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Controllability and Observability of Linear Discrete-Time Systems with Network Induced Variable Delay ⋆

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Abstract: This paper studies the controllability and observability of discrete-time systems with network-induced variable delays. Since controllability and observability are structural properties of systems, which are first checked before control design, we study if a controllable (resp. observable) non-delayed system can loose these properties if we augment the model with particular pure input-output variable delays caused by a situation of overload in the networked control architecture. We start our approach with a discrete-time multivariable linear time-invariant system with non-equal network-induced delays on control signals (inputs) and measures (outputs). The considered delays may only remain constant or increase with unitary increments. We prove that if a non-delayed system is controllable (resp. observable), then the network-delayed system is controllable (resp. observable) despite the monotonically-increasing delay values in each input/output channel. This general powerful result ensures further implementation of model-based predictive control strategies based on state observers methods for the considered model of networked control systems.

Keywords: Networked control systems, linear systems, controllability, observability

1. INTRODUCTION

A networked control system is a control system whose sensors, actuators, and controllers are interconnected over a shared communication network (NCS). The main advantage of this NCS configuration is the reduction of complexity, weight and volume with respect to the point-to-point wiring. It allows also improving the flexibility and the modularity of the overall system: new sensors, actuators or controllers can be added to the system with no major change in its structure. However, using a network introduces modifications of the temporal characteristics of control and measure signals, due to the communication constraints. In the literature, different models of communication constraints have been studied (Hespanha et al. (2007)), such as network induced delays (Nilsson (1998)), information loss (Schenato et al. (2007)), data rate limitations (Nair et al. (2007)) and medium access constraints (Ben Gaid et al. (2006)).

The use of networked control systems in industrial environment is rapidly spreading. In the general case of spatially distributed control systems, the transmission delays affecting the components of control commands or measurements vectors are different and time-varying. In deterministic control networks, those delays can occur as a consequence of a situation of network overload, resulting, for example, from a component fault.

Controllability and observability are structural properties of systems, which are first checked before control design. Various aspects of the controllability of discrete-time linear systems with delay were considered by several authors (Klamka (1977); Watanabe (1984); Phat (1989)), where mathematical conditions for investigating the controllability were stated. It is easy to see that for a given fixed delay structure, a linear time invariant system may be put into a standard state space representation with an extended state vector. However, it is not straightforward to see that under variable delay, the controllability and observability properties of the original system will be preserved.

In this paper, we consider a particular model of networked control systems operating under overload conditions. In this model, the delays affecting the transmission of controls or measures may only increase in unitary increments or remain constant. This assumption of increasingly-varying delays ensures that no loss of information occurs. Under these assumptions, we prove that the input/output increasingly-varying delay, which is induced by the overloaded network, does not change the controllability and observability properties of the original non-delayed plant. This result is established thanks to the Popov-Belevitch-

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Hautus criterion and the Artstein transform. Compared to non-delayed systems (Richard (2003); Niculescu (2000)), the controllability and observability of delayed systems presents differences related with state variables, time variables and the realization of control law. The state variables depend on a function defined on a time interval whose length is equal to the maximum input-channel delay. This article extends the results from Ionete et al. (2006) to the case of variable network-induced delays.

The paper is structured as follows: In Section 2 we introduce some mathematical preliminaries for modeling time-varying delays. Section 3 studies the controllability problem of input-delayed systems. In the fourth section, the dual problem, observability, is treated in the same manner but relative to varying output delays. The conclusion represents the final section.

2. PROBLEM FORMULATION

The original non-delayed discrete-time linear system is described by the state space representation

$$x_{k+1} = Ax_k + Bu_k \tag{1a}$$

$$y_k = Cx_k \tag{1b}$$

where A, B and C are respectively $n \times n$, $n \times m$ and $p \times n$ matrices and

$$u_k = \left[u_k^1 \; u_k^2 \; \ldots \; u_k^m \right]^T, y_k = \left[y_k^1 \; y_k^2 \; \ldots \; y_k^p \right]^T.$$

Control commands as well as plant measures are sent through a share communication network which induces different time-varying delays in each transmission channel. The non-uniform time-varying delays are measured in multiples of the sampling period. Using notations from (Marinescu and Bourlès (2000)), we can define:

$$z^{-G(k)} = \operatorname{diag}\left(z^{-g_1(k)}, z^{-g_2(k)}, ..., z^{-g_m(k)}\right),$$
$$z^{-H(k)} = \operatorname{diag}\left(z^{-h_1(k)}, z^{-h_2(k)}, ..., z^{-h_{p(k)}}\right).$$

where z^{-1} is the shift (one step delay) operator.

