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Optimal Trajectories for Homing Navigation with Bearing Measurements

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Abstract: In this paper we examine the geometry of a number of navigation schemes for guiding a pursuer from a fixed initial position to a final target position. We explicitly characterize the optimal pursuer trajectories for the given problems in terms of minimizing the error in an unbiased estimate of the target position from successive bearing measurements made along the trajectory.

1. INTRODUCTION

The problem of optimal navigation and guidance involves steering a mobile *pursuer* from an initial position to a final position while minimizing some performance index, i.e. the terminal miss distance; e.g. see Lin (1991); Zarchan (1994); Ben-Asher and Yaesh (1998). In many cases the target is also mobile and in some cases the mobile target explicitly attempts to evade the pursuer. In such navigation problems, it is typical to explore the actual control system design for a specified vehicle kinematic model (or class of vehicle model); e.g. see Stevens and Lewis (1992). The actual state (i.e. position and/or velocity etc.) of the target destination is usually assumed to be measured by the pursuer in a recursive fashion. Thus, an optimal control law should be designed around the form of measurement available (e.g. typically target range and/or bearing measurements are available). In many instances, it is typical to neglect the effect of measurement noise and assume perfect target state estimation when designing the controller, e.g. Savkin et al. (2003).

The state estimate of the target destination in a given navigation problem is critical in minimizing such performance indexes as miss distance etc. For example, suppose we are given an optimal control law that can provably obtain a miss distance of zero. Then we obtain noisy measurements of the target destination and apply such a control law. Ultimately, the miss distance achieved in practice will be equal to the estimation error at the terminal time. Hence, the performance of the target state estimator is crucial in realizing the best performance for any provably optimal guidance law; see Zarchan (1994). Indeed, this is hardly surprising given the duality of estimation and control.

In this paper we examine the *effects of the actual navigation trajectory* on the error bounds of any (unbiased) state estimator which takes successive (noisy) bearing measurements along a trajectory. In particular, we focus on the navigation problem which involves a mobile pursuer and a stationary target destination. The pursuer measures the bearing to the target destination as discussed in Nardone et al. (1984); Gavish and Weiss (1992). Actually, we consider the problem of characterizing the optimal pursuer trajectory which permits the smallest target estimation error with initial and final (pursuer) position constraints. This problem is clearly significant in navigation and guidance scenarios where the goal is to steer a mobile pursuer from an initial position to a final (target) position while estimating the target state and minimizing some controller performance index, i.e. the terminal miss distance. Related work has been done in bearing-only tracking and localization where the objective is either to improve estimator performance in the mean-square-error sense or to permit/improve target state observability for nonlinear filtering applications; e.g. see Fogel and Gavish (1988); Becker (1993); Jauffret and Pillon (1996); Bishop et al. (2007a,b).

The immediate goal is to explicitly characterize those trajectories which permit the smallest error (in the mean-square sense) in an (unbiased) estimate of the target destination. For practicality and simplicity we explore the problem in discrete time which further permits an intuitive reasoning about the derived results. The characterization is inherently described in geometric terms. To the best our knowledge, no other work explicitly attempts to characterize such navigation and guidance problems.

We emphasize the fact that we are not deriving specific control laws or estimators in this paper. Rather, we are characterizing the trajectory for a guidance problem where the pursuer must estimate the target state using (noisy) bearing measurements and minimize some (controller) performance index, i.e. miss distance. This characterization is independent of the estimator or control law employed (although we assume unbiased estimation) and yields a lower bound on the variance (or mean-square-error) for such navigation problems with discrete-step pursuer movements. In the next section we formally introduce the problem.

2. NOTATION AND PROBLEM DEFINITION

We consider a single stationary target and a single mobile pursuer located in $\mathbb{R}^2$. Time is indexed by the symbol k and the initial problem time is when $k = 0$. With no loss of generality we assume the system (and measurement) sample time is given by $t_s = 1$ time unit such that $k \in \mathbb{N}$. The target's position is given by

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$$\mathbf{x} = [x_1 \ x_2]^{\mathrm{T}} \in \mathbb{R}^2 \tag{1}$$

where x_1 and x_2 denote the target's position in the traditionally denoted x and y. The pursuer's location at time k is given by

$$\mathbf{s}(k) = [s_1(k) \ s_2(k)]^{\mathrm{T}} \in \mathbb{R}^2 \tag{2}$$

where s_1 and s_2 denote the pursuer's position in the traditionally denoted x and y directions respectively. The pursuer location can change over time and we place no restrictions on the sensor's movement capabilities in this paper. The range at time k between the sensor $\mathbf{s}(k)$ and the target $\mathbf{x}$ is given by $r(k) = \|\mathbf{x} - \mathbf{s}(k)\|$.

