Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Optimal Estimation and Regulator:Risk-Sensitive Method for Systems of First Degree ⋆

Ma. Aracelia Alcorta Garcia *

* *Department of Physical and Mathematical Sciences-Autonomous University of Nuevo Leon Apdo Postal 143,Cd. Universitaria, San Nicolás de los Garza N. L., México, C.P. 66451,(e-mail: aalcorta@fcfm.uanl.mx)*

Abstract: The algorithms for the optimal filter and control have been obtained for polynomial systems of first grade. For the filter, two cases are presented: systems with disturbances in L^2 and systems with Brownian motion and parameter ϵ multiplying the diffusion term, in state and observations equations. The performance of this algorithms is verified and compared with the optimal Kalman-Bucy filter through an example. Besides the solution to the optimal control Risk-Sensitive problem for stochastic system, taking quadratic value function as solution of PDE HJB is obtained. These Risk-Sensitive control algorithms are compared with the L-Q control algorithms through a numerical example, using quadratic-exponential cost function to be minimized. The optimal risk-sensitive filter and control algorithms show better performance for large values of the parameter ϵ.

1. INTRODUCTION

Since the linear optimal filter was obtained by Kalman and Bucy (60's), numerous works are based on it. I could mention some as M. V. Basin et al. [2003], M. V. Basin et al. [2003], F. L. Lewis [1992], V. S. Pugachev and I.N. Sinitsyn [2001], S. S.-T. Yau [1994], of the variety of all those. More than thirty years ago, Mortensen R. E. Mortensen [1968] introduced a deterministic filter model which provides an alternative to stochastic filtering theory. In this model, errors in the state dynamics and the observations are modeled as deterministic "disturbance functions," and a mean-square disturbance error criterion is to be minimized. In this case, special conditions for the existence, continuity and boundedness of $f(x(t))$ in the state equation, which is considered nonlinear, and for the linear function $h(x(t))$ in the observation equation, are given. A concept of the deterministic estimator, which is introduced more recently by McEneaney W. M. McEneaney [1998], is reviewed and applied to system with disturbances in L^2, where $f(x)$ has a nonlinear form in the dynamics of the system and linear observations. Since the optimal linear control problem has been solved in 60's H. Kwakernaak, R. Sivan [1972], W. H. Fleming, R. W. Rishel [1975], the basis of the optimal control theory is Dynamic Programming equation or Hamilton-Jacobi-Bellman equation W. H. Fleming, R. W. Rishel [1975], and the maximum principle of Pontryagin L. S. Pontryagin, et al. [1962]. A long tradition of the optimal control design for nonlinear systems (see, for example, E. G. Albrekht [1962], A. Haime and R. Hamalainen [1975], E. B. Lee and L. Marcus [1967], D. L. Lukes [1969], A. P. Willemstein [1977]) has been developed. The problem statement in robust H_∞ approach uses a dynamical model of the form

$$\dot{X}(t) = f(X(t)) + w(t) \tag{1}$$

where X is the state, $f(x(t))$ represents the nominal dynamics, $w(t) \in L^2$ is a deterministic process. This model is in contrast to the the diffusion model

$$dX_t = f(X(t))dt + \sqrt{\epsilon}dW(t) \tag{2}$$

where W is a Brownian motion. In W.H. Fleming et al. [2001] and W. M. McEneaney [2004] are presented these models where $f(X(t))$ takes nonlinear form. This paper presents an application of the algorithms obtained in W.H. Fleming et al. [2001] and W. M. McEneaney [2004] for singular form of $f(x(t))$. The goal of this work is to obtain the optimal filter and control risk-sensitive equations for these models, when $f(x(t))$ and $h(x(t))$ take a polynomial of first grade form in the state and observation equation respectively. The performance of the risk-sensitive optimal filter and control (stochastic case) algorithms is checked doing a comparison to the algorithms of the optimal Kalman-Bucy filter and traditional control through an example, for large values of ϵ. Following the theory of control and estimation, other method used in stochastic systems, is the finite time horizon case. In this method is considerate the risk-averse stochastic problem and its solution is obtained taking in account a value function which is a viscosity solution to the dynamic programming equation (H-J-B)W. M. McEneaney [2005], W.H. Fleming et al. [1992]. Since H_∞ control was originally formulated in the frequency domain, most of the results have been for the infinite time horizon problem. But the finite time horizon case is of interest in itself, because some applications

⋆ The author would like to thank W. M. McEneaney and Efrain Alcorta Garcia for helpful discussions. UCMEXUS-CONACyT Foundation for financial support under Posdoctoral Research Fellowship Program, UANL-PAICYT num. CA1480-07, and CONACYT num. 52930.

