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Transfer Matrix Approach to the Triangular Decoupling of General Neutral Multi-Delay Systems

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Abstract: The following major aspects of the problem of Input–Output Triangular Decoupling (TD) for general neutral multi–delay systems, via proportional realizable state feedback, are resolved for the first time: The necessary and sufficient conditions for the problem to have a realizable solution and the general analytical expressions of the proportional realizable TD controller matrices. The conditions and the solution of the controller matrices are computed using a finite step pure algebraic approach.

1. INTRODUCTION

1.1 Time delay systems

Time delay systems are of great importance particularly in describing complex and/or distributed processes, where transition phenomena (mass/energy transfer) take place, as well as distributed manufacturing systems (Gu et al.,2003). Here we study the general class of linear neutral multi–delay differential systems

$$\sum_{j=1}^{q_0}\tilde{E}_j\dot{x}\left(t-\sum_{i=1}^{q}q_{j,i}\tau_i\right)=\sum_{j=1}^{q_0}\tilde{A}_jx\left(t-\sum_{i=1}^{q}q_{j,i}\tau_i\right)+\sum_{j=1}^{q_0}\tilde{B}_ju\left(t-\sum_{i=1}^{q}q_{j,i}\tau_i\right) \tag{1a}$$

$$\sum_{j=1}^{q_0}\bar{C}_jy\left(t-\sum_{i=1}^{q}q_{j,i}\tau_i\right)=\sum_{j=1}^{q_0}\tilde{C}_jx\left(t-\sum_{i=1}^{q}q_{j,i}\tau_i\right) \tag{1b}$$

where $x(t)\in\mathbb{R}^n$ denotes the vector of state variables, $u(t)\in\mathbb{R}^m$ the vector of control inputs, $y(t)\in\mathbb{R}^m$ the vector of performance outputs, τ_i ($i=1,\ldots,q$) are positive real numbers denoting point delays, and $q_{j,i}$ ($j=1,\ldots,q_0$; $i=1,\ldots,q$) is a finite sequence of integers with regard to i and j. The quantities q and q_0 are positive integers. Clearly, if the quantity $\sum_{i=1}^{q}q_{j,i}\tau_i$ is negative then it denotes prediction. The real matrices $\tilde{E}_j$, $\tilde{A}_j$, $\tilde{B}_j$ have n rows while the real matrices $\bar{C}_j$, $\tilde{C}_j$ have m rows. Without loss of generality, the delays $\tau_1,\ldots,\tau_q\in\mathbb{R}$ are considered to be rationally independent (Koumboulis et al., 2005), namely linearly independent among themselves over the field of rational numbers, i.e. there are no rational numbers (dependence coefficients) expressing one delay as a linear combination of the others. If the delays were not rationally independent, namely one delay was linearly dependent to the others then, after dividing each independent delay by the denominator of the respective dependence coefficient (rational number), a new set of independent delays would occur. So, the dependent delay could be substituted by a combination (via integer coefficients) of the new independent delays, thus yielding an equivalent description of system (1) involving at the most $q-1$ delays. If there were more than one dependent delays then a number of dependence relations would occur. In this case each independent delay should be divided by the least common multiplier of the denominators of the dependence coefficients (rational numbers) multiplying the particular independent delay in the dependence relations. This way a new set of independent delays is derived. For the special case of rationally dependent delays where all delays are multiple of one, with dependence coefficients being integers, the delays are called commensurate (Jacubow and Bayoumi, 1977). Note that for the special case where $\tilde{E}_1=I_n$, $\tilde{E}_j=0$ ($j\neq1$), $\bar{C}_1=I_m$, $\bar{C}_j=0$ ($j\neq1$), $q=1$ and $q_{j,1}=j-1$ the case of retarded delay systems is derived (Jacubow and Bayoumi, (1977), Rekasius and Milzarek, (1977), Kono, (1983), Liu, (1989), Sename and Lafay, (1993), Sename et al. (1995), Sename and Lafay, (1997)), where I_k is the k–dimensional unitary matrix. For the special case where $q=1$ and $q_{j,1}=j-1$ with $\tilde{E}_1=I_n$ and $\bar{C}_1=I_m$ the standard category of neutral delay systems is derived, see (Picard et al., 1998). The case of regular systems without delays is also covered by the description in (1) with $q_{j,1}=0$, $\tilde{E}_1=I_n$, $\tilde{E}_j=0$ ($j\neq1$), $\tilde{A}_j=0$ ($j\neq1$), $\tilde{B}_j=0$ ($j\neq1$), $\bar{C}_1=I_m$, $\bar{C}_j=0$ ($j\neq1$), $\tilde{C}_j=0$ ($j\neq1$).

In the frequency domain the forced behavior of the system (1) is governed by the following algebraic system of equations

$$s\tilde{E}\left(\mathbf{e}^{-s\mathbf{T}}\right)X(s)=\tilde{A}\left(\mathbf{e}^{-s\mathbf{T}}\right)X(s)+\tilde{B}\left(\mathbf{e}^{-s\mathbf{T}}\right)U(s) \tag{2a}$$

$$\bar{C}\left(\mathbf{e}^{-s\mathbf{T}}\right)Y(s)=\tilde{C}\left(\mathbf{e}^{-s\mathbf{T}}\right)X(s) \tag{2b}$$

where, $X(s) = \mathfrak{L}_-\{x(t)\}$, $U(s) = \mathfrak{L}_-\{u(t)\}$, $Y(s) = \mathfrak{L}_-\{y(t)\}$ with $\mathfrak{L}_-\{\bullet\}$ denoting the Laplace transform of the argument signal, while

$$\tilde{E}\left(\mathbf{e}^{-s\mathbf{T}}\right) = \sum_{j=1}^{q_0} \tilde{E}_j \exp\left[-s\left(\sum_{i=1}^{q} q_{j,i}\tau_i\right)\right],$$

$$\tilde{A}\left(\mathbf{e}^{-s\mathbf{T}}\right) = \sum_{j=1}^{q_0} \tilde{A}_j \exp\left[-s\left(\sum_{i=1}^{q} q_{j,i}\tau_i\right)\right],$$

$$\tilde{B}\left(\mathbf{e}^{-s\mathbf{T}}\right) = \sum_{j=1}^{q_0} \tilde{B}_j \exp\left[-s\left(\sum_{i=1}^{q} q_{j,i}\tau_i\right)\right],$$

$$\bar{C}\left(\mathbf{e}^{-s\mathbf{T}}\right) = \sum_{j=1}^{q_0} \bar{C}_j \exp\left[-s\left(\sum_{i=1}^{q} q_{j,i}\tau_i\right)\right],$$

$$\tilde{C}\left(\mathbf{e}^{-s\mathbf{T}}\right) = \sum_{j=1}^{q_0} \tilde{C}_j \exp\left[-s\left(\sum_{i=1}^{q} q_{j,i}\tau_i\right)\right]$$