The true azimuth bearing $\phi(k)$ between the sensor and the target at time k is measured positive counter-clockwise from the traditionally denoted x-axis. The true bearing is assumed to obey

$$\phi(\mathbf{x})(k) = \arctan\left(\frac{x_2 - s_2(k)}{x_1 - s_1(k)}\right) \tag{3}$$

such that the measured bearing obeys the following model

$$\widehat{\phi}(k) = \phi(\mathbf{x})(k) + e(k) \tag{4}$$

where $e(k)$ is the so-called measurement *error* with $e(k) \sim \mathcal{N}(0, \sigma_\phi^2)$. We assume the bearing measurements are uncorrelated over time. We define the measurement vector for $k \in \{0, \ldots, M\}$ measurements by

$$\widehat{\mathbf{y}} = \mathbf{y}(\mathbf{x}) + \mathbf{e} = [\phi(0) \ \ldots \ \phi(M)]^{\mathrm{T}} + [e(0) \ \ldots \ e(M)]^{\mathrm{T}} \tag{5}$$

such that $\widehat{\mathbf{y}} \sim \mathcal{N}(\mathbf{y}(\mathbf{x}), \mathbf{R}_\phi)$ where $\mathbf{R}_\phi = \sigma_\phi^2 \mathbf{I}$. Without loss of generality let us assume further that the coordinate system is rotated and scaled such that

$$\mathbf{x} = [1 \ 0]^{\mathrm{T}} \in \mathbb{R}^2 \quad \text{and} \quad \mathbf{s}(0) = [0 \ 0]^{\mathrm{T}} \in \mathbb{R}^2 \tag{6}$$

and hence $r(0) = 1$ and $\phi(\mathbf{x})(k) = 0$. Consider the following problem and assumption.

Informally, the guidance problem considered in this paper is that of navigating a pursuer from an initial position $\mathbf{s}(0) \in \mathbb{R}^2$ to an estimate of the target position $\mathbf{x}$ given by $\widehat{\mathbf{x}}$ and given noisy bearing measurements $\widehat{\phi}(k)$, $\forall k \in \{0, \ldots, N-1\}$. The objective of this paper is to characterize and analyze the optimal discrete-time trajectory for a pursuer such that the estimate of $\mathbf{x}$ given by $\widehat{\mathbf{x}}$ is optimal (in the mean-square-error sense). In this case we expect the navigation error, i.e. miss distance, to be minimized in the same mean-square-error sense. We assume the following on the pursuer movements.

Assumption 1. Suppose there exists a pursuer with a given initial position $\mathbf{s}(0) \in \mathbb{R}^2$ *which can measure the bearing* $\widehat{\phi}(k)$ *of a desired target destination given by* $\mathbf{x} \in \mathbb{R}^2$. *Suppose further that the pursuer moves in discrete steps and reaches the target destination (or an estimate of the target) in a fixed number* N *of steps. Now assume there exists* $N-1$ *evenly distributed concentric circles centered at the target (or an estimate of the target) position such that the intersection of the* $N-1$ *circles with the line joining the initial pursuer position and the target position divide the line into* N *evenly spaced sections; e.g. see Figure 1. The pursuer navigation scheme dictates that each successive pursuer step is restricted to lie on the subsequent circle centered at the target, see Figure 1.*

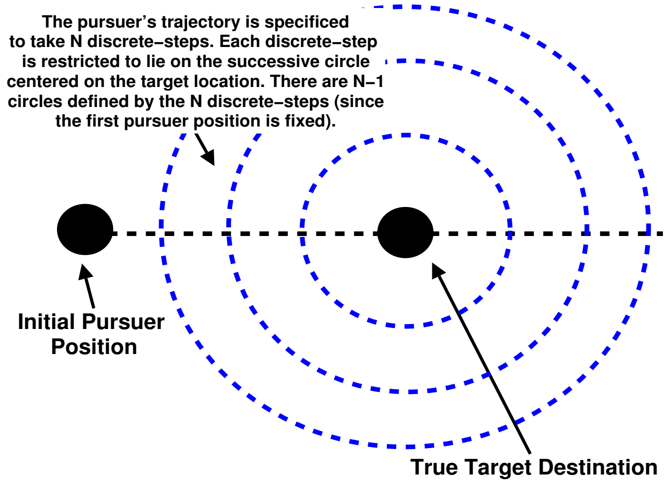

Fig. 1. *The pursuer can only move via discrete steps which must lie on the concentric circles centered at the final target position (or it's estimate). The bearing measurements taken at each step determine the accuracy of the final estimate of the target destination given by* $\widehat{\mathbf{x}}$ *and only* $N-1$ *steps can be arbitrarily chosen. The first step is given at time* $k = 0$ *and the final step coincides with the pursuer reaching the (estimated) target destination. Each successive step must lie on the subsequent circle centered at the target. Hence, the pursuer-target range at each step is independent of the relative target bearing at that step and the pursuer-target range decreases evenly with the number* N. *In the figure we illustrate the scenario with* $N = 4$ *which implies that there exists* $N - 1 = 3$ *discrete-steps for the pursuer to optimally determine. The recurring pattern is clear with different* N *values.*

The assumption placed on the pursuer movements results in a *homing* strategy since the pursuer iteratively moves closer to the target. Indeed, given the assumed coordinate system arrangements we get have

$$r(k) = \|\mathbf{x} - \mathbf{s}(k)\| = \frac{N-k}{N} \tag{7}$$

which follows from simple geometry and under the original assumption of (6) with $r(0) = 1$ and $\phi(\mathbf{x})(k) = 0$. Since we are examining bearing-only navigation, the relationship (7) is particularly useful since we can describe the entire pursuer-target geometry at any time k solely in terms of the bearing $\phi(k)$, the known constant N and the current step number k. With the given assumption on the discrete pursuer steps we state the two principal objectives of this paper. The first is to characterize the optimal pursuer trajectory for an N-step ahead navigation planning scheme. That is, a navigation scheme that involves planning all N-steps ahead and considers the entire bearing history. A formal statement of the first characterization problem is as follows.