The following assumptions are made:

Assumption 1. The controller can obtain all the measures from the sensors and can send all the control commands to the plant. In our framework, we consider that all the control commands or measures reach their targets after a variable time-delay.

Assumption 2. We assume that the induced delays, which are time-varying, may only remain constant or increase. Such delays are called *monotonically-increasing time-varying.* This situation may occur in deterministic networks which are subject to an overload situation and where a fair scheduling strategy is implemented.

Assumption 3. The network-induced variable delays may only remain constant or increase with unitary increments. Indeed, supposing the contrary, we can obtain the following time evolution (Table 1).

Table 1. Consequences of a non-unitary increase in the delay

Time instant	k	$k+1$	$k+2$
Delay	2	2	4
Control	u_{k-2}	u_{k-1}	u_{k-2}

From Table 1, we observe that if we allow a two sampling time delay increase at moment $k+2$, we will use for control an older command: u_{k-2}. This is pointless since we have already received a more recent command, u_{k-1}.

3. CONTROLLABILITY PROBLEM STATEMENT

3.1 Preliminaries

We first address on the problem of controllability in presence of time-varying delays verifying assumptions 1,2 and 3. To this end, we consider the input-delayed system defined by

$$x_{k+1} = Ax_k + Bu_{k-G(k)}, \tag{2}$$

where

$$u_{k-G(k)} = \left[u_{k-g_1(k)}^1 \; u_{k-g_2(k)}^2 \; \cdots \; u_{k-g_m(k)}^m \right]^T.$$

The column representation of the input matrix B is

$$B = \left[c_B^1 \; c_B^2 \; \ldots \; c_B^m \right], \tag{3}$$

where c_B^j is the j^{th} column of B, $j \in \{1, \ldots, m\}$. In the following, we will consider that the variable delay in each input channel is bounded by a value M, defined by

$$M = \sup_{k \in \mathbb{N}} \left\{ \max_{i \in \{1,2,...,m\}} \{g_i(k)\} \right\}.$$

This assumption is natural since an infinite value for M leads necessarily to uncontrollability. First, consider all possible input signals with bounded delay

$$\{u_k, u_{k-1}, ..., u_{k-M}\}.$$

The discrete system with network-induced variable delays (2) has the following equivalent representation

$$x_{k+1} = Ax_k + \sum_{j=0}^{M} B_j(k) u_{k-j}. \tag{4}$$

In the above representation, $B_j(k)$ contains at instant k exactly the columns of the input matrix B that multiply the components of the control input vector having a delay of j steps. The other columns are zero. For example, assume that $m = 5$ (i.e. system (2) have 5 input channels). If at instant k, input channels 1 and 3 of system (2) have a delay of 2 steps, then matrix $B_2(k)$ will be defined as

$$B_2(k) = \left[c_B^1, 0_{n\times1}, c_B^3, 0_{n\times1}, 0_{n\times1}\right].$$

Of course, if we do not have a two steps delay in an input channel, $B_2(k) = 0_{n\times m}$. As a consequence, we have the equality

$$\sum_{j=0}^{M} B_j(k) = B. \tag{5}$$

A given column c_B^i ($i \in \{1, 2, ..., m\}$) from B may only be located in a unique matrix $B_j(k)$ ($j \in \{1, 2, ..., M\}$) at instant k.

The above representation is the model of a linear time-varying system. In this model, the time-varying delays affecting the control inputs are transferred to time-varying input matrices that multiply the current and the previous control inputs. It is important to underline the link between the variable delay of a given input channel and the inclusion of the corresponding column from input matrix B in different matrices $B_j(k)$.

Matrix A is the state matrix of system (1a), which does not contain delays in the state. It is obtained by discretizing a continuous-time linear time-invariant system. For that reason, matrix A is invertible.