and where $\mathbf{T} = \left[\tau_1 \ \cdots \ \tau_q\right]$, $\mathbf{e}^{-s\mathbf{T}} = \left[\exp\left(-s\tau_1\right) \ \cdots \ \exp\left(-s\tau_q\right)\right]$, where $\exp[\cdot] = e^{[\cdot]}$ denotes the exponential of the argument quantity. The elements of the matrices $\tilde{E}\left(\mathbf{e}^{-s\mathbf{T}}\right)$, $\tilde{A}\left(\mathbf{e}^{-s\mathbf{T}}\right)$, $\tilde{B}\left(\mathbf{e}^{-s\mathbf{T}}\right)$, $\bar{C}\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $\tilde{C}\left(\mathbf{e}^{-s\mathbf{T}}\right)$, are multivariable polynomials of $e^{-s\tau_1},\dots,e^{-s\tau_q}$ (or more compactly of $\mathbf{e}^{-s\mathbf{T}}$). It is important to note that a characteristic of general neutral time delay systems is (Picard et al., 1998) $\det\left[\tilde{E}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right] \not\equiv 0$ and $\det\left[\bar{C}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right] \not\equiv 0$. Hence, the algebraic system of equations (2) may equivalently be written as

$$sX(s) = A\left(\mathbf{e}^{-s\mathbf{T}}\right)X(s) + B\left(\mathbf{e}^{-s\mathbf{T}}\right)U(s) \tag{3a}$$

$$Y(s) = C\left(\mathbf{e}^{-s\mathbf{T}}\right)X(s) \tag{3b}$$

Where

$$A\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[\tilde{E}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\tilde{A}\left(\mathbf{e}^{-s\mathbf{T}}\right),$$

$$B\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[\tilde{E}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\tilde{B}\left(\mathbf{e}^{-s\mathbf{T}}\right),$$

$$C\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[\bar{C}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\tilde{C}\left(\mathbf{e}^{-s\mathbf{T}}\right).$$

The elements of $A\left(\mathbf{e}^{-s\mathbf{T}}\right)$, $B\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $C\left(\mathbf{e}^{-s\mathbf{T}}\right)$, are multivariable rational functions of $e^{-s\tau_1},\dots,e^{-s\tau_q}$ (or more compactly of $\mathbf{e}^{-s\mathbf{T}}$). $\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)$ denotes the set of these multivariable rational functions

1.2 Proportional feedback time delay controller

Several types of controllers involving time delays have been proposed in the literature. Here we focus on frequency domain controllers. One general class of controllers is that of a static feedback where the elements of the feedback matrices belong to the field of rational functions of the delay operator, which may or may not require prediction (Jacubow and Bayoumi, (1977), Rekasius and Milzarek, (1977)). The controller is static in the sense that it does not incorporate derivatives of the variables to be measured or of the external command. Another class is that of static feedback with elements over the ring of polynomials of the delay operator which of course does not require any prediction (Kono, (1983), Liu, (1989), Sename and Lafay, (1993), Conte et al., (1998), Conte and Perdon., (1998)). A third class of feedback laws is a static feedback with elements over the field of rational functions of the delay operator restricted not to be predictive (Koumboulis et al., (2005), Jacubow and Bayoumi, (1977), Sename et al., (1995), Sename and Lafay, (1997), Koumboulis and Panagiotakis, (2005)). Also dynamic feedback laws have been proposed in the literature (Kono, (1983), Liu, (1989), Picard et al., (1998), Conte et al., (1998)). The elements of the feedback matrices are proper rational functions of the complex variable with coefficients being rational functions (Picard et al., 1998) or polynomials (Kono, (1983), Liu, (1989), Conte et al., (1998)) of the delay operator. In the case of coefficients being rational functions of the delay operator the realizability of the controller is guaranteed through the properness with regard to the delay operator. It is important to mention that the aforementioned classes of time delay feedback laws have been used to control retarded systems (Jacubow and Bayoumi, (1977), Rekasius and Milzarek, (1977), Kono, (1983), Liu, (1989), Sename and Lafay, (1993), Sename et al., (1995), Sename and Lafay, (1997), Conte et al., (1998), Conte and Perdon., (1998)), while others have been used to control neutral systems (Jacubow and Bayoumi, (1977), Picard et al., (1998)).

Here we consider the most general class of proportional controllers, not involving continuous time dynamics, but involving delays (i.e. "discrete time dynamics" of the delays $\tau_1 \ \cdots \ \tau_q$) to be

$$U(s) = F\left(\mathbf{e}^{-s\mathbf{T}}\right)X(s) + G\left(\mathbf{e}^{-s\mathbf{T}}\right)\Omega(s) \tag{4}$$

where $\Omega(s)$ is the Laplace transform of the $m \times 1$ vector of external inputs $\omega(t)$ and where the elements of $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $G\left(\mathbf{e}^{-s\mathbf{T}}\right)$ are multivariable rational functions of $e^{-s\tau_1},\dots,e^{-s\tau_q}$ (or more compactly of $\mathbf{e}^{-s\mathbf{T}}$). It is noted that even though the elements of the matrices $A\left(\mathbf{e}^{-s\mathbf{T}}\right)$, $B\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $C\left(\mathbf{e}^{-s\mathbf{T}}\right)$ are not restricted to be realizable rational functions of $e^{-s\tau_1},\dots,e^{-s\tau_q}$, the implementability of the controller requires that the elements of $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $G\left(\mathbf{e}^{-s\mathbf{T}}\right)$ must be realizable (i.e. $\lim_{\substack{s\to\infty \\ s\in\mathbb{R}_+}}\left[F\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]$: finite and $\lim_{\substack{s\to\infty \\ s\in\mathbb{R}_+}}\left[G\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]$: finite). Finally to ensure the independence of the external inputs the following necessary condition is derived $\det\left[G\left(\mathbf{e}^{-s\mathbf{T}}\right)\right] \not\equiv 0$.

1.3 Problem Formulation

Here, the design goal is that of I/O Triangular Decoupling (TD), namely to derive a closed loop system with triangular and invertible transfer function matrix. The TD problem is formally stated as follows

$$C\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[sI_n - A\left(\mathbf{e}^{-s\mathbf{T}}\right) - B\left(\mathbf{e}^{-s\mathbf{T}}\right)F\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1} \times B\left(\mathbf{e}^{-s\mathbf{T}}\right)G\left(\mathbf{e}^{-s\mathbf{T}}\right) = \operatorname{triang}\left\{h_{i,j}\left(s,\mathbf{e}^{-s\mathbf{T}}\right)\right\} \quad (5)$$

$h_{i,i}\left(s,\mathbf{e}^{-s\mathbf{T}}\right) \not\equiv 0 \quad (i=1,\ldots,m)$. Here, $\operatorname{triang}\{\bullet\}$ denotes an $m \times m$ lower triangular matrix with elements belonging to a field or a vector space. The (i,j) element of this matrix is equal to zero if $i<j$. Hence, it holds

$$\operatorname{triang}\left\{h_{i,j}\left(s,\mathbf{e}^{-s\mathbf{T}}\right)\right\} = \begin{bmatrix} h_{1,1}\left(s,\mathbf{e}^{-s\mathbf{T}}\right) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ h_{m,1}\left(s,\mathbf{e}^{-s\mathbf{T}}\right) & h_{m,2}\left(s,\mathbf{e}^{-s\mathbf{T}}\right) & \cdots & h_{m,m}\left(s,\mathbf{e}^{-s\mathbf{T}}\right) \end{bmatrix} \quad (6)$$