978-1-1234-7890-2/08/$20.00 © 2008 IFAC **10.3182/20080706-5-KR-1001.3726**

as aerospace trajectory guidance, control and navigation are naturally in finite time. A future work is to obtain the risk-sensitive filtering and control algorithms when $f(x(t))$ takes other polynomial forms, as quadratic or cubic, and do a comparison with the polynomial filtering and control algorithms. This work is organized as follows: The problem statement for system with disturbances in L^2 and for systems with Brownian motion is presented in Section 2.1 and 2.2. In Section 3 is given the general form of the optimal risk-sensitive stochastic nonlinear control problem. The solution for each case is given in Section 4 and 5. In Section 6 an numerical example is solved applying the risk-sensitive optimal filter algorithms and the algorithms of Kalman-Bucy optimal filter; risk-sensitive and traditional control. And the section 7 are the conclusions.

2. FILTERING PROBLEM STATEMENT

2.1 Deterministic case

For the first case, the state to be estimated $x(t)$ has differential equation (1) where $w(t)$ is the state disturbance and $x(0) = x_o$. The observation equation is given by:

$$y(t) = h(x(t)) + v(t) \tag{3}$$

where $v(t) \in \mathrm{L}^2$ is the observation disturbance. f_x, h_x bounded is assumed throughout. Here h_x is the matrix of partial derivatives of h and in the same form for Z_x. Taking $f(x(t)) = A(t) + A_1(t)x(t),\ h(x(t)) = E(t) + E_1(t)x(t)$, with $A(t) \in R^n, A_1(t), \in M_{n\times n}, E(t) \in R^p, E_1(t) \in M_{n\times p}$ where $M_{i\times j}$ denotes the field of matrices of dimension $i\times j$. The following system equations is obtained:

$$\dot{x}(t) = A(t) + A_1(t)x(t) + w(t), \tag{4}$$
$$y(t) = E(t) + E_1(t)x(t) + v(t).$$

Taking in account the state equation (1) and the observation equation (3), replacing the observation trajectory $y.$ by an accumulated observation trajectory: $Y_t = \int_0^t y_s ds\cdot$ In W.H. Fleming et al. [2001], you can see that taking in account the accumulate observations, the function $J(T, x; w)$ has the form:

$$\mathcal{J}(T, x; w.) = -\{\phi(x_0) + \int_0^T [\frac{1}{2}|w(t)|^2 + \tag{5}$$
$$\frac{1}{2}|h(x(t))|^2 + (Y(t)\cdot h(x(t)))_x \cdot (f(x(t)) + w(t))]dt\},$$

$Z(T, x) = sup_w \mathcal{J}(T, x; w)$ then, the value function associated is given by:

$$W(T, x) = Y(T)\cdot h(x) + Z(T, x), \tag{6}$$

it is shown W.H. Fleming et al. [2001] that $Z(T, x)$ is continuous, and that Z is a viscosity solution of the dynamic programming PDE:

$$-Z_T - f\cdot Z_x - \frac{1}{2}|h|^2 - (Y(T)\cdot h)_x\cdot f + \frac{1}{2}|(Z_x + \tag{7}$$
$$(Y(T)\cdot h)_x)|^2 = 0, \quad Z(0, x) = -\phi(x).$$

As was proposed in W. M. McEneaney [1998], and taking in account W.H. Fleming et al. [2001], in this case, $W(T, x)$ takes the form $W(T, x) = \frac{1}{2}(X - C(T)^{\mathcal{Z}})^T Q(T)^{\mathcal{Z}}(X - C(T)^{\mathcal{Z}}) + \rho(T)^{\mathcal{Z}} + \frac{1}{2}\int_0^T /y(t)/^2 dt$, where $C(T)$ denotes the estimate vector, Q_T is a quadratic, positive definite symmetric matrix and ρ is a parameter with values in final time T. The filtering problem is to find the best estimate of the state $x(t)$, which minimizes the quadratic criterion (5), where $Z(T, x)(6)$, is a viscosity solution of (7).

2.2 Filtering Stochastic case

Consider the stochastic model formed by (2), in which $X(t)$ denotes the state process. $Y(t)$ denotes a continuous accumulated observation process which is represented by.

$$dY(t) = (E(t) + E_1(t)x(t))dt + \sqrt{\epsilon}d\tilde{B}(t), \quad Y(0) = 0 \tag{8}$$

where ϵ is a parameter and B and $\tilde{B}$ are independent Brownian motions in themselves and both are independent of the initial state $X(0)$. X_0 has probability density $k_\epsilon exp(-\epsilon^{-1}\phi(x(0)))$ for some constant k_ϵ. The rest of the paper are verify assumptions (A1)-(A5) (from W.H. Fleming et al. [2001]). Besides, it is assumed that

$$q^\epsilon(0, x) = exp(-\epsilon^{-1}\phi(x)) \tag{9}$$
$$q^\epsilon(T, x) = p^\epsilon(T, x)exp[\epsilon^{-1}Y(T)\cdot h(x)]$$

where $p^\epsilon(T, x)$ is called pathwise unnormalized filter density. Taking log transform: $Z^\epsilon(T, x) = \epsilon log p^\epsilon(T, x)$, which satisfies the nonlinear parabolic PDE

$$\frac{\partial Z^\epsilon}{\partial T} = \frac{\epsilon}{2}tr(Z^\epsilon_{xx}) + A^\epsilon\cdot Z^\epsilon + \frac{1}{2}Z^\epsilon_x\cdot Z^\epsilon_x + B^\epsilon \tag{10}$$