Problem 1. Characterize the optimal discrete-time N*-step trajectory for a pursuer navigating to a target location* $\mathbf{x}$ *while planning all* N*-steps ahead and using bearing-only measurements of* $\mathbf{x}$. *The optimality is defined such that the error in an (unbiased) estimate of* $\mathbf{x}$ *given by* $\widehat{\mathbf{x}}$ *is minimized in the mean square error sense. Moreover, such that the error in the final destination* $\|\mathbf{x} - \mathbf{s}(N)\|$ *is minimized in the same sense.*

This first objective and characterization involves deriving the globally optimal trajectory.

The second objective of this paper is to characterize the optimal pursuer trajectory for a one-step look ahead navigation scheme. That is, a navigation scheme which involves planning only one-step ahead. A formal statement of the second characterization problem is as follows.

Problem 2. Characterize the best discrete-time N-step trajectory for a pursuer which navigates with a one-step ahead planning strategy using bearing-only measurements to the target $\mathbf{x}$ such that the error in an (unbiased) estimate of $\mathbf{x}$ given by $\widehat{\mathbf{x}}$ is minimized in the mean-squared-error sense. Moreover, such that the error in the final destination $\|\mathbf{x}-\mathbf{s}(N)\|$ is also minimized in the same mean-squared-error sense.

The second objective and characterization is an important and practical special case.

The difference between the two problems is subtle but straightforward when one considers that in the first problem the entire trajectory is derived at time $k = 0$. This results in a globally optimal trajectory in the sense defined. In the second problem, the pursuer plans only one-step ahead and thus considers only the bearing history up to and including the current step. The resulting trajectory is suboptimal when compared to the trajectory derived in the first problem. However, the one-step plan ahead navigation scheme is justified from a practical control point of view.

3. CRAMER-RAO BOUNDS AND OPTIMAL TRAJECTORIES

The navigation trajectory for a mobile pursuer effects the potential accuracy of a state estimation algorithm which in turn effects the optimal miss distance that can be achieved. For a given estimator, it is the goal of this section to outline a metric with which to assess the optimality of a given navigation trajectory. In this paper, the navigation trajectory is characterized for unbiased and efficient estimation algorithms. For an *unbiased* estimate $\widehat{\mathbf{x}}$ of $\mathbf{x}$ the Cramer-Rao inequality states that

$$\mathbb{E}\left[(\widehat{\mathbf{x}}-\mathbf{x})(\widehat{\mathbf{x}}-\mathbf{x})^T\right] \geq \mathcal{I}(\mathbf{x})^{-1} \triangleq \mathcal{C}(\mathbf{x}) \tag{8}$$

where $\mathcal{I}(\mathbf{x})$ is the Fisher information matrix. If $\mathcal{I}(\mathbf{x})$ is singular then (generally) no unbiased estimator for $\mathbf{x}$ exists with a finite variance; see Van Trees (1968). If (8) holds with equality then the estimator is called *efficient* and the parameter estimate $\widehat{\mathbf{x}}$ is unique; see Van Trees (1968).

The matrix $\mathcal{I}(\mathbf{x})$ quantifies the amount of information that the observable random vector $\widehat{\mathbf{y}}$ carries about the unobservable parameter $\mathbf{x}$. We know that the Fisher information matrix is given by

$$\mathcal{I}(\mathbf{x}) = \nabla_{\mathbf{x}}\mathbf{y}(\mathbf{x})^{\mathrm{T}}\mathbf{R}_{\phi}^{-1}\nabla_{\mathbf{x}}\mathbf{y}(\mathbf{x}) \tag{9}$$

under the assumption of Gaussian measurement errors. The variance of the sum of independent random variables is equal to the sum of the variances and thus the general Fisher information matrix for $k \in \{0,\ldots,M\}$ measurements is given by

$$\mathcal{I}(\mathbf{x}) = \frac{1}{\sigma_\phi^2}\sum_{k=0}^{M}\frac{1}{r(k)^2}\begin{bmatrix} \sin^2(\phi(k)) & -\dfrac{\sin(2\phi(k))}{2} \\ -\dfrac{\sin(2\phi(k))}{2} & \cos^2(\phi(k)) \end{bmatrix} \tag{10}$$

where k indexes over the discrete pursuer movements. We know that $\det(\mathcal{I}(\mathbf{x}))$ is inversely proportional to the uncertainty area of an unbiased (efficient) estimate of $\mathbf{x}$ given by $\widehat{\mathbf{x}}$ Van Trees (1968). We will use $\det(\mathcal{I}(\mathbf{x}))$ to analyze the optimal geometrical trajectory for a pursuer navigating from a given initial position to a desired final position in a fixed number of discrete steps given by N. Note that there are only $N-1$ bearing measurements made since the final step will coincide with the pursuer arriving at the estimated destination $\widehat{\mathbf{x}}$.

We firstly consider the N-step look ahead navigation scenario and we completely characterize the points which must lie on the optimal trajectory for estimating and navigating to a particular target destination. Following this we explore the practically important case of one-step look ahead navigation where the pursuer plans only the next step considering the entire bearing history up to and including the current step.

We state the following important result which outlines the specific optimization problems which form the basis of the analysis in this paper.

