_2)$ such that $|t_1 - t_2| < \beta$ and $|u^a(t)| < N$ for all $t \in [t_1, t_2]$ and the following conditions are satisfied:

(i) $|s(t) - s(t_1)| < C$ for all $t \in [t_1, t_2]$,
(ii) $s(\tau) = 0$ for some $\tau \in (t_1, t_2)$,
(iii) $|q(t) - q(t_1)| < \sigma$ for all $t \in [t_1, t_2]$,

where $|\cdot|$ is the appropriate Euclidean norm.

Proof. Again, the proof is a consequence of the Mean Value Theorem for vector-valued functions and that $\frac{ds}{dt}$ is quadratically dependent upon w whereas $\frac{dq}{dt}$ is linearly dependent on w. If we choose w large enough for arbitrarily small time then s can be driven to rest while keeping q within an ϵ-neighborhood of the initial conditions.

Similar statements hold for $q \in M^-$ and $q \in M^{+/-}$ however we omit them for the sake of brevity. Now we state our main result.

Theorem 12. Let $\Sigma = \{M, \mathbb{G}, V, \mathcal{Y}, U\}$ be a simple mechanical system underactuated by one control. If the critical points $q^\star$ of $det(B(q))$ satisfy the following conditions:

(i) $det(B(q^\star)) - det(B(\Phi^{Y_a}_\delta(q^\star)) < 0$ for all $q^\star \in cl(M^+)$,

(ii) if m is odd, $det(B(q^\star)) - det(B(\Phi^{Y_a}_\delta(q^\star)) > 0$ for all $q^\star \in cl(M^-)$,

(iii) if m is even, $det(B(q^\star)) - det(B(\Phi^{Y_a}_\delta(q^\star)) < 0$ for all $q^\star \in cl(M^-)$,

and if for all $q \in M^\emptyset$

(iv) $det(B(q)) - det(B(\Phi^{Y_a}_\delta(q)) \neq 0$,

for some $a = 1, \ldots, m$ and $|\delta| << 1$ then Σ is ϵ-stabilizable to rest from an arbitrary initial configuration and velocity.

Proof. It follows from Lemma 10 and Lemma 11 that it is sufficient to show that if the regions M^+, M^- or $M^\emptyset$ exist the sufficient conditions $(i) - (iv)$ imply that these regions are not invariant. Let us examine condition (i). This implies that the critical points in M^+ cannot be a local minima. If this is the case then we can simply flow along a vector field parallel to $\mathcal{Y}$ that always reduces $det(B)$ which will eventually drive the system out of $cl(M^+)$ and into either M^- or $M^{+/-}$. A similar procedure can be followed for initial conditions contained in M^-. Condition (iv) allows the system to move off of $M^\emptyset$.

4. APPLICATION

4.1 Control analysis for the roller racer

This section contains the application of the preceding theory to the roller racer (RR). It has been shown that the roller racer cannot be brought to rest using F_1 as the single control input [8], [5]. Let us simplify our calculations by assuming $m = 1$, $l_1 = 1$, $l_2 = 1$, $I_1 = 1$ and $I_2 = 1$. Recall that the geometric model of the RR includes a rank two constraint distribution H. The projection of the control vector field $\mathbb{G}^\sharp(F^1)$ onto H is given by:

$$\begin{aligned} Y = &\frac{1}{2}\cos(\theta)(1-\cos(\psi))\sin(\psi)\frac{\partial}{\partial x} \\ &+\frac{1}{2}(1-\cos(\psi))\sin(\theta)\sin(\psi)\frac{\partial}{\partial y} \\ &+\frac{1}{4}(-2\cos(\psi)+\cos(2\psi)-3)\frac{\partial}{\partial\theta} \\ &+\frac{1}{4}(4\cos(\psi)-\cos(2\psi)+5)\frac{\partial}{\partial\psi}. \end{aligned}$$