with initial data $Z^\epsilon_x(0, x) = -\phi(x)$. The risk-sensitive optimal filter problem consists in found the estimate C^ϵ_T, of the state $x(t)$ through verification that

$$Z^\epsilon(T, x) = \frac{1}{2}(x - C(T)^\epsilon)^T Q(T)^\epsilon(x - C(T)^\epsilon) + \tag{11}$$
$$\rho(T)^\epsilon - Y(T)\cdot h(x(t))$$

is a viscosity solution of (10), and $Q(T)^\epsilon$ is Riccati matrix equation ($Q(T)^\epsilon$is symmetric matrix). In W.H. Fleming et al. [2001] it is proved, that the equation (10)(stochastic case) converges to the equation (7)(deterministic case) as ϵ goes to zero. Substituting $f(x(t)), h(x(t))$ in (2) as in deterministic case (4), the next stochastic equations system is obtained:

$$dX(t) = A(t) + A_1(t)X(t) + \sqrt{\epsilon}dB(t) \tag{12}$$
$$dY(t) = E(t) + E_1(t)X(t) + \sqrt{\epsilon}d\tilde{B}(t)$$

where $A(t), A_1(t), E(t), E_1(t)$ are as in (4).

3. FILTERING SOLUTION

Taking in account the system of state and observation equation (4), the partial derivatives of (6) are obtained. Upon substituting into (7) and collecting x terms, the next filter equation is obtained, where $C(T)^{\mathcal{Z}}$ denotes the estimator of $x(t)$, and it is the solution of the following differential equation:

$$dC(T)^{\mathcal{Z}} = (A(t) + A_1(t)C(T)^{\mathcal{Z}})dt - Q(T)^{\mathcal{Z}^{-1}}E_1(t)(dY(t) - (E_1(t)^T C(T)^{\mathcal{Z}} + E(t))dt),\ C(0)^{\mathcal{Z}} = c(o)^{\mathcal{Z}}. \quad (13)$$

where the symmetric matrix $Q(T)^{\mathcal{Z}}$ is obtained collecting x^2 terms and is the solution of the next Riccati matrix equation:

$$\dot{Q}(T)^{\mathcal{Z}} = -A_1(t)Q(T)^{\mathcal{Z}} - Q(T)^{\mathcal{Z}}A_1^T + (Q(T)^{\mathcal{Z}})^2 - E_1(t)^T E_1(t),\ \ Q(0)^{\mathcal{Z}} = q(o)^{\mathcal{Z}}. \quad (14)$$

Here $Q(T)^\epsilon$ is a symmetric matrix and the initial condition $Q(0)^\epsilon = q(o)^\epsilon$ is found from the equilibrium condition $\dot{Q}(T)^\epsilon = 0$. Where if $\overline{Q}$ is one solution. Then, should be $q(0)^\epsilon \leq \overline{Q}$, for which $\ddot{Q}(T)^\epsilon(\overline{Q}) < 0$, where $\overline{Q}$ is one of two equilibrium points.Taking $Z^\epsilon(T,x)$ and following the steps of the deterministic case, the equation for the optimal risk-sensitive stochastic estimator is the same obtained in the deterministic case.

4. CONTROL PROBLEM STATEMENT

The following stochastic risk-sensitive control problem has dynamics:

$$dX(t)^\epsilon = f(t, X(t)^\epsilon, u(t))dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dB(t), \quad (15)$$

$X(s)^\epsilon = x$. The quadratic cost criterion is given by

$$I^\epsilon(s,x,u) = \epsilon log E_{s,x} exp\{\frac{1}{\epsilon}[\int_s^T L(t, X(t)^\epsilon, u(t))dt + \psi(X(T)^\epsilon)]\}.$$

where $f(t, X(t)^\epsilon, u(t))$ is a nonlinear function which represents the nominal dynamics with control $u(t)$ taking values in $U \in \mathbf{R}^l$ and $\{B,\ F\}$ is an m-dimensional Brownian motion on the probability space (Ω, F, P) where F_0 contains all the P-negligible elements of F. ϵ will be a measure of the risk-sensitivity and scales the diffusion term in (15) above so that the cost below will remain bounded (for each x as a function of ϵ), $0 \leq s \leq T < \infty, T$ is a fixed terminal time, $L(t, X(t)^\epsilon, u(t))$ is the quadratic running cost, and $\psi(X(T))$ is the quadratic terminal cost. The following costs functions are defined:

$$A^\epsilon(s,x,u,\omega) = \int_s^T L(t, X(t), u)dt + \psi(X(T)),$$

$$J^\epsilon(s,x,u) = E_{s,x} exp[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega)], \quad (16)$$

so that

$$I^\epsilon(s,x,u) = \epsilon log J^\epsilon(s,x,u) = \epsilon log E_{s,x} exp[\frac{1}{\epsilon}A^\epsilon(s,x,u,\omega)].$$