Theorem 1. Let $M \leq N-1$ and N be arbitrarily given and suppose that $r(k) = \frac{N-k}{N}$ and $\phi(0) = 0$. Moreover, with no loss of generality let $\sigma_\phi^2 \equiv 1$. Then the following optimization problems are equivalent:

(i) $\max_{\phi(0),\ldots,\phi(M)} \ \det(\mathcal{I}(\mathbf{x}))$;

(ii) $\min_{\phi(0),\ldots,\phi(M)} \ \left(\sum_0^M \frac{\sin(2\phi(k))}{r(k)^2}\right)^2 + \left(\sum_0^M \frac{\cos(2\phi(k))}{r(k)^2}\right)^2$;

(iii) $\min_{\phi(0),\ldots,\phi(M)} \ \left|\sum_{k=0}^M \frac{1}{r(k)^2}e^{2\mathrm{j}\phi(k)}\right|^2, \ \mathrm{j}=\sqrt{-1}$;

(iv) $\max_{\phi(0),\ldots,\phi(M)} \ \sum_{\mathcal{S}} \frac{\sin^2(\phi(j)-\phi(i))}{r(i)^2 r(j)^2}, \mathcal{S} = \{\{i,j\}\}$;

where in (iv) we define $\mathcal{S} = \{\{i,j\}\}$ to be the set of all combinations of i and j with $i,j \in \{0,\ldots,M\}$ and $j > i$ such that $|\mathcal{S}| = \binom{M+1}{2}$.

Notice that the optimization problems in Theorem 1 are solely in terms of the bearings $\phi(k)$ since the range has been parameterized in terms of the number of steps N and the current step number k as a result of our problem definition; e.g. see equation (7). Moreover, note also that as a result of the problem definition we find that $\phi(k)$ explicitly defines the location of $\mathbf{s}(k)$.

Proof. Note that (iii) follows from (ii) by Euler's formula. Hence, we need to show the equivalence of (ii) and (i), and (iv) and (i). We easily find that (10) can be rewritten as

$$\mathcal{I}(\mathbf{x}) = \begin{bmatrix} \displaystyle\sum_{k=0}^{M}\frac{(1-\cos(2\phi(k)))}{2r(k)^2} & \displaystyle -\sum_{k=0}^{M}\frac{\sin(2\phi(k))}{2r(k)^2} \\ \displaystyle -\sum_{k=0}^{M}\frac{\sin(2\phi(k))}{2r(k)^2} & \displaystyle\sum_{k=0}^{M}\frac{(1+\cos(2\phi(k)))}{2r(k)^2} \end{bmatrix} \tag{11}$$

where $M \leq N-1$ is the number of discrete steps which are to be determined beginning at $k=0$. Hence the Fisher information determinant is given by

$$\det(\mathcal{I}(\mathbf{x})) = \frac{1}{4}\left[\left(\sum_{k=0}^{M}\frac{1}{r(k)^2}\right)^2 - \left(\sum_{k=0}^{M}\frac{\cos(2\phi(\mathbf{x})(k))}{r(k)^2}\right)^2 - \left(\sum_{k=0}^{M}\frac{\sin(2\phi(\mathbf{x})(k))}{r(k)^2}\right)^2\right] \quad (12)$$

which directly implies the equivalence of (ii) and (i). Note also that $\mathbf{R}_\phi = \sigma_\phi^2\mathbf{I}$ and thus $\mathbf{R}_\phi^{-1} = 1/\sigma_\phi^2\mathbf{I}$ which under our assumption of $\sigma_\phi^2 \equiv 1$ implies $\mathbf{R}_\phi = \mathbf{R}_\phi^{-1} = \mathbf{I}$. Let $\mathbf{G} = \nabla_\mathbf{x}\mathbf{y}(\mathbf{x})$ such that from (9) we also find

$$\det(\mathcal{I}(\mathbf{x})) = \det\left(\mathbf{G}^\mathrm{T}\mathbf{G}\right) = \sum_{m=1}^{\binom{M+1}{2}} \det(\mathbf{G}_m)^2 \quad (13)$$

using the Cauchy-Binet formula; see e.g. Horn and Johnson (1985). Here $\mathbf{G}_m$ is a 2×2 minor of $\mathbf{G}$ taken from the set of minors indexed by $\mathcal{S} = \{\{i,j\}\}$. All 2×2 minors of $\mathbf{G}$ can be given as

$$\mathbf{G}_\mathcal{S} = \begin{bmatrix} \frac{1}{r(i)}\cos(\phi(i)) & \frac{1}{r(i)}\sin(\phi(i)) \\ \frac{1}{r(j)}\cos(\phi(j)) & \frac{1}{r(j)}\sin(\phi(j)) \end{bmatrix} \quad (14)$$

where in fact $\mathcal{S} = \{\{i,j\}\}$ with $|\mathcal{S}| = \binom{M+1}{2}$ can be considered the set of all combinations of i and j with $j > i$. Now the equivalence of (iv) and (i) follows easily by trigonometry. $\square$

Solving the problems in Theorem 1 leads to the $M \leq N-1$ optimal discrete-step positions given that $\phi(0) = 0$ and $\mathbf{s}(0)$ are fixed. The optimality is in the sense that if this particular trajectory were taken, then the best estimate of $\widehat{\mathbf{x}}$ of $\mathbf{x}$ given the $M+1$ bearing measurements is achieved.

4. OPTIMAL TRAJECTORIES WITH N-STEP AHEAD PLANNING

In this section we explicitly characterize the optimal trajectories which solve Problem 1, where the pursuer considers all N-steps ahead when planning it's navigation trajectory. Given N and the relationship on $r(k) = \frac{N-k}{N}$ we have the following result which will be used subsequently.