The elements of the triangular matrix in (5) belong to $\mathbb{R}_e\left(s,\mathbf{e}^{-s\mathbf{T}}\right)$, i.e. they are rational functions of s with numerator and denominator polynomials having coefficients that they are multivariable rational functions of $\mathbf{e}^{-s\mathbf{T}}$. It is reminded that even though the elements of the matrices $A\left(\mathbf{e}^{-s\mathbf{T}}\right)$, $B\left(\mathbf{e}^{-s\mathbf{T}}\right)$, $C\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $h_{i,j}\left(s,\mathbf{e}^{-s\mathbf{T}}\right)(i \le j)$ are not restricted to be realizable rational functions of $e^{-s\tau_1},\ldots,e^{-s\tau_q}$, the implementability of the controller requires that the elements of $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $G\left(\mathbf{e}^{-s\mathbf{T}}\right)$ must be realizable (i.e. $\lim_{\substack{s\to\infty \\ s\in\mathbb{R}_+}}\left[F\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]$: finite and $\lim_{\substack{s\to\infty \\ s\in\mathbb{R}_+}}\left[G\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]$: finite).

TD is a very attractive design problem (Wang, (1972), Descusse and Lizarzaburu, (1979), Koumboulis et al., (1991), Koumboulis, (1996)). It appears to have some of the advantages of diagonal decoupling. One of these is that each output is still controlled by only one external input. Furthermore, TD has the distinct advantage of being applicable to a wider class of systems as compared to that of diagonal decoupling. For time delay systems TD has not as yet been studied. For the special case of commensurate retarded delay systems the diagonal decoupling problem has been studied (indicatively see the results in (Rekasius and Milzarek, 1977), (Sename and Lafay, 1997)). For diagonal decoupling of standard neutral commensurate delay systems of standard form, some first results have been presented in (Jacubow and Bayoumi, 1977). In (Conte et al., 1998) and (Conte and Perdon., 1998) the diagonal decoupling problem has been studied for normal systems with system and controller matrices having their elements in a Noetherian ring. In (Paraskevopoulos et al., 2005) the combined problem of diagonal decoupling with disturbance rejection has been studied for general neutral multi–delay differential systems. Before closing the presentation of results for diagonal decoupling, it is important to mention that all the aforementioned results focus towards a realizable controller, namely a controller not involving predictions.

Here, the TD problem of general neutral multi–delay systems, via proportional realizable state feedback is studied for the first time. Using a new algebraic technique providing simple and elegant results, the necessary and sufficient solvability conditions for the problem to have a solution are established and the general solution of the realizable controllers solving the problem is derived.

2. PRELIMINARY DEFINITIONS

A multivariable rational function of $\exp\left(-s\tau_1\right),\ldots,\exp\left(-s\tau_q\right)$ is realizable (Rekasius and Milzarek, (1977), Kono, (1983)) if no predictors are required for its realization, i.e. the limit of the rational function, for s being a positive real number and for s tending to infinity, is finite. The set of multivariable realizable rational functions of $\exp\left(-s\tau_1\right),\ldots,\exp\left(-s\tau_q\right)$, denoted by $\mathbb{R}_r\left(\mathbf{e}^{-s\mathbf{T}}\right)$, is clearly a ring.

Lemma 2.1 (Koumboulis et al., 2005): Let $\Theta\left(\mathbf{e}^{-s\mathbf{T}}\right) \in \left[\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{\alpha\times\beta}$ be of full row rank over $\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)$, then there exist two invertible matrices, let $\Theta_M\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right)$, with $\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right)$ bi–realizable, i.e. $\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is realizable ($\lim_{\substack{s\to\infty \\ s\in\mathbb{R}_+}}\left\{\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}$: finite) and $\left[\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}$ is realizable ($\lim_{\substack{s\to\infty \\ s\in\mathbb{R}_+}}\left\{\left[\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\right\}$: finite), having the property

$$\Theta_M\left(\mathbf{e}^{-s\mathbf{T}}\right)\Theta\left(\mathbf{e}^{-s\mathbf{T}}\right)\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[I_a \mid 0\right]. \quad \square$$

Explicit formulae for $\Theta_M\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right)$ are given in (Koumboulis et al., 2005). The above transformation, being of great importance for the study of realizability issues for time delay systems, is called *right bi–realizable unitarizing transformation* (Koumboulis et al., 2005). The matrices $\Theta_M\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $\Theta_\Xi\left(\mathbf{e}^{-s\mathbf{T}}\right)$ are called (Koumboulis and Panagiotakis, 2005) the **L**eft and the **R**ight **M**ultiples with regard to the **R**ight birealizable unitarizing transformation of the full row rank, over $\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)$, matrix $\Theta\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and they are denoted by $\operatorname{LMR}\left\{\Theta\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}$ and $\operatorname{RMR}\left\{\Theta\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}$, respectively.

Following the definition of the LMR and the RMR of a full row matrix, the **L**eft and the **R**ight Multiples of a full column rank matrix (over $\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)$), coming from a **L**eft Bi–realizable Unitarizing Transformation, denoted by $\operatorname{LML}\{\bullet\}$ and $\operatorname{RML}\{\bullet\}$ respectively (with $\operatorname{LML}\{\bullet\}$ being bi–realizable) can be defined. Hence, for the full column rank matrix $\Theta^{\mathrm{T}}\left(\mathbf{e}^{-s\mathbf{T}}\right)$, we may define

$\mathrm{LML}\left\{\Theta^{\mathrm{T}}\left(\mathbf{e}^{\text{-sT}}\right)\right\} = \begin{bmatrix} 0 & I_{\beta-\alpha} \\ I_\alpha & 0 \end{bmatrix} \Theta_{\Xi}^{\mathrm{T}}\left(\mathbf{e}^{\text{-sT}}\right)$ and $\mathrm{RML}\left\{\Theta^{\mathrm{T}}\left(\mathbf{e}^{\text{-sT}}\right)\right\} = \Theta_M^{\mathrm{T}}\left(\mathbf{e}^{\text{-sT}}\right)$. Then, it holds that $\mathrm{LML}\left\{\Theta^{\mathrm{T}}\left(\mathbf{e}^{\text{-sT}}\right)\right\}\Theta^{\mathrm{T}}\left(\mathbf{e}^{\text{-sT}}\right)\mathrm{RML}\left\{\Theta^{\mathrm{T}}\left(\mathbf{e}^{\text{-sT}}\right)\right\} = \begin{bmatrix} 0 \\ I_a \end{bmatrix}$.