Taking in account that the controller $u(t)$ is minimizing, and $w \in \mathbf{R}^n$ is a maximizing control, the next value functions are considered:

$$V^\epsilon(s,x) = inf_{u \in A_{s,\nu}} I^\epsilon(s,x,u) \quad (17)$$

where $A_{s,v}$ is the set of progressively measurable controls with values in U.

$$\varphi^\epsilon(s,x) = inf_{u \in A_{s,\nu}} J^\epsilon(s,x,u) \quad (18)$$

It is showed in W. M. McEneaney [2004] that under certain conditions, when $f(t, X(t), u(t))$ is a nonlinear function, V^ϵ is a viscosity solution of the dynamical programming equation

$$0 = V_s^\epsilon + \frac{\epsilon}{2\gamma^2}\sum V_{x_i x_j}^\epsilon + min_{u \in U}\{f(t, X(t)^\epsilon, u(t))\nabla_x V^\epsilon + L(t, X(t), u(t)) + \frac{1}{2\gamma^2}\nabla V^{\epsilon T}\nabla V^\epsilon\},\ V^\epsilon(x(t),T) = \psi(X(T)). \quad (19)$$

The following lemma shows that when $f(t, X(t)^\epsilon, u(t)) = A(t) + A_1(t)X(t)^\epsilon + u(t)$, V^ϵ is a viscosity solution of the dynamical programming equation(19). Taking $V^\epsilon = \epsilon log\varphi^\epsilon$, and substituting in (19) it is obtained the equation for φ^ϵ :

$$0 = \varphi_s^\epsilon + \frac{\epsilon}{2\gamma^2}\sum \varphi_{x_i x_j}^\epsilon + min_{u \in U}\{f(t, X(t)^\epsilon, u(t))\nabla_x\varphi^\epsilon + L(t, X(t), u(t))\phi^\epsilon\}, \varphi^\epsilon(x(t),T) = \psi(X) \quad (20)$$

The optimal control problem is to show that V^ϵ is a viscosity solution to the dynamic programming equation (19) when $f(t, X(t)^\epsilon, u(t))$ is polynomial of first grade, to find the optimal control which minimize the quadratic criterion J and find the optimal trajectory x^*, substituting u^* in to the state equation. The conditions for f, L, φ, U proposed in W. M. McEneaney [2004] are true when $f(t, X(t)^\epsilon, u(t))$ takes this form. As in W. M. McEneaney [2004], "cut off" problem is important, because the possibility unbounded functions f, L and ψ are replaced by bounded counterparts f^k, L^k and ψ^k in (19) and (20). The next lemma provides of proof that $V^{\epsilon,k}$ is the unique, bounded, classical solution to (19), taking in account that $f(t, X(t), u(t))$ polynomial of first degree, the proof for $f(t, X(t), u(t))$ nonlinear can see in W. M. McEneaney [2004].

Lemma The solution to (19) is the value function $V^{\epsilon,k}$ and the solution to (20) is the value function $\varphi^{\epsilon,k}$. An admissible feedback solution exists which yields the minimum. Furthermore, $V^{\epsilon,k}$ is the unique, bounded, classical solution to (19).

Proof: The result for φ is proved. The result for V follows. Let φ a solution of (20). First we show $\varphi(s,x) \leq J^{\epsilon,k}(s,x,u)$, for all $u \in A_{s,v}$ and $(s,x) \in Q_T = [0,T] \times \mathbf{R}^n$. For a fixed F_t- progressively measurable control, the solution to the stochastic differential equation, x^k (which would be denoted as x throughout the remainder of this proof) is a continuous semi-martingale, with in fact, square-integrable martingale part. Thus, since $\varphi \in C^{1,2}$, we can apply Itô's rule to yield:

$$\varphi^k(t,x(t)) = \varphi^k(s,x) + \int_s^t (\frac{\partial\varphi^k(s,x(r))}{\partial s} + \frac{\partial\varphi(s,x(r))}{\partial x(r)} \times f^k(x_r) + \frac{\epsilon}{4\gamma^2}\frac{\partial^2\varphi}{\partial x(r)^2})dr + \sqrt{\frac{\epsilon}{2\gamma^2}}\int_s^t \frac{\partial\varphi^k(s,x(r))}{\partial s}dB(r), \quad (21)$$

if

$$\frac{\partial\varphi^k(s,x(r))}{\partial s} = \varphi^k(r),\ \frac{\partial\varphi^k(s,x(r))}{\partial x(r)} = \nabla\varphi^k,\ \frac{\partial^2\varphi^k(s,x(r))}{\partial x(r)_i \partial x(r)_j} = \triangle\varphi^k,$$

then

$$\varphi^k(t,x(t)) = \varphi^k(s,x) + \int_s^t (\varphi^k(r)(r,x(r)) + \bigtriangledown\varphi^k(r,x(r))f^k(r,$$

$$x(r),u(r)) + \frac{\epsilon}{4\gamma^2}\bigtriangleup\varphi^k(r,x(r)))dr + \sqrt{\frac{\epsilon}{2\gamma^2}}\int_s^t \bigtriangledown\varphi^k(r,x_r)dB(r),$$

where

$$\int\bigtriangledown\varphi(r,x_r)dB(r) = \sum\int\varphi^k_{x_i}(r,x_r)dB(r)^{(i)}. \quad (22)$$