Lemma 1. Consider the optimization problems in Theorem 1 with $M = N-1$ and $N > 2$. Then, $\phi(j) = \phi(i) \pm \pi/2$ for some j and for all $i \in \{0,\ldots,N-1\}/\{j\}$ is the globally optimal solution if and only if $1/r(j)^2 > \sum_i 1/r(i)^2$ for the same j and for all other $i \in \{0,\ldots,N-1\}/\{j\}$. This solution implies that all $\phi(i)$ for $i \in \{0,\ldots,N-1\}/\{j\}$ are separated by $c\pi$ with $c \in \{0,1\}$. If $N = 2$ then $\phi(j) = \phi(i) \pm \pi/2$ for $i,j \in \{0,1\}$ is the globally optimal solution to the optimization problems given in Theorem 1.

Proof. The proof of Lemma 1 is omitted for brevity. $\square$

The initial position $\mathbf{s}(0) = [0\ 0]^\mathrm{T}$ and the terminal constraint $\mathbf{s}(N) = \mathbf{x} = [1\ 0]^\mathrm{T}$ are fixed. Thus, there exists $N-1$ discrete optimal pursuer trajectory points indexed by $k \in \{1,\ldots,N-1\}$ to determine.

Proposition 1. Suppose that N is arbitrarily fixed and $\mathbf{s}(0) = [0\ 0]^\mathrm{T}$ and $\mathbf{x} = [1\ 0]^\mathrm{T}$ are given. Suppose also that $\mathbf{s}(N)$ coincides with the pursuer arriving at the target destination or a target estimate. The range is given by $r(k) = \frac{N-k}{N}$, $\forall k \in \{0,\ldots,N-1\}$ and $\phi(0) = 0$ with no loss of generality. Then the optimal pursuer trajectory is completely characterized by the bearing history given by $\phi(k) = 0$ or $\phi(k) = \pi$, $\forall k \in \{1,\ldots,N-2\}$ and $\phi(N-1) = \pm\pi/2$.

Proof. The first part of the proof involves showing that $1/r(N-1)^2 > \sum_i 1/r(i)^2$ for $i \in \{0,\ldots,N-2\}$ where $r(N-1) = \frac{1}{N}$ and $r(i) = \frac{N-i}{N}$. Hence, we note that $1/r(N-1)^2 = N^2$ and $1/r(i)^2 = N^2/(N-i)^2$ for $i \in \{0,\ldots,N-2\}$. Then clearly

$$N^2 > N^2\sum_{i=0}^{N-2}\frac{1}{(N-i)^2} \quad (15)$$

since the summation obeys $\sum_{i=0}^{N-2}\frac{1}{(N-i)^2} < 1$. Now it follows from Lemma 1 that $\phi(N-1) = \pm\pi/2$ and all other $\phi(i)$ for $i \in \{0,\ldots,N-2\}$ are separated by $c\pi$ where $c \in \{0,1\}$ and the proof is complete. $\square$

Proposition 1 completely characterizes the globally optimal pursuer trajectories for N-step plan ahead guidance schemes under the given problem definition.

4.1 Examples for N-Step Look Ahead Navigation

Proposition 1 defines a number of optimal trajectories. Here, two example cases of an optimal N-step plan ahead navigation trajectory are examined.

The first example case is given in Figure 2 and is basically a straight line based trajectory with the last step forcing a $\frac{\pi}{2}$ bearing separation such that $\phi(N-1) = \pm\frac{\pi}{2}$. The second example case is given in Figure 3 and is a spiraling trajectory. The optimality of both cases are equivalent. Both are globally optimal in the sense defined.

Note that in Figure 3 we also plot an approximating continuous spiraling trajectory which touches on each of the optimal discrete step points except for the last point. These Archimedean spiral trajectories are shown to illustrate the intuitive reasoning behind considering such characterizations. Moreover, the Archimedean spirals illustrate a very close (continuous) approximation to the required optimal trajectory.

No other trajectory yields a better (unbiased) estimate of the target position at time $k = N-1$ than the example trajectories given in Figures 2 and 3 assuming the constraint on the pursuer movements introduced in this paper. These trajectories might seem surprising but occur as a result of the dominance of the final pursuer-target range in the given objective function, i.e. the Fisher information determinant.

The globally optimal trajectory for N-step look ahead guidance is not unique and different controllers might handle different trajectories with better or worse efficiency in terms of other controller performance measures, e.g. time taken, controller energy etc. As stated, we are not designing controllers but rather characterizing the optimal trajectories in the sense that the variance in any (unbiased)

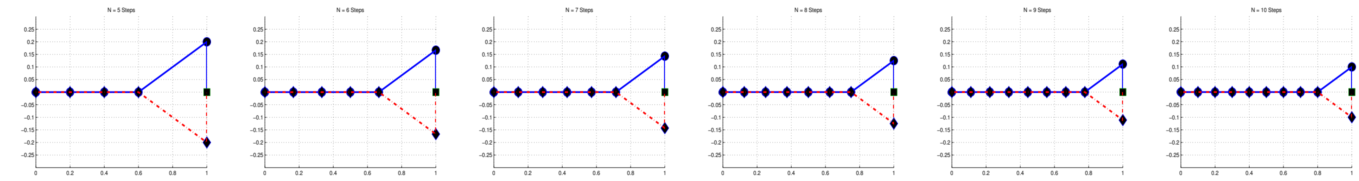

Fig. 2. *Two examples of optimal trajectories for N-step ahead navigation when $N \in [5, 10]$. There are two trajectories shown in each figure with the $k = N-1$ position defined by two different yet equivalent bearings $\phi(N-1) = \pm\pi/2$. The optimal trajectories shown are not unique and any trajectory which satisfies the conditions given in Proposition 1 is equivalently optimal.*