3. SOLUTION OF THE TD PROBLEM VIA CONTROLLERS POSSIBLY INVOLVING PREDICTORS

From the definition of the problem, namely from the design equation (5), it can readily be concluded that the system (3) must be invertible i.e. the following necessary condition must be satisfied $\det\left[C\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} B\left(\mathbf{e}^{\text{-sT}}\right)\right] \not\equiv 0$. Then, following the methodology presented in (Koumboulis, 1996) for systems without delays and based on the necessary condition $\det\left[C\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} B\left(\mathbf{e}^{\text{-sT}}\right)\right] \not\equiv 0$, leading to the condition $c_1\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} B\left(\mathbf{e}^{\text{-sT}}\right) \not\equiv 0$, the following definitions can be introduced

$$\rho_1 = \min\left\{j : c_1^{(j)}\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right) \not\equiv 0,\ j \in \{0,1,\ldots,n-1\}\right\},$$
$$C_1^*\left(\mathbf{e}^{\text{-sT}}\right) = c_1^{(\rho_1)}\left(\mathbf{e}^{\text{-sT}}\right),$$
$$c_1^{(j)}\left(\mathbf{e}^{\text{-sT}}\right) = c_1\left(\mathbf{e}^{\text{-sT}}\right)\left[A\left(\mathbf{e}^{\text{-sT}}\right)\right]^j \quad (j = 0,1,\ldots,n-1)$$

where, $c_i\left(\mathbf{e}^{\text{-sT}}\right)$ is the $i-\mathrm{th}$ row of $C\left(\mathbf{e}^{\text{-sT}}\right)$ $(i = 1,\ldots,m)$. Clearly it holds that $\mathrm{rank}_e\left[C_1^*\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right)\right] = 1$ and $\mathrm{Rank}_e\begin{bmatrix} C_1^*\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} B\left(\mathbf{e}^{\text{-sT}}\right) \\ c_2\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} B\left(\mathbf{e}^{\text{-sT}}\right) \end{bmatrix} = 2$, where the operator $\mathrm{rank}_e[\cdot]$ denotes the rank of the argument matrix over the field $\mathbb{R}_e\left(\mathbf{e}^{\text{-sT}}\right)$ and where $\mathrm{Rank}_e[\cdot]$ denotes the rank of the argument matrix over the field $\mathbb{R}_e\left(s,\mathbf{e}^{\text{-sT}}\right)$. The respective definitions for $i = 2,\ldots,m$ will be presented inductively, following the respective definitions in (Koumboulis, 1996). If

$$\mathrm{rank}_e\left[C_{i-1}^*\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right)\right] = i-1,\ (i = 2,\ldots,m)$$

$$\mathrm{Rank}_e\begin{bmatrix} C_{i-1}^*\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} B\left(\mathbf{e}^{\text{-sT}}\right) \\ c_i\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]B\left(\mathbf{e}^{\text{-sT}}\right) \\ \vdots \\ c_m\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]B\left(\mathbf{e}^{\text{-sT}}\right) \end{bmatrix} = m, \quad (i = 2,\ldots,m)$$

we may define for $i = 2,\ldots,m$

$$C_i^*\left(\mathbf{e}^{\text{-sT}}\right) = \begin{bmatrix} c_1^{(\rho_1)}\left(\mathbf{e}^{\text{-sT}}\right) \\ \vdots \\ c_i^{(\rho_i)}\left(\mathbf{e}^{\text{-sT}}\right) \end{bmatrix},$$

$$N_{i-1}\left(\mathbf{e}^{\text{-sT}}\right) = \left[C_{i-1}^*\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right)\right]^{\mathrm{T}} \times\left\{\left[C_{i-1}^*\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right)\right]\left[C_{i-1}^*\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right)\right]^{\mathrm{T}}\right\}^{-1}$$

$$c_i^{(0)}\left(\mathbf{e}^{\text{-sT}}\right) = c_i\left(\mathbf{e}^{\text{-sT}}\right)$$

$$c_i^{(j)}\left(\mathbf{e}^{\text{-sT}}\right) = c_i^{(j-1)}\left(\mathbf{e}^{\text{-sT}}\right)\left[I_m - B\left(\mathbf{e}^{\text{-sT}}\right)N_{i-1}\left(\mathbf{e}^{\text{-sT}}\right)C_{i-1}^*\left(\mathbf{e}^{\text{-sT}}\right)\right] \times A\left(\mathbf{e}^{\text{-sT}}\right) \quad (j = 1,2,\ldots)$$

$$\rho_i = \begin{cases} \min\left\{j : \mathrm{rank}_e\begin{bmatrix} C_{i-1}^*\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right) \\ c_i^{(j)}\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right) \end{bmatrix} = i,\ j = 0,\ldots,n-1\right\} \\ n-1 \ \mathit{if}\ \mathrm{rank}_e\begin{bmatrix} C_{i-1}^*\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right) \\ c_i^{(j)}\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right) \end{bmatrix} = i-1, \\ \qquad \forall j \in \{0,\ldots,n-1\} \end{cases}$$

Based on the above definitions and similarly to (Koumboulis, 1996) the following properties can readily be proven

$$M\left(s,\mathbf{e}^{\text{-sT}}\right)C\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} B\left(\mathbf{e}^{\text{-sT}}\right) = C_m^*\left(\mathbf{e}^{\text{-sT}}\right)\left[sI_n - A\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} B\left(\mathbf{e}^{\text{-sT}}\right) \quad (7a)$$

$$\mathrm{rank}_e\left[C_m^*\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right)\right] = m \quad (7b)$$

where $M\left(s,\mathbf{e}^{\text{-sT}}\right) = \prod_{i=1}^{m}\prod_{j=0}^{\rho_i-1} M_{ij}\left(s,\mathbf{e}^{\text{-sT}}\right)$ and where

$$M_{ij}\left(s,\mathbf{e}^{\text{-sT}}\right) = \begin{bmatrix} I_{i\text{-}1} & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & I_{m\text{-}i} \end{bmatrix}\begin{bmatrix} I_{i\text{-}1} & 0 & 0 \\ -\overline{c}_i^{(j)}\left(\mathbf{e}^{\text{-sT}}\right) & 1 & 0 \\ 0 & 0 & I_{m\text{-}i} \end{bmatrix};$$

$$\overline{c}_i^{(j)}\left(\mathbf{e}^{\text{-sT}}\right) = c_i^{(j)}\left(\mathbf{e}^{\text{-sT}}\right)B\left(\mathbf{e}^{\text{-sT}}\right)N_{i-1}\left(\mathbf{e}^{\text{-sT}}\right).$$

For the special case where $\rho_i = 0$ the following expression is derived

$$\prod_{j=0}^{\rho_i-1} M_{ij}\left(s,\mathbf{e}^{\text{-sT}}\right) = I_m$$

Using (7a), the equation (5) can be expressed more compactly as follows

$$C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[sI_n - A\left(\mathbf{e}^{-s\mathbf{T}}\right) - B\left(\mathbf{e}^{-s\mathbf{T}}\right)F\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1} B\left(\mathbf{e}^{-s\mathbf{T}}\right)G\left(\mathbf{e}^{-s\mathbf{T}}\right) = M\left(s,\mathbf{e}^{-s\mathbf{T}}\right)\operatorname{triang}\left\{h_{i,j}\left(s,\mathbf{e}^{-s\mathbf{T}}\right)\right\}$$

$$;h_{i,i}\left(s,\mathbf{e}^{-s\mathbf{T}}\right) \not\equiv 0 \quad (8)$$

Define

$$A_C\left(\mathbf{e}^{-s\mathbf{T}}\right) = A\left(\mathbf{e}^{-s\mathbf{T}}\right) - B\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1} C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)A\left(\mathbf{e}^{-s\mathbf{T}}\right)$$

$$\Delta\left(\mathbf{e}^{-s\mathbf{T}}\right) = B\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}$$