Since the lemma 2.3.1 in W. M. McEneaney [2004] states that the solution to (19) is bounded, from Ladyženskaja et al. O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva [1968], theorem 5.3.1 that $|\bigtriangledown V|$ is also bounded. Consequently the solution to (20) satisfies $|\varphi|, |\bigtriangledown\varphi|$ bounded also. Then exists a such that $|\bigtriangledown\varphi| \le a$ for all $(s,x) \in Q(T)$ which implies $E\int_s^T |\bigtriangledown\varphi|^2 dt \le a^2 T$, and consequently $\int_s^t \bigtriangledown\varphi(r,X(r))\cdot dB(r)$ is a square-integrable martingale. Thus $\varphi(t,X(t))$ is also a continuous semi-martingale. Since L^k is bounded, $\varepsilon_t \equiv exp[\frac{1}{\epsilon}\int_s^t L^k(r,X(r),u(r))dr]$ is a continuous semi-martingale with zero martingale part. Therefore applying the stochastic integration by parts formula to the product $\varepsilon_t\varphi(t,x(t))$ to yield:

$$\varepsilon_t\varphi^k(t,X(t)) - \varphi^k(s,x) = \int_s^t \varepsilon(r)[\frac{\partial\varphi^k}{\partial r} + \frac{\partial\varphi^k}{\partial x}\cdot f^k(x(t)) +$$

$$\frac{\epsilon}{4\gamma^2}\frac{\partial^2\varphi}{\partial x^2}]dr + \int_s^t \varphi^k(r,x(r))\frac{1}{\epsilon}L^k(r,X(r),u(r))\varepsilon(r)dr$$

$$+\sqrt{\frac{\epsilon}{2\gamma^2}}\int_s^t \varepsilon(r)\frac{\partial\varphi^k}{\partial x}dB(t).$$

Using PDE (20) to eliminate the first two terms on the right, we have:

$$\varepsilon_t\varphi^k(t,X(t)) - \varphi^k(s,x) \ge \sqrt{\frac{\epsilon}{2\gamma^2}}\int_s^t \varepsilon(r)\frac{\partial\varphi^k}{\partial x}dB(t).$$

Since L^k and $\bigtriangledown\varphi^k$ are bounded, so is $\varepsilon_r\bigtriangledown\varphi(r,X(r))$. Thus, by the same argument as above, $\int_s^t \varepsilon(r)\bigtriangledown\varphi^k(r,x(r))dB(r)$ is a square-integrable martingale. Therefore, taking $t = T$,

$$\varphi^k(s,x) \le E_{s,x}[\varepsilon_T\varphi(T,X(T))],$$

substituting ε and the terminal condition in (20):

$$E_{s,x}[\varepsilon_T\varphi(T,X(T))] = E_{s,x}exp\{\frac{1}{\epsilon}[\int_s^T L^k(t,X(t),u(t))dt +$$

$$\psi(X(T))]\} = J^{\epsilon,k}(s,x,u). \quad (23)$$

Now suppose there exists $u^* \in A_{s,v}$ such that

$$u^* \in argmin_{u\in U}[f^k(t,X(t)^*,v)\bigtriangledown\varphi(t,X(t)^*) + \quad (24)$$

$$\frac{1}{\epsilon}L^k(t,X(t),u(t))\varphi(t,X^*)], \forall t \in [s,t].$$

Then the equality in the above is right, and consequently,

$$\varphi(s,x) = J^{\epsilon,k}(s,x,u^*).$$

It is easily seen from W. H. Fleming, R. W. Rishel [1975], Appendix B, that exists a Borel measurable function $g(t,x)$ such that:

$$g(t,x) \in argmin_{u\in U}[f^k(t,x(t),u)\varphi^k_{x_i}(x,t) +$$

$$\frac{\varphi^k(t,x)}{\epsilon}L^k(t,X(t),u(t))], \forall(t,x) \in Q(T). \quad (25)$$

Consider the SDE:

$$dX(t) = f^k(t,X(t),g(t,X(t)))dt + \frac{\epsilon}{2\gamma^2}dB(t). \quad (26)$$

By Veretennikov A. J. Veretennikov [1981], Theorem 1, it has a unique strong solution for any reference probability system, ν. Letting $u^* = g(t,x(t))$ for the strong solution yields $u^* \in A_{s,\nu}$. Therefore

$$\varphi(s,x) = min_{u\in A_{s,\nu}} J^{\epsilon,k}(s,x,u) = \varphi^{\epsilon,k}(s,x)$$

To prove uniqueness claim suppose there exists another bounded, classical solution, $\tilde{\varphi}$. Then, by the same proof as above, it is the value function $\varphi^{\epsilon,k}$. The result for V follows similarly. ◊.