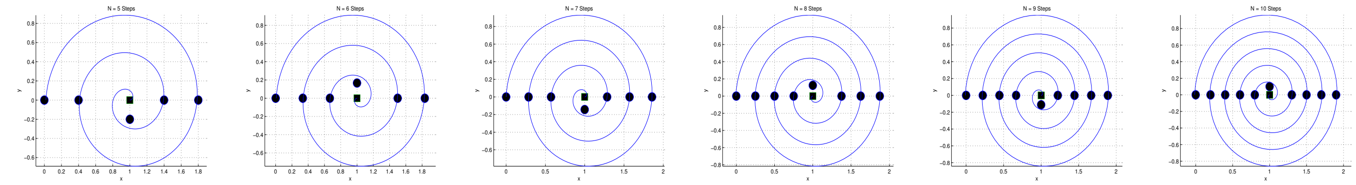

Fig. 3. *An optimal trajectory for N-step ahead navigation when $N \in [5, 10]$. In this case we show only one trajectory with the $k = N-1$ position defined by $\phi(N-1) = \pm\pi/2$ depending in whether N is odd or even. The remaining bearing alternate between $\phi(i) = 0$ and $\phi(i) = \pi$ thus facilitating the spiraling trajectory. Again, the optimal trajectories shown are not unique but are intuitively appealing from an illustrative point of view and a controller design point of view. Any trajectory which satisfies the conditions given in Proposition 1 is equivalently optimal. The Archimedean spiral is plotted in the figures to illustrate the spiraling pattern and also to illustrate a very close approximation for a continuous trajectory.*

estimate $\widehat{\mathbf{x}}$ is minimized. This implies the miss-distance in the guidance problem is minimized in the same sense.

5. OPTIMAL TRAJECTORIES WITH ONE-STEP PLANNING

Assume that N is given and recall that $r(k) = \frac{N-k}{N}$ is the known range of the pursuer from the target destination at time $k \in \{0, \ldots, N-1\}$. The following optimization result is used to characterize the optimal trajectories for a one-step ahead planning guidance scheme.

Theorem 2. Suppose N is arbitrarily fixed and $\mathbf{s}(0)$ and $\mathbf{x}$ are fixed such that $r(0) = 1$ and $\phi(0) = 0$. Then at each time k the following optimization problem

$$\phi(k+1) = \underset{\phi(k+1)}{\operatorname{argmax}} \sum_{i=0}^{k} \frac{\sin^2(\phi(k+1) - \phi(i))}{(N-i)^2} \tag{16}$$

leads to the optimal, one-step ahead, value for $\phi(k+1)$ and subsequently $\mathbf{s}(k+1)$.

The next step of the trajectory, i.e. at step $k+1$, is calculated at the current step, i.e. at step k, given the bearing history, i.e. $\phi(0)$ up to and including $\phi(k)$. Theorem 2 defines a single variable optimization problem.

Proof. Suppose an arbitrary time k is given such that to optimally plan one-step ahead we solve the problems in Theorem 1 with $M = k+1$. Thus, from Theorem 1 part (iv) we obtain the following problem at time k

$$\underset{\phi(k+1)}{\operatorname{argmax}} \sum_{\mathcal{S}} \frac{\sin^2(\phi(j) - \phi(i))}{r(i)^2 r(j)^2} \tag{17}$$

where we define $\mathcal{S} = \{\{i, j\}\}$ to be the set of all combinations of i and j with $i, j \in \{0, \ldots, k+1\}$ and $j > i$ such that $|\mathcal{S}| = \binom{M+1}{2}$. However, at time k we know $\phi(0)$ through to $\phi(k)$ and we consider these as constants. Moreover, $r(k) = \frac{N-k}{N}$ such that the only variable in the relevant optimization problem is $\phi(k+1)$. Factoring out the relevant terms involving step $k+1$ in the optimization problem leads to the equivalent optimization problem

$$\underset{\phi(k+1)}{\operatorname{argmax}} \quad \frac{1}{r(k+1)^2} \sum_{i=0}^{k} \frac{\sin^2(\phi(k+1) - \phi(i))}{r(i)^2} + \sum_{\mathcal{S}^*} \frac{\sin^2(\phi(m) - \phi(l))}{r(l)^2 r(m)^2} \tag{18}$$

where now we define $\mathcal{S}^* = \{\{l, m\}\}$ to be the set of all combinations of l and m with $l, m \in \{0, \ldots, k\}$ and $m > l$ such that $|\mathcal{S}^*| = \binom{M}{2}$. Note that the second summation term is a constant which plays no part in solving the optimization problem. Consequently, we neglect this term such that substituting $r(k) = \frac{N-k}{N}$ into the first term leads to the optimization problem in Theorem 2. $\square$

Corollary 1. Suppose N is arbitrarily fixed and $\mathbf{s}(0)$ and $\mathbf{x}$ are fixed such that $r(0) = 1$ and $\phi(0) = 0$. Then there exists two potential optimal values for $\phi(1)$ given by $\phi(1) = \pm\pi/2$ and these in turn define $\mathbf{s}(1)$.