Note that

$$C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[sI_n - A_C\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\Delta\left(\mathbf{e}^{-s\mathbf{T}}\right) = \frac{I_m}{s}$$

Also define

$$G_c\left(\mathbf{e}^{-s\mathbf{T}}\right) = C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)G\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad (9a)$$

$$F_c\left(\mathbf{e}^{-s\mathbf{T}}\right) = C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)F\left(\mathbf{e}^{-s\mathbf{T}}\right) + C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)A\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad (9b)$$

$$P\left(s,\mathbf{e}^{-s\mathbf{T}}\right) = \left[sM\left(s,\mathbf{e}^{-s\mathbf{T}}\right)\operatorname{triang}\left\{h_{i,j}\left(s,\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]^{-1} \quad (9c)$$

Making use of the definitions following (8) the design equation (8) takes the following equivalent form

$$sP\left(s,\mathbf{e}^{-s\mathbf{T}}\right)C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[sI_n - A_C\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\Delta\left(\mathbf{e}^{-s\mathbf{T}}\right) - \Gamma\left(\mathbf{e}^{-s\mathbf{T}}\right) + \Phi\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[sI_n - A_C\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\Delta\left(\mathbf{e}^{-s\mathbf{T}}\right) = 0 \quad (10)$$

where

$$\Gamma\left(\mathbf{e}^{-s\mathbf{T}}\right) = \begin{bmatrix}\gamma_1\left(\mathbf{e}^{-s\mathbf{T}}\right)\\ \vdots \\ \gamma_m\left(\mathbf{e}^{-s\mathbf{T}}\right)\end{bmatrix} = \left[G_c\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1} \quad (11a)$$

$$\Phi\left(\mathbf{e}^{-s\mathbf{T}}\right) = \begin{bmatrix}\varphi_1\left(\mathbf{e}^{-s\mathbf{T}}\right)\\ \vdots \\ \varphi_m\left(\mathbf{e}^{-s\mathbf{T}}\right)\end{bmatrix} = \left[G_c\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}F_c\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad (11b)$$

From (10) and the condition $\det\left[C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right] \not\equiv 0$, it is observed that $P\left(s,\mathbf{e}^{-s\mathbf{T}}\right)$ is a proper matrix with regard to s. The quantity in the left–hand side of (10) is a rational function of s with coefficients rational functions of $\exp\left(-s\tau_1\right) \ \cdots \ \exp\left(-s\tau_q\right)$. With respect to s the order of this rational function is n. Making use of the fact that s and $\exp\left(-s\tau_1\right) \ \cdots \ \exp\left(-s\tau_q\right)$ are independent functions it is concluded that the quantity in the left–hand side of (10) is equal to zero if and only if the first n coefficients of its expansion in negative power series of s, being rational functions of $\exp\left(-s\tau_1\right) \ \cdots \ \exp\left(-s\tau_q\right)$, are equal to zero.

Making use of this last remark the following equations governing the general form of the controller matrices proposed in (Koumboulis, 1996) are derived

$$\gamma_i\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[p_{0,i,1}\left(\mathbf{e}^{-s\mathbf{T}}\right), p_{0,i,2}\left(\mathbf{e}^{-s\mathbf{T}}\right), \cdots, p_{0,i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right), 0_{1\times(m-i)}\right] \quad (i=1,\ldots,m) \quad (12a)$$

$$\varphi_i\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[\Delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad A_C\left(\mathbf{e}^{-s\mathbf{T}}\right)\Delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad \cdots \quad \left[A_C\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{n-1}\Delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right)\right] = 0 \quad (i=1,\ldots,m) \quad (12b)$$

where $p_{k,i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is the $ij-th$ element of the lower triangular matrix $P_k\left(\mathbf{e}^{-s\mathbf{T}}\right)$ $k=0,\ldots,2n$ and $P\left(s,\mathbf{e}^{-s\mathbf{T}}\right) = P_0\left(\mathbf{e}^{-s\mathbf{T}}\right)s^0 + P_1\left(\mathbf{e}^{-s\mathbf{T}}\right)s^{-1} + \ldots$, and where $\Delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[\delta_{i+1}\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad \delta_{i+2}\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad \cdots \quad \delta_m\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]$ with $\delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right)$ denoting the $i-$th column of the matrix $\Delta\left(\mathbf{e}^{-s\mathbf{T}}\right)$.

The invertibility of $\Gamma\left(\mathbf{e}^{-s\mathbf{T}}\right)$ coming from the problem definition is now translated to $p_{0,i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right) \not\equiv 0 \quad (i=1,\ldots,m)$.

The solution of the controller matrices proposed in (Koumboulis, 1996) does not facilitate the derivation of the conditions under which there exist realizable controllers solving the problem. To circumvent this difficulty, an alternative general solution possibly involving predictors is proposed.

To derive the new general solution of $\varphi_i\left(\mathbf{e}^{-s\mathbf{T}}\right)$ the following definitions are made:

$$\sigma_i = n - \operatorname{rank}\left[\Delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad A_C\left(\mathbf{e}^{-s\mathbf{T}}\right)\Delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad \cdots \quad \left[A_C\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{n-1}\Delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right)\right] \quad (i=0,1,\ldots,m-1),$$

$$\sigma_m = n$$

The controllability indices of $\left(A_C\left(\mathbf{e}^{-s\mathbf{T}}\right), \Delta\left(\mathbf{e}^{-s\mathbf{T}}\right)\right)$, from m to 1, are denoted by υ_i ($\upsilon_i = \sigma_i$-σ_{i-1}; $i=1,\ldots,m$). Since $C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\Delta\left(\mathbf{e}^{-s\mathbf{T}}\right) = I_m$ it is observed that $\upsilon_i \geq 1$ ($i=1,\ldots,m$). Furthermore, it can readily be observed that $\sigma_i = n - \upsilon_{i+1} - \cdots - \upsilon_m$ ($i=0,\ldots,m-1$). Define

$$R_i\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[\delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right) \quad \cdots \quad \left[A_C\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{\upsilon_i - 1}\delta_i\left(\mathbf{e}^{-s\mathbf{T}}\right)\right] \quad (i=1,\ldots,m)$$

$$S\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[\begin{array}{c|c}\operatorname{LML}\left\{S_2\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\} & 0 \\ \hline 0 & I_{n-\sigma_2}\end{array}\right] \cdots \left[\begin{array}{c|c}\operatorname{LML}\left\{S_m\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\} & 0 \\ \hline 0 & I_{n-\sigma_m}\end{array}\right]$$

where

$$S_m\left(\mathbf{e}^{-s\mathbf{T}}\right)=R_m\left(\mathbf{e}^{-s\mathbf{T}}\right)$$
$$S_i\left(\mathbf{e}^{-s\mathbf{T}}\right)=\Pi_{i+1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\cdots\Pi_m\left(\mathbf{e}^{-s\mathbf{T}}\right)R_i\left(\mathbf{e}^{-s\mathbf{T}}\right),\quad (i=m-1,\ldots,1)$$

and where

$$\Pi_i\left(\mathbf{e}^{-s\mathbf{T}}\right)=\left[I_{\sigma_{i-1}}\mid 0_{\sigma_{i-1}\times(n-\sigma_{i-1})}\right]\mathrm{LML}\left\{S_i\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\ (i=m,\ldots,1)$$

It can readily be observed that $S\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is the product of bi–realizable matrices. Using all above definitions the following theorem will be established.