5. CONTROL SOLUTION

Taking in account that $f(t,X(t)^\epsilon,u(t)) = A(t)+A_1(t)X(t) + b(t)u(t)$ and substituting in (15), the next state equation is obtained:

$$dX(t)^\epsilon = (A(t) + A_1(t)X(t) + b(t)u(t))dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dB(t), \quad (27)$$

$X^\epsilon_s = x$, where $X(t), \quad A(t) \in \mathbf{R}^n, A_1(t) \in M_{nxn}$, where M_{nxn} denotes the field of matrices of dimension nxn, and $B(t)$ is as in (15). If $L(t,X^\epsilon_t,u) = X(t)^2+u(t)^2, \psi(X(T)) = X(T)^2$, the quadratic cost criterion has the form:

$$J^{\epsilon,k}(s,x,u) = \epsilon log E_{s,x}exp[\frac{1}{\epsilon}[\int_s^T (X(t)^2 + u(t)^2)dt + (X(T)^\epsilon)^2]]$$

As is proposed in W. M. McEneaney [1998], in this case, the value function

$$V^\epsilon(s,X) = \frac{1}{2}(X(t) - C(s))^T P(s)(X(t) - C(s)) + r(s)$$

$(C(s), P(s), r(s)$ are functions of $s \in [0,T], C(s) \in \mathbf{R}^n, P(s)$ is a symmetric negative defined matrix of dimension nxn and $r(s)$ is a scalar function) as a viscosity solution of the dynamic programming equation

$$0 = V_s + \frac{\epsilon}{2\gamma^2}\sum V_{x_i x_j} + min_{u\in U}\{(A(t) + \quad (28)$$

$$A_1(t)X(t) + u(t))\nabla_x V + X(t)^2 + u(t)^2 + \frac{1}{2\gamma^2}\nabla V^T\nabla V\}$$

$V(x(t),T) = \psi(X)$ where V_s, V_x are the partial derivatives of V respect to s,x respectively and ∇V is the gradient of V. Then the partial derivatives of V^ϵ are obtained and substituting these in (28), when $f(t,x(t),u(t))$ is polynomial of first degree, following the steps as in the filter solution:

$$\dot{P}(s) = P(s)^T(\frac{b^T b}{2} - \frac{1}{\gamma^2})P(s) - A_1^T(t)P(s) - P(s)A_1(t) - 2I$$

$$\dot{C}(s) = A_1^T(t)C(s) + 2C(s)^T P(s)^{-1} + A(t), \quad (29)$$

where $P(T) = I$, $C(T) = 0$, the optimal control law which minimizes the quadratic criterion is given by:

$$u^*(t) = -\frac{1}{2}b^T(t)P(s)(X - C(s)) \quad (30)$$

6. APPLICATIONS

6.1 Risk-sensitive optimal filter

For the dynamical system (12), if $f(x(t)) = 1 - 0.1x(t)$, $h(x(t)) = 1 + x(t)$, the following stochastic state and observation equation are obtained:

$$dx(t) = (1 - 0.1x(t))dt + \sqrt{\epsilon}dB(t), \quad (31)$$
$$dy(t) = (1 + x(t))dt + \sqrt{\epsilon}d\bar{B}(t)$$

where $x(t) \in R, B(t), \bar{B}(t)$ are independent Brownian motions, $\epsilon = 1000000$. Proposing (11) as a viscosity solution of (10), getting the derivatives $Z_x^\epsilon, Z_{xx}^\epsilon, \frac{\partial Z^\epsilon}{\partial T}$ of (11) and substituting in (10), the following equations are obtained for the estimate $C(T)^\epsilon$ and for the symmetric matrix $Q(T)$, which are equivalent to substituting the corresponding values in (13) and (14):

$$\dot{Q}(T) = 0.2Q(T) + Q^2(T) - 1 \quad (32)$$
$$dC(T)^\epsilon = (1 - (0.1)C(T)^\epsilon)dt - \frac{1}{Q(T)}(dY(T) - C(T)^\epsilon dt)$$

The last equations (32) are simulated using $MatLab7$. The initial conditions for the simulation are $x(0) = y(0) = 0$, $Q(0)^\epsilon = -0.0001$, $C(T)^\epsilon = 1000$, $T = 10seg$. The graph of the absolute values of the difference between state $x(t)$, and the estimate $C(T)^\epsilon$: $error = |x(t) - C(T)^\epsilon|$, is shown in Figure 1.

6.2 Kalman-Bucy optimal filter equations.

Applying the Kalman-Bucy optimal filter algorithms R. E. Kalman and R. S. Bucy [1961] to the state equations (31), the equations for the estimate vector $m(t)$ and symmetric covariance matrix $P(t)$ are obtained:

$$dm(t) = (-0.1m(t) + 1)dt + \frac{P}{\epsilon}(dY - (m(t) + 1)dt)$$
$$\dot{P}(t) = -0.2P(t) + \epsilon - \frac{P^2(t)}{\epsilon}$$

This system of equations is simulated with the initial conditions: $m(0) = 1000$, $P(0) = 10000$. The graph of the absolute value of the difference between state $x(t)$, and the estimate $m(t)$, that is: $error = |x(t) - m(t)|$, can be seen in Figure 2.