In order to generate the optimal trajectory for one-step ahead navigation under the considered problem assumptions, we provide the following recursive algorithm.

Algorithm 1. (One-Step Ahead Trajectory Derivation)

(1) Initialize $k = 0,\ \phi(0) = 0,\ r(0) = 1$
(2) Choose $\phi(1) = +\pi/2$ or $\phi(1) = -\pi/2$
(3) $\phi_i(k+1) = \operatorname{argmin}_{\phi(k+1)} \ \sum_{i=0}^{k} \frac{\sin^2(\phi(k+1) - \phi(i))}{(N-i)^2}$
(4) $k = k+1$, Goto Step 3

Algorithm 1 specifies how the optimal trajectory is actually computed recursively beginning with the optimization and decision on the value of $\phi(1)$. Only the relevant bearing history given by $\phi(0)$ through to $\phi(k)$ is needed in determining the next step position at time $k+1$.

5.1 One-Step Plan Ahead Trajectories with $N \in \{2,3\}$

Consider firstly the case of $N = 2$ and recall that with no loss of generality, the coordinate system is arranged such that $r(0) = 1$ and $\phi(0) = 0$. Hence, as stated previously, there are only $N-1$ discrete points to optimally determine in a one-step look ahead fashion. From Theorem 1 part (iv) we obtain the following optimization problem for $\phi(1)$ in terms of the bearing history,

$$\phi(1) = \underset{\phi(1)}{\text{argmax}} \quad \frac{1}{r(0)^2 r(1)^2} \sin^2(\phi(1) - \phi(0)) \tag{19}$$

which, upon making the substitution $r(0) = 1$, $\phi(0) = 0$ and $r(k) = \frac{N-k}{N}$, leads to

$$\phi(1) = \underset{\phi(1)}{\text{argmax}} \quad 4\sin^2(\phi(1)) \tag{20}$$

Clearly when $N = 2$, the optimization problem (20) is easily solved analytically or numerically. Solving (20) gives $\phi(1) = \pm\pi/2$ where the optimal value of $\phi(1)$ now restricts the position of $\mathbf{s}(1)$ to one of two possible locations, i.e. $\phi(1) = +\pi/2 \Rightarrow \mathbf{s}(1) = [1, -0.5]$ or $\phi(1) = -\pi/2 \Rightarrow \mathbf{s}(1) = [1, 0.5]$. The two locations are determined as a consequence of the assumption (i.e. constraint) that each pursuer position is subsequently shifted one-step ahead to the next circle centered at the target (out of the $N-1$ evenly spaced circles), e.g. as in Figure 1. The two resulting trajectories for $N = 2$ are shown in Figure 4 in the arranged coordinate system.

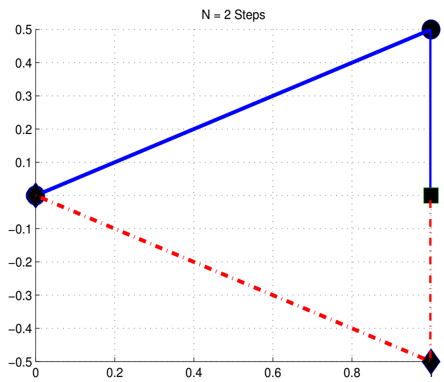

Fig. 4. *Two optimal trajectories for one-step ahead navigation when $N = 2$. At each step k the pursuer considers the bearing history up to and including that step k and plans ahead only one-step.*

Now consider the case of $N = 3$ with $r(0) = 1$ and $\phi(0) = 0$. It is easy to derive the following optimization problem for $\phi(1)$ in terms of the bearing history, i.e. in terms of $\phi(0)$,

$$\phi(1) = \underset{\phi(1)}{\text{argmax}} \quad \frac{9}{4}\sin^2(\phi(1)) \tag{21}$$

Solving the maximization (21) gives $\phi(1) = \pm\pi/2$. In this case the value of $\phi(1)$ restricts the position of $\mathbf{s}(1)$ to one of two possible locations which are different to those found for the $N = 2$ case despite the same bearing values. The range $r(k) = \frac{N-k}{N}$ at each time k is obviously different when N is different.

In this section we are considering a one-step ahead planning scheme such that when $N = 3$ and $k = 1$ we can assume knowledge of $\phi(0) = 0$ and $\phi(1) = \pm\pi/2$. Thus, we derive the following optimization problem for $\phi(2)$ from Theorem 1 part (iv),

$$\begin{aligned}\phi(2) = \underset{\phi(2)}{\text{argmax}} \quad & \frac{1}{r(0)^2 r(2)^2} \sin^2(\phi(2) - \phi(0)) + \\ & \frac{1}{r(1)^2 r(2)^2} \sin^2(\phi(2) - \phi(1)) + \\ & \quad \frac{1}{r(0)^2 r(1)^2} \sin^2(\phi(1) - \phi(0))\end{aligned} \tag{22}$$

where $r(k) = \frac{N-k}{N}$ and since we know the bearing history $\phi(0) = 0$ and $\phi(1) = \pm\pi/2$ there is only one unknown variable $\phi(2)$. That is, we have derived a single variable optimization problem. Making the relevant substitutions leads to

$$\phi(2) = \underset{\phi(2)}{\text{argmax}} \ \frac{27}{2}\sin^2(\phi(2) \pm \frac{\pi}{2}) + 9\sin^2(\phi(2)) \tag{23}$$

Solving the maximization (23) is again analytically and numerically simple. The problem is easily solved with $\phi(2) = 0$ or $\phi(2) = \pi$. Note that the trajectory is not generally unique. Any optimal solution to the problem (23) given the specific bearing history will permit an optimal trajectory. In the examples shown, we illustrate the most intuitive trajectories from a controller design point of view. Two example optimal trajectories for $N = 3$ and one-step look ahead navigation are shown in Figure 5.