Theorem 3.1: The TD problem is solvable if the condition $\det\left[C\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[sI_n-A\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]\not\equiv 0$ is satisfied, and the general analytic expressions of the proportional controller matrices possibly involving predictions are:

$$G\left(\mathbf{e}^{-s\mathbf{T}}\right)=\left[C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\left[\mathrm{triang}\left\{p_{0,i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]^{-1}\quad (13a)$$
$$F\left(\mathbf{e}^{-s\mathbf{T}}\right)=\left[C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}$$
$$\times\left\{\left[Q_0\left(\mathbf{e}^{-s\mathbf{T}}\right)\mid\mathrm{triang}\left\{q_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]S\left(\mathbf{e}^{-s\mathbf{T}}\right)-C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)A\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\quad (13b)$$

where the only free parameters are the elements of the arbitrary matrix $Q_0\left(\mathbf{e}^{-s\mathbf{T}}\right)\in\left[\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{1\times\sigma_0}$, the scalars $p_{0,i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)$ $(i=1,2,\ldots,m\ \ j=1,\ldots,i)$ (being arbitrary over $\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)$ with the restriction $p_{0,i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right)\not\equiv 0$) and the vectors $q_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\in\left[\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{1\times v_j}$ $(i=1,2,\ldots,m\ \ j=1,\ldots,i)$ with arbitrary elements over $\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)$.

Proof: From (12a) and (11a) the general solution for $G_c\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is $G_c\left(\mathbf{e}^{-s\mathbf{T}}\right)=\left[\mathrm{triang}\left\{p_{0,i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]^{-1}$. Substituting this last relation in (9a) formula (13a) is proven to be the general solution for $G\left(\mathbf{e}^{-s\mathbf{T}}\right)$. Regarding the general solution for $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$ we will first determine the general solution for $\varphi_i\left(\mathbf{e}^{-s\mathbf{T}}\right)$, $i=1,\ldots,m$. Equation (12b) can be reduced as follows

$$\varphi_i\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[R_{i+1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\mid R_{i+2}\left(\mathbf{e}^{-s\mathbf{T}}\right)\mid\cdots\mid R_m\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]=0\quad (i=1,\ldots,m-1)$$

while the solution for $\varphi_m\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is given by the relation

$$\varphi_m\left(\mathbf{e}^{-s\mathbf{T}}\right)=\left[\tau_{m,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \ \tau_{m,1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \cdots\ \tau_{m,m}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]$$

where $\tau_{m,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\in\left[\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{1\times v_j}$ $(j=1,2,\ldots,m)$ and $\tau_{m,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\in\left[\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{1\times\sigma_0}$ are arbitrary row vectors over $\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)$. Using the definitions just before Theorem 3.1 the general solution of $\varphi_i\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is expressed for $i=1,\ldots,m-1$ as follows

$$\varphi_i\left(\mathbf{e}^{-s\mathbf{T}}\right)=\left[\tau_{i,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \ \tau_{i,1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \cdots\ \tau_{i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \mid\ 0\ \cdots\ 0\right]$$
$$\times\left[\begin{array}{c|c}\mathrm{LML}\left\{S_{i+1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\} & 0\\ \hline 0 & I_{n-\sigma_{i+1}}\end{array}\right]\cdots\left[\begin{array}{c|c}\mathrm{LML}\left\{S_m\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\} & 0\\ \hline 0 & I_{n-\sigma_m}\end{array}\right]$$

where $\tau_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\in\left[\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{1\times v_j}$ $(i=1,\ldots m-1;\ j=1,\ldots,i)$ and $\tau_{i,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\in\left[\mathbb{R}_e\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{1\times\sigma_0}$ $(i=1,\ldots m-1)$ are arbitrary. The arbitrary vector $\left[\hat{q}_{i,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \ \hat{q}_{i,1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \cdots\ \hat{q}_{i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \mid\ 0\ \cdots\ 0\right]$ for $i=1,\ldots,m-1$ and $i=m$ is defined to be

$$\left[\tau_{i,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \ \tau_{i,1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \cdots\ \tau_{i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \mid\ 0\ \cdots\ 0\right]=$$
$$=\left[\hat{q}_{i,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \ \hat{q}_{i,1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \cdots\ \hat{\mathbf{q}}_{i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \mid\ 0\ \cdots\ 0\right]$$
$$\times\left[\begin{array}{c|c}\mathrm{LML}\left\{S_2\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\} & 0\\ \hline 0 & I_{n-\sigma_2}\end{array}\right]\cdots\left[\begin{array}{c|c}\mathrm{LML}\left\{S_i\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\} & 0\\ \hline 0 & I_{n-\sigma_i}\end{array}\right]$$
$$\left[\tau_{m,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \ \tau_{m,1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \cdots\ \tau_{m,m}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]$$
$$=\left[\hat{q}_{m,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \ \hat{q}_{m,1}\left(\mathbf{e}^{-s\mathbf{T}}\right)\ \cdots\ \hat{q}_{m,m}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]\mathbf{S}\left(\mathbf{e}^{-s\mathbf{T}}\right).$$

Substitution of the general solution

$$\Phi\left(\mathbf{e}^{-s\mathbf{T}}\right)=\left[\hat{Q}_0\left(\mathbf{e}^{-s\mathbf{T}}\right)\mid\mathrm{triang}\left\{\hat{q}_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]S\left(\mathbf{e}^{-s\mathbf{T}}\right)$$

(where $\hat{Q}_0\left(\mathbf{e}^{-s\mathbf{T}}\right)=\begin{bmatrix}\hat{q}_{1,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\\ \vdots\\ \hat{q}_{m,0}\left(\mathbf{e}^{-s\mathbf{T}}\right)\end{bmatrix}$) in (11b), yields

$$F_c\left(\mathbf{e}^{-s\mathbf{T}}\right)=G_c\left(\mathbf{e}^{-s\mathbf{T}}\right)\Phi\left(\mathbf{e}^{-s\mathbf{T}}\right)=$$
$$=\left[\mathrm{triang}\left\{p_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]^{-1}\left[\hat{Q}_0\left(\mathbf{e}^{-s\mathbf{T}}\right)\mid\mathrm{triang}\left\{\hat{\mathbf{q}}_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]S\left(\mathbf{e}^{-s\mathbf{T}}\right)$$
$$=\left[Q_0\left(\mathbf{e}^{-s\mathbf{T}}\right)\mid\mathrm{triang}\left\{q_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]S\left(\mathbf{e}^{-s\mathbf{T}}\right).$$

Substituting this last relation into (9b), the relation (13b) is derived to be the general solution for $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$. □

4. SOLUTION OF THE TD PROBLEM VIA REALIZABLE SOLUTION

4.1 Necessary and sufficient conditions

To solve the problem at hand, namely the TD problem via realizable controllers, the following definitions are made

$$B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right) = \mathrm{LMR}\left\{C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\},$$
$$B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right) = \mathrm{RMR}\left\{C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\};$$

Note that $B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left\{C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right) = I_m$.