6.3 Optimal R-S Stochastic Control

Give the next linear stochastic state equation:

$$dx(t) = (1 + 0.1x(t) + u(t))dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dB(t) \quad (33)$$
$$L(t, X(t), u(t)) = X(t)^2 + u(t)^2; \psi(x(T)) = x(T)^2$$

where $A(t) = 1, A_1(t) = 0.1$, $\epsilon = 0.01, \gamma = 2$. The value of γ is obtained (as equilibrium point of (35)). The quadratic cost criterion takes the form:

$$J(s, x(t), u(t)) = \epsilon log E_{s,x} exp(\frac{1}{\epsilon}\int_s^T (x(t)^2 + u(t)^2)dt + x(T)^2) \quad (34)$$

Substituting the values of A, A_1 into the equations (29), and (30), are obtained the next equations in reverse time:

$$\frac{dP}{dt} = -0.2P(s) - 2 + p(s)^2(\frac{1}{2} - \frac{1}{\gamma^2})$$
$$\frac{dC(s)}{dt} = 1 + (0.1)C(s) + 2\frac{C(s)}{P(s)} \quad (35)$$
$$u* = -\frac{1}{2}P_s(x - C(s)).$$

The system (35), is stable if $|\gamma| \geq 1.40$. The final conditions in $T = 5seg$ are: $P(5) = 1, C(5) = 0$, the initial condition for $x(0) = 1; \gamma = 2$. Solving this system of equations (35), the values of the optimal control law u^*, the optimal trajectory: $\dot{x}^* = (1 + (0.1)x(t) - 1/2P_s(x - C(s)) + \sqrt{(\epsilon/2\gamma^2)}dB(t)$, are obtained, substituting the optimal control u^* in to the state equation (33). The value of the criterion quadratic to be minimized J at time T is obtained for each value of ϵ. The graphics of the state $x(t)$, the optimal control $u(t)$, the criterion J can be seen in the Figure 4, for $\epsilon = 1000$.Table 1 illustrate the values of J for some values of the parameter ϵ. The value of J was approximated using Monte Carlo method.

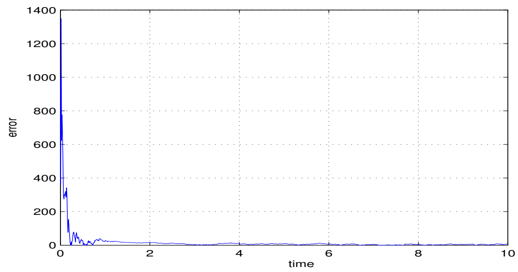

Fig. 1. Graphs of the absolute values of the difference between $x(t)$ and the linear r-s estimate C_T.

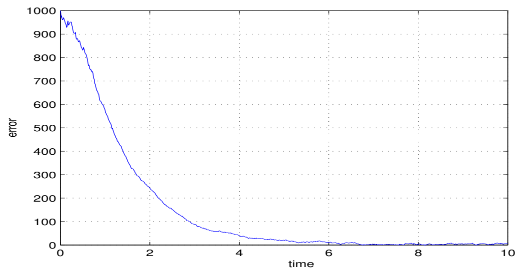

Fig. 2. Graphs of the absolute values of the difference between $x(t)$ and the linear K-B estimate m .

6.4 Optimal Linear Quadratic Control

Taking in account the state equation (33), the traditional optimal non homogeneous control W. H. Fleming, R. W. Rishel [1975] is obtained: $u(t)^* = R^{-1}b^T(t)(Q(t)x(t) + p(t))$, where $Q(t)$ is the solution of the gain equation:

$$\dot{Q(t)} = -0.2Q(t) + 1 - Q^2(t)$$

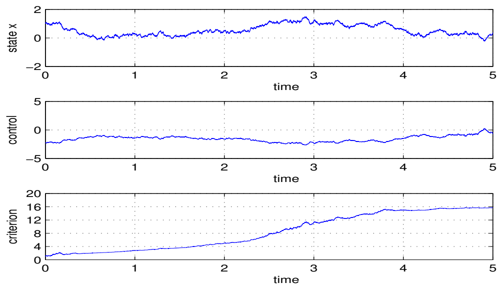

Fig. 3. Graphs of the optimal state variable $x(t)$, optimal control $u(t)^*$ and criterion J for L-Q control.

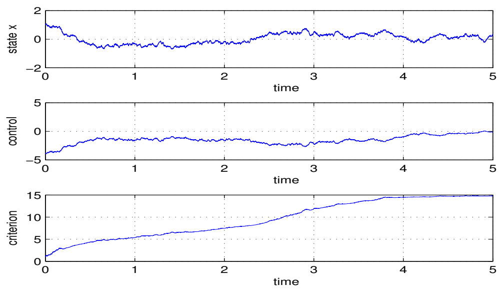

Fig. 4. Graphs of the state variable $x(t)$, optimal control $u(t)$, and criterion $J(t)$ for risk-sensitive control.