We construct the single element in the orthonormal set $\mathcal{Y}^\perp = \{Y^\perp\}$ given $\mathcal{Y}$ by:

$$\begin{aligned} Y^\perp = &\frac{\cos(\theta)}{\sqrt{\frac{4}{\cos(\psi)+1}-1}}\frac{\partial}{\partial x} + \frac{\sin(\theta)}{\sqrt{\frac{4}{\cos(\psi)+1}-1}}\frac{\partial}{\partial y} \\ &+\frac{\tan\left(\frac{\psi}{2}\right)}{\sqrt{\frac{4}{\cos(\psi)+1}-1}}\frac{\partial}{\partial\theta}. \end{aligned}$$

Next, we compute the coefficients of the intrinsic real-valued symmetric bilinear form B. The single coefficient is

$$\begin{aligned} B_{11} &= -\frac{1}{2}\langle\langle\langle Y : Y\rangle, Y^\perp\rangle\rangle \\ &= \frac{(\cos(\psi)-3)\sin^2(\psi)}{4\sqrt{\frac{4}{\cos(\psi)+1}-1}} \end{aligned}$$

Now we expand the expression $w^T B w$ to get

$$B_{ij}w^i(t)w^j(t) = \frac{(\cos(\psi)-3)\sin^2(\psi)}{4\sqrt{\frac{4}{\cos(\psi)+1}-1}}w^1(t)w^1(t) \quad (7)$$

Equation (7) is a quadratic function in w^1. Figure 3 reveals that B is negative definite for all ψ which is equivalent to $M = M^-$.

It then follows from Theorem 9 that RR given the single control force F^1 cannot be driven to rest from an arbitrary initial configuration and velocity.

4.2 Control analysis for the snakeboard

This section contains the application of the preceding theory to the snakeboard (SB). Let us simplify our calculations by assuming $m = 1$, $M = 1$, $l = 1$, $J_r = 1$ and

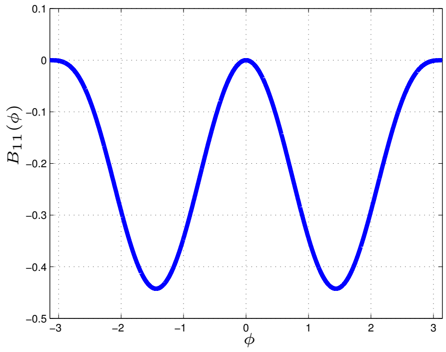

Fig. 3. Plot of $B_{11}(\phi)$ vs. ϕ

$J_w = 1$. Recall that the geometric model of the SB includes a rank three constraint distribution H. The projection of the control vector fields $\mathbb{G}^\sharp(F^1)$ and $\mathbb{G}^\sharp(F^2)$ onto H is given by:

$$\begin{aligned} Y_1 &= \cos(\theta)\tan(\phi)\frac{\partial}{\partial x} + \sin(\theta)\tan(\phi)\frac{\partial}{\partial y} \\ &\quad + -\tan^2(\phi)\frac{\partial}{\partial \theta} + \sec^2(\phi)\frac{\partial}{\partial \psi}, \\ Y_2 &= \frac{\partial}{\partial \phi}. \end{aligned}$$

We construct the single element in the orthonormal set $\mathcal{Y}^\perp = \{Y^\perp\}$ given $\mathcal{Y} = \{Y_1, Y_2\}$ by:

$$\begin{aligned} Y^\perp &= \cos(\theta)\cos(\phi)\frac{\partial}{\partial x} + \cos(\phi)\sin(\theta)\frac{\partial}{\partial x} \\ &\quad - \sin(\phi)\frac{\partial}{\partial \theta}. \end{aligned}$$