Theorem 4.1: The TD problem for general neutral multi–delay systems is solvable, via a proportional realizable controller, if and only if the following conditions are satisfied:

(i) $\det\left[C\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[sI_n - A\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}B\left(\mathbf{e}^{-s\mathbf{T}}\right)\right] \not\equiv 0$, and

(ii) $\left\{B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)A\left(e^{-sT}\right)\left[S\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\right\}_{i,j}$ ($i = 1,\ldots,m-1$ $j = \sigma_i + 1,\ldots,n$) are realizable, where the symbol $\{\bullet\}_{i,j}$ denotes the $(i,j)-th$ element of the argument matrix.

Proof: Condition (i) is the necessary and sufficient condition for the solvability of the problem via controllers, involving possibly predictors. To establish the necessary and sufficient conditions via realizable controllers, assume that (i) holds and then rewrite (13a) as follows $G\left(\mathbf{e}^{-s\mathbf{T}}\right) = B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[\mathrm{triang}\left\{p_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]^{-1}$. Since $B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is bi–realizable and $B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is lower triangular and invertible, $G\left(\mathbf{e}^{-s\mathbf{T}}\right)$ can always be made realizable by appropriate choice of $p_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)$. The general form of such a choice is $\left[\mathrm{triang}\left\{p_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]^{-1} = \left[B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\left[\mathrm{triang}\left\{k_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]$ where the scalars $k_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)$ $(i = 1,\ldots,m \;\; j = 1,\ldots i)$ are arbitrary over $\mathbb{R}_r\left(\mathbf{e}^{-s\mathbf{T}}\right)$ with $k_{i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right) \not\equiv 0$. With regard to $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$, rewrite (13b) as follows:

$$\begin{aligned} F\left(\mathbf{e}^{-s\mathbf{T}}\right) &= B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right) \\ &\times\left[Q_0\left(\mathbf{e}^{-s\mathbf{T}}\right) \mid \mathrm{triang}\left\{q_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]S\left(\mathbf{e}^{-s\mathbf{T}}\right) \\ &-B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)A\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[S\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}S\left(\mathbf{e}^{-s\mathbf{T}}\right) \end{aligned}$$

Since $B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $S\left(\mathbf{e}^{-s\mathbf{T}}\right)$ are both bi–realizable, the realizability of $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$ depends entirely upon the realizability of $B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[Q_0\left(\mathbf{e}^{-s\mathbf{T}}\right) \mid \mathrm{triang}\left\{q_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]$ $-B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)A\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[S\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}$.

The matrix $B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is an invertible lower triangular matrix and hence $B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[Q_0\left(\mathbf{e}^{-s\mathbf{T}}\right) \mid \mathrm{triang}\left\{q_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]$ has the block triangular form of $\left[Q_0\left(\mathbf{e}^{-s\mathbf{T}}\right) \mid \mathrm{triang}\left\{q_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]$. Thus, condition (ii) is proven to be necessary and sufficient for the existence of a realizable $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$. □

4.2 General form of the realizable controller matrices

In this subsection and under the assumption that the system satisfies the conditions of Theorem 4.1, the set of all realizable controller matrices $G\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $F\left(\mathbf{e}^{-s\mathbf{T}}\right)$, let $G_r\left(\mathbf{e}^{-s\mathbf{T}}\right)$ and $F_r\left(\mathbf{e}^{-s\mathbf{T}}\right)$, will be derived. To this end, consider the $m \times n$ matrix

$$W\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[W_0\left(\mathbf{e}^{-s\mathbf{T}}\right) \;\; W_1\left(\mathbf{e}^{-s\mathbf{T}}\right) \;\; \cdots \;\; W_m\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]$$

where the $m \times v_i$ submatrices $W_i\left(\mathbf{e}^{-s\mathbf{T}}\right)$ $(i = 1,\ldots,m)$ are

$$W_i\left(\mathbf{e}^{-s\mathbf{T}}\right) = \left[\begin{array}{c|c} 0_{(i-1)\times(i-1)} & 0 \\ \hline 0 & I_{m-i+1} \end{array}\right]B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right) \times A\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[S\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\left[\begin{array}{c} 0_{(\sigma_{i-1})\times v_i} \\ \hline I_{v_i} \\ \hline 0_{(n-\sigma_i)\times v_i} \end{array}\right]$$

while the $m \times \sigma_0$ matrix $W_0\left(\mathbf{e}^{-s\mathbf{T}}\right)$ is defined as follows

$$W_0\left(\mathbf{e}^{-s\mathbf{T}}\right) = B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)A\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[S\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{-1}\left[\begin{array}{c} I_{\sigma_0} \\ \hline 0_{(n-\sigma_0)\times\sigma_0} \end{array}\right]$$

Theorem 4.2: If conditions (i)–(ii) of Theorem 4.1 are satisfied, then the general analytic expressions of the proportional realizable controller matrices solving the TD problem are

$$G_r\left(\mathbf{e}^{-s\mathbf{T}}\right) = B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left[\mathrm{triang}\left\{k_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right] \tag{14a}$$

$$\begin{aligned} F_r&\left(\mathbf{e}^{-s\mathbf{T}}\right) \\ &= B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)\left\{\left[\Lambda_0\left(\mathbf{e}^{-s\mathbf{T}}\right) \mid \mathrm{triang}\left\{\lambda_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\}\right]S\left(\mathbf{e}^{-s\mathbf{T}}\right)\right\} \\ &\quad + B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)W\left(\mathbf{e}^{-s\mathbf{T}}\right)S\left(\mathbf{e}^{-s\mathbf{T}}\right) \\ &\quad - B_R^*\left(\mathbf{e}^{-s\mathbf{T}}\right)B_L^*\left(\mathbf{e}^{-s\mathbf{T}}\right)C_m^*\left(\mathbf{e}^{-s\mathbf{T}}\right)A\left(\mathbf{e}^{-s\mathbf{T}}\right) \end{aligned} \tag{14b}$$

where the only free parameters are the elements of the arbitrary realizable matrix $\Lambda_0\left(\mathbf{e}^{-s\mathbf{T}}\right) \in \left[\mathbb{R}_r\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{m\times\sigma_0}$, the scalars $k_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right)$ $(i = 1,\ldots,m \;\; j = 1,\ldots i)$ being arbitrary over $\mathbb{R}_r\left(\mathbf{e}^{-s\mathbf{T}}\right)$ with the restriction $k_{i,i}\left(\mathbf{e}^{-s\mathbf{T}}\right) \not\equiv 0$ and the vectors $\lambda_{i,j}\left(\mathbf{e}^{-s\mathbf{T}}\right) \in \left[\mathbb{R}_r\left(\mathbf{e}^{-s\mathbf{T}}\right)\right]^{1\times v_j}$ $(i = 1,\ldots,m \;\; j = 1,\ldots i)$ with arbitrary realizable elements over $\mathbb{R}_r\left(\mathbf{e}^{-s\mathbf{T}}\right)$.