ϵ	J(r-s control)	J(trad. control)
0.1	9.875	7.9152
1	9.8982	8
10	10	8.278
100	10.5616	9.4371
1000	14.8716	15.774
10000	53.7456	62.5252
100000	429.1228	477.4969

Table 1. Values of J, for some ϵ values, with algorithms risk-sensitive and L-Q control

and $p(t)$ is the solution of: $\dot{p}(t) = -Q - 0.1p(t) - Q(t)p(t)$ with final conditions: $Q(5) = -2, p(5) = 0$ The optimal trajectory takes the form: $dx(t) = (1 + (0.1)x(t) + (Q(t)x(t) + p(t)))dt + \sqrt{(\epsilon/2\gamma^2)}dB(t)$. The quadratic criterion to be minimized is the same in both controls. The graphics of the state, optimal control and criterion, for $\epsilon = 1000$ can see in Figure 3.

7. CONCLUSION

This paper presents the optimal solutions to the risk-sensitive optimal control and filtering problems for stochastic first degree polynomial systems with Gaussian white noises, an exponential-quadratic criterion to be minimized, and intensity parameters multiplying the white noises, using using quadratic value functions as solutions to the corresponding Hamilton-Jacobi-Bellman equations. The optimal filter risk-sensitive algorithms and Kalman-Bucy optimal filter are obtained, and compared. When ϵ grows, the estimate risk-sensitive converges in less time to the real value than the Kalman-Bucy estimate, as shown in Figure 1 and 2. The optimal control risk-sensitive algorithms and traditional optimal control are obtained, and compared, using the criterion exponential-quadratic of Risk-Sensitive method. When ϵ takes small values ($0.1 \leq \epsilon < 1000$), the performance of L-Q control is verify, when ϵ grow, the performance of Risk-Sensitive control is verify (values of J are lowest for $\epsilon \geq 1000$). You can see it in Table 1, and in Figure 3 and 4 for $\epsilon = 1000$.

REFERENCES

E. G. Albrekht. On the Optimal Stabilizacion of Nonlinear Systems. J. Appl. Maath.Mech.**25**, 1254–1266, 1962.

M. V. Basin et al. Optimal Filtering and Control for Third Degree Polynomial Systems. *Dynamics of Continuous Discrete and Impulsive Systems*,**10** 663–680, 2003.

M. V. Basin et al. Optimal Filtering for Bilinear Systems and its Application to Terpolymerization Process State. Proc. IFAC 13th. Symposium System Identification-SYSID-2003.467–472, 2003.

W. H. Fleming et al. Robust limits of Risk Sensitive NonLinear Filters. Math. Control Signals and Systems, **14**, 2001, 109–142.

W. H. Fleming et al. Risk-sensitive control and differential games. Springer-Verlag, New York, 1992, pp. 185–197.

W. H. Fleming, R. W. Rishel. Deterministic and Stochastic Optimal Control. Springer,New York, USA, 1975.

A. Haime and R. Hamalainen. On the Nonlinear Regulator Problem. J. Opt. Theory and Appl., **16**, 3–4, 1975.

R. E. Kalman and R. S. Bucy. New results in linear filtering and prediction theory. ASME J. Basic Eng. Ser. D, **83**, 1961.

H. Kwakernaak, R. Sivan. Linear Optimal Control Systems. Wiley, New York, USA, 1972.

O.A. Lady*ž*enskaja, V.A. Solonnikov, N.N. Ural'ceva. linear and Quasi-Linear Equations for Parabolic Type, American Mathematical Society,Providence, RI, 1968.

E. B. Lee and L. Marcus. Foundations of the Optimal Control Theory Systems. Wiley, New York, USA. 1967.

F. L. Lewis, Applied Optimal Control and Estimation, Prentice Hall Englewood Cliffs, New Jersey, EUA, 1992.

D. L. Lukes. Optimal Regulation of Nonlinear Dynamic Systems", JSIAM j. Control Opt., **7**, 75–100, 1969.

W. M. McEneaney Robust H-∞ filtering for nonlinear systems. Systems and Control Letters, 315–325, 1998.

W. M. McEneaney. Max-Plus Methods for Nonlinear Control and Estimation. Ed. Board, USA, 2005.

W.M.McEneaney. Max-Plus eigenvector representations for solution of nonlinear H_∞ problems:error analysis. SIAM J. Control and Opt.,**43**, 379–412, 2004.

R. E. Mortensen. Maximum Likelihood Recursive Nonlinear Filtering. J. Optim. Theory Appl. 386–394, 1968.

L. S. Pontryagin, et al. The Mathematical Theory of Optimal Processes. Fizmalit, New York, USA, 1962.

V. S. Pugachev and I.N. Sinitsyn. Stochastic Systems Theory and Applications. World Scientific, Singapore, 2001.

A. J. Veretennikov. On strong Solutions and Explicit Formulas for Solutions of Stochastic Integral Equations. Math. USSR Sbornik,**39**, 387-403, 1981.

A. P. Willemstein. Optimal Regulation of Nonlinear Dynamical Systems in a Finite Interval. SIAM J. Control Opt.**15**, 1050–1069, 1977.

S. S.-T. Yau. Finite-Dimensional Filters with Nonlinear Drift. I: A Class of Filters Including Both Kalman-Bucy and Benes Filters. J. Math. Systems Estimation and Control,181–203, 1994.