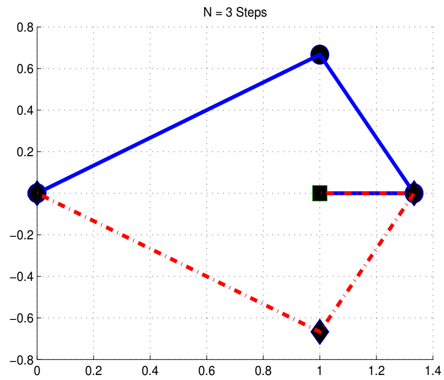

Fig. 5. *Two example optimal trajectories for one-step ahead navigation when $N = 3$. At each step k the pursuer considers the bearing history up to and including that step k and plans ahead only one-step.*

It is important to note that the optimization at each step k is a single variable optimization problem. Indeed, the optimization problem is clearly straightforward to solve analytically or numerically via a simple algorithm. A recursive (symbolic) computer aided algorithm is also straightforward to implement for this problem since the first and second derivatives of the single variable equation are defined and easily obtained.

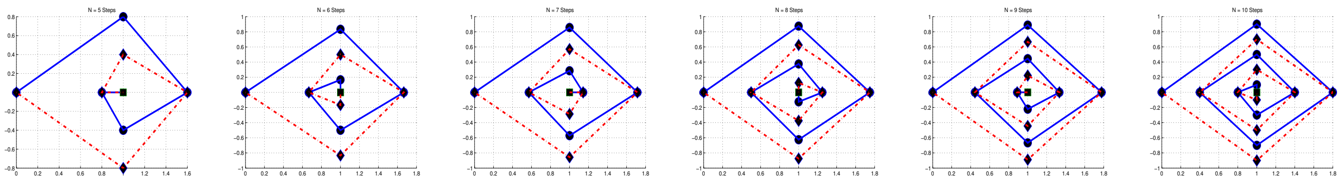

Fig. 6. *Two example optimal trajectories for one-step ahead navigation with $N \in [5, 10]$. At each step k the pursuer considers the bearing history up to and including that step k and plans ahead only one-step.*

Finally, consider the case of $N = 4$ with $r(0) = 1$ and $\phi(0) = 0$. It is easy to derive the following optimization problem for $\phi(1)$ in terms of the bearing history, i.e. in terms of $\phi(0)$,

$$\phi(1) = \underset{\phi(1)}{\text{argmax}} \quad \frac{16}{9} \sin^2(\phi(1)) \tag{24}$$

Solving the maximization (24) gives $\phi(1) = \pm\pi/2$. Again, the value of $\phi(1)$ restricts the position of $\mathbf{s}(1)$ to one of two possible locations.

Notice that the guidance problem considered in this paper is a strict homing guidance problem. At each time $k + 1$ the pursuer is strictly closer to the target than at time k.

5.2 Additional Examples and Further Discussions

Consider the optimization problem described in Theorem 2 or the recursive Algorithm 1 which generates the optimal trajectory for one-step look ahead navigation. Applying Algorithm 1 in practice is straightforward and numerically solving the relevant optimization problem is also straightforward and robust. That is, the single variable optimization problem considered in this case has a bounded parameter input, i.e. $\phi(k) \in [0, 2\pi)$ or $\phi(k) \in [-\pi, \pi)$, and is generally easy to solve numerically with an arbitrarily high accuracy. In this section we *analytically* employed Algorithm 1 to derive the optimal trajectories for $N \in [5, 10]$.

We applied Algorithm 1 for $N \in [5, 10]$ and generated the two most intuitive optimal trajectories. The resulting optimal discrete-time trajectories are given in Figure 6.

No other trajectory will yield a better (unbiased) estimate of the target position at time $k = N - 1$ then the example trajectories given in Figure 6 assuming the constraint on the pursuer movements introduced in this paper and assuming one-step look ahead navigation.

6. CONCLUSION

We provided a characterization of optimal trajectories for N-step and one-step look ahead guidance and navigation using bearing measurements. No directly similar work exists in the literature which attempts to explicitly characterize such pursuer-target geometries. We defined the problem to make it tractable. However, the practicality is not lost and the problem definition is still general and useful. We also provided a number of illustrative examples that show the practicality of the results provided.

The objective of this work was not to derive explicit control laws or estimators but rather to characterize optimal trajectories. Similarly to the localization geometry characterization, the analysis given in this paper inherently assumes an unbiased and efficient estimator is used to estimate the target location. However, the estimation technique used in practice is likely to be biased Nardone et al. (1984); Gavish and Weiss (1992). Nonetheless, many localization algorithms are actually designed with unbiasedness in mind and with a goal of achieving the Cramer-Rao lower bound.

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