Proof: The general solution for $G\left(\mathbf{e}^{\text{-sT}}\right)$ comes from the proof of Theorem 4.1. Using the proof of Theorem 4.1 the degrees of freedom yielding realizable solution for $F\left(\mathbf{e}^{\text{-sT}}\right)$ are given by the following formula for $i = 1,\ldots,m$

$$\begin{bmatrix} 0_{(i-1)\times v_i} \\ q_{i,i}\left(\mathbf{e}^{\text{-sT}}\right) \\ \vdots \\ q_{m,i}\left(\mathbf{e}^{\text{-sT}}\right) \end{bmatrix} = \left[B_L^*\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} \begin{bmatrix} 0_{(i-1)\times v_i} \\ \lambda_{i,i}\left(\mathbf{e}^{\text{-sT}}\right) \\ \vdots \\ \lambda_{m,i}\left(\mathbf{e}^{\text{-sT}}\right) \end{bmatrix} +$$

$$+\left[B_L^*\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} \begin{bmatrix} 0_{(i-1)\times(i-1)} & 0 \\ 0 & I_{m-i+1} \end{bmatrix}$$

$$\times B_L^*\left(\mathbf{e}^{\text{-sT}}\right) C_m^*\left(\mathbf{e}^{\text{-sT}}\right) A\left(\mathbf{e}^{\text{-sT}}\right) \left[S\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} \begin{bmatrix} 0_{(\sigma_{i-1}-\sigma_0)\times v_i} \\ I_{v_i} \\ 0_{(n-\sigma_i+\sigma_0)\times v_i} \end{bmatrix}$$

$$Q_0\left(\mathbf{e}^{\text{-sT}}\right) = \left[B_L^*\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} \Lambda_0\left(\mathbf{e}^{\text{-sT}}\right) + C_m^*\left(\mathbf{e}^{\text{-sT}}\right) A\left(\mathbf{e}^{\text{-sT}}\right) \left[S\left(\mathbf{e}^{\text{-sT}}\right)\right]^{-1} \times \begin{bmatrix} I_{\sigma_0} \\ 0_{(n-\sigma_0)\times\sigma_0} \end{bmatrix}$$

Substituting the above formula to (13b) and using the definition of $W\left(\mathbf{e}^{\text{-sT}}\right)$, the formula for $F_r\left(\mathbf{e}^{\text{-sT}}\right)$ in (14b) is derived. □

5. CONCLUSIONS

The following aspects of the TD problem for general neutral multi–delay systems, via proportional realizable state feedback, have been established for the first time: The necessary and sufficient conditions for the problem to have a solution and the general analytical expressions of the realizable TD controller matrices. The derived results are simple and elegant thus facilitating their extension to other familiar design problems in the field. This way, for the solution of the TD problem for general neutral multi–delay systems, the presence of delays does not significantly increase the order of multiplicity of the solution.

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REFERENCES

Conte, G., and A.M. Perdon (1998) 'The block decoupling problem for systems over a ring', *IEEE Trans. Automat. Cont.*, **43**, pp. 1600–1604.

Conte, G., A.M. Perdon and A. Lombardo (1998). 'Dynamic Feedback Decoupling Problem for Delay Differential Systems via systems over rings', *Mathematics and Computers in Simulation*, **45**, pp. 235–244.

Descusse, J., and R. Lizarzaburu (1979). 'Triangular Decoupling and pole placement in linear multivariable systems: a direct algebraic approach', *Int. J. Control*, **30**, pp. 139–152.

Gu, K., V. Kharitonov, and J. Chen, (2003) '*Stability of Time–Delay Systems*', Birkhauser, Boston.

Jacubow, R.P., and M.M. Bayoumi (1977). 'Decoupling of differential–difference systems using state feedback', *Int. J. Systems Sci*, **8**, pp.587–599.

Kono, M. (1983). 'Decoupling and arbitrary coefficient assignment in time delay systems', *Syst. Control Lett.*, **3**, pp. 349 –354.

Koumboulis, F.N, P.N. Paraskevopoulos, K.G. Tzierakis and R.E. King (1991). 'Disturbance rejection with simultaneous triangular decoupling for linear time invariant systems', *Proc. 30th IEEE Conf. On Decision and Control*, Brighton, U.K., pp. 581–582.

Koumboulis, F.N. (1996). 'Input–output triangular decoupling and data sensitivity', *Automatica*, **32**, pp. 569–573.

Koumboulis, F.N., G.E. Panagiotakis and P.N. Paraskevopoulos (2005). 'Exact Model Matching of Left Invertible Neutral Time Delay Systems', *Proc. of the 13th Mediterranean Conf. on Control and Automation (2005 ISIC–MED),* Limassol, Cyprus, pp. 1548–1555.

Koumboulis, F.N., and G.E. Panagiotakis (2005). 'Exact Model Matching and Disturbance Rejection for general linear time delay systems via measurement output feedback', *Proc. of the 10th IEEE Int. Conf. on Emerging Technologies and Factory Automation (ETFA 2005)*, Catania, Italy, pp.545–554.

Liu, M.Z. (1989). 'Decoupling and coefficient assignment for (A, B, C, D) time delay systems', *Int. J. Control*, **50**, pp. 1089–1101.

Paraskevopoulos, P.N., F.N. Koumboulis and G.E. Panagiotakis (2005) 'Disturbance Rejection with Simultaneous Decoupling of Neutral Time Delay Systems', *Proc. of the Int. Conf. on Computational Intelligence for Modeling Control and Automation (CIMCA 2005)*, Vienna, Austria, **I**, pp. 350–357.

Picard, P., J.F. Lafay and V. Kucera (1998). 'Model Matching for linear systems with delays and 2–D Systems', *Automatica*, **34**, pp. 183–191.

Rekasius, Z.V., and R.L. Milzarek, (1977). 'Decoupling without prediction of systems with delays', *JACC, pt. II,* pp. 1470–1475.

Sename, O., and J.F. Lafay (1993). 'A sufficient condition for static decoupling without prediction of linear time–invariant systems with delays', *Proc. of the 2nd European Control Conference (ECC 93),* Groningen, the Netherlands, **2**, pp. 673–678.

Sename, O., and J.F. Lafay (1997). 'Decoupling of Square linear systems with delays', *IEEE Trans. Automat. Cont.*, **42**, pp. 736–742.

Sename, O, R. Rabah and J.F. Lafay (1995). 'Decoupling without prediction of linear systems with delays: A structural approach', *Syst. Cont. Letters*, **25**, pp. 387–395.

Wang, S.H. (1972). 'Relationship between triangular decoupling and invertibility', *Int. J. Control*, **15**, pp 395–399.