Next, we compute the coefficients of the intrinsic real-valued symmetric bilinear form B. The nonzero symmetric coefficients are

$$\begin{aligned} B_{12} = B_{21} &= -\frac{1}{2}\langle\langle\langle Y_1 : Y_2\rangle, Y^\perp\rangle\rangle \\ &= -\frac{\sec(\phi)}{2}, \end{aligned}$$

Now we expand the expression $w^T B w$ to get

$$B_{ij} w^i(t) w^j(t) = -\sec(\phi)\, w^1(t) w^2(t). \tag{8}$$

Equation (8) is a multivariate function of $w^1(t)$ and $w^2(t)$ with degree 2. The graph of this function is a saddle away from $\phi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. This follows from the second derivative test:

$$\left(\frac{\partial^2 Q_{12}}{\partial w^1 \partial w^1}\right)\left(\frac{\partial^2 Q_{12}}{\partial w^2 \partial w^2}\right) - \left(\frac{\partial^2 Q_{12}}{\partial w^1 \partial w^2}\right)^2 = -\sec^2(\phi).$$

By Theorem 12, it is vacuously true that given the input distributions $\mathcal{Y} = \{Y_1, Y_2\}$ and away from $\phi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ SB is ϵ-STR from an arbitrary initial configuration and velocity.

5. FUTURE WORK

We plan to use the intrinsic real-valued symmetric bilinear form as a basic framework for motion planning. We also seek to extend our results to constrained mechanical systems underactuated by an arbitrary number of controls. This will involve characterizing coordinate invariant properties of the intrinsic *vector-valued* symmetric bilinear form that allow motion in the input foliation.

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REFERENCES

[1] F. Bullo and A. Lewis. *Geometric Control of Mechanical Systems.* Springer Science+Business Media, New York, NY, 2005.

[2] F.Bullo and A.Lewis. *Low-order controllablility and kinematic reductions for affine connection control systems.* SIAM Journal on Control and Optimization 44(3), pages 885-908, 2005.

[3] F. Bullo and M. Zefran. *On mechanical control systems with nonholonomic constraints and symmetries*, Systems and Control Letters 45(2):133-143, 2002.

[4] F. Bullo and M. Zefran. *Modeling and controllability for a class of hybrid mechanical systems*, IEEE Transactions on Robotics and Automation 18(4):563-573, 2002.

[5] S.Chitta, P.Cheng, E.Frazzoli and V.Kumar. *RoboTrikke: a novel undulatory locomotion system.* Proceedings of the 2005 IEEE ICRA, 1597-1602, 2005.

[6] R. Hirschorn and A. Lewis. *Geometric local controllability: second-order conditions.* Proceedings of the 41st IEEE CDC, 1:368-369, December 2002.

[7] F. Bullo, J. Cortes, A. Lewis, and S. Martinez *Vector-valued quadratic forms in control theory* Sixty Open Problems in Mathematical Systems and Control Theory, V. Blondel and A. Megretski (Eds.), Princeton University Press, 315-320, 2004.

[8] P.S.Krishnaprasad and D.P.Tsakiris. *Oscillations, $SE(2)$-snakes and motion control: a study of the roller racer.* Dynamics and Stability of Systems, 16(4):347-397, 2001.

[9] A. Lewis *Simple mechanical control systems with constraints.* IEEE Transactions on Automatic Control, 45(8):1420-1436, August 2000.

[10] A. Lewis. *Affine connections and distributions with applications to nonholonomic mechanics.* Reports on Mathematical Physics, 42(1):135-164,1998.

[11] A. Lewis and R. Murray. *Configuration controllability of simple mechanical control systems.* SIAM Journal on Control and Optimization, 35(3):766-790,1997.

[12] A.Lewis, J.P.Ostrowski, J.Burdick and R.Murray. *Nonholonomic mechanics and locomotion: the snakeboard example.* Proceedings of the 1994 IEEE ICRA, 2391-2400, 1994.

[13] M.P. do Carmo. *Riemannian Geometry.* Birkhauser, Boston, MA, English edition, 1992.