We define the state evolution matrix

$$\Phi(k,h) = A^{k-h}.$$

If $\{u_k\}_{k\in\mathbb{Z}}$ is the control function, then $u[k]$, for a fixed k, denotes the restriction of $\{u_k\}_{k\in\mathbb{Z}}$ to the limited set

$$u[k] = \{u_{k-M},\dots,u_{k-1}\}.$$

Definition 1. The pair $(x_k, u[k])$ is referred to as the *absolute state* of system (4) at moment k.

It is common to say that the control function u steers the absolute state $(x(k_0),\nu_{k_0})$ to $(x(k_1),\nu_{k_1})$ during $\{k_0,k_0+1,\dots,k_1\}$ if $u[k_0]=\nu_{k_0}$, $u[k_1]=\nu_{k_1}$ and the solution of (4) with $x_{k_0}=x(k_0)$ satisfies $x_{k_1}=x(k_1)$. The controllability of system (4) is formally defined using the notion of absolute controllability (Olbrot (1972)).

Definition 2. System (4) is called *absolute controllable* on $\{k_0,k_0+1,\dots,k_1\}$ (with $k_1-k_0>M$) if for any prescribed absolute states $(x_{k_0},u[k_0])$ and $(x_{k_1},u[k_1])$, there exists a control function steering the first to the second during $\{k_0,k_0+1,\dots,k_1\}$.

In (Artstein (1982)) was introduced the so called “Artstein transform” for continuous-time linear time-variant systems with delays in command. The discrete-time analogue of this transform was established in Ji (2006). Using this transform, the discrete-time variant is defined by

$$z_k = x_k + \sum_{j=0}^{M}\sum_{i=k-j}^{k-1}\Phi(k+1,i+1+j)\,B_j(i+j)\,u_i.$$

Applying the above transform to (4), we obtain

$$z_{k+1} = Az_k + \left(\sum_{j=0}^{M}\Phi(k+1,k+1+j)\,B_j(k+j)\right)u_k.$$

Consequently, we obtain a non-delayed system

$$z_{k+1} = Az_k + F(k)u_k, \tag{6}$$

where

$$F(k) = \sum_{j=0}^{M}A^{-j}B_j(k+j).$$

The time evolution of the linear time-varying system (6) from the initial state z_0 is described by

$$z_k = A^k z_0 + \sum_{i=0}^{k-1}A^{k-i-1}F(i)\,u_i. \tag{7}$$

An important property of the Artstein transform is that it preserves the controllability. This result was established in the continuous-time case in Artstein (1982). The following Theorem treats the discrete-time case, and establishes that the “non-delayed” system (6) and the input-delayed system (4) are equivalent from the controllability point of view.

Theorem 1. The system (4) is absolute controllable on $\{k_0,\dots,k_2\}$ $(k_2-k_0>M)$ if and only if (6) is controllable on $\{k_0,\dots,k_2-M\}$.

Proof. Let x_{k_0} and $u[k_0]$ be specified at moment k_0. They determine an initial state z_{k_0} from the Artstein transform The un-delayed system (6) is controllable on $\{k_0,\dots,k_2-M\}$ if and only if any state z_{k_2} at moment k_2 can be reached from z_{k_0} by specifying $u[k_2]$. In fact, steering z_{k_0} to z_{k_2} with a prescribed control $u(k)$ on $k\in\{k_1,\dots,k_2\}$ $(k_0<k_1<k_2)$ is equivalent to steering z_{k_0} to z_{k_1} on $k\in\{k_0,\dots,k_1\}$ with

$$z(k_1) = \Phi(k_1,k_2)z(k_2) + \sum_{i=k_1-1}^{k_2}\Phi(k_1-1,i)B(i)u_i.$$

However, specifying z_{k_2} and $u[k_2]$ determines x_{k_2} from the Artstein transform. Therefore, the absolute state $(x_{k_2},u[k_2])$ can be arbitrarily specified if and only if (6) is controllable on $\{k_0,\dots,k_2-M\}$. This proves that the discrete-time Artstein transform establishes equivalence between the controllability of the delayed system (4) and the un-delayed system (6). □

Finally, we mention the algebraic Popov-Belevitch-Hautus (PBH) controllability criterion for LTI discrete systems, which states the following equivalence

$$\operatorname{rk}\left[B, AB,\dots,A^{n-1}B\right]=n \Leftrightarrow \forall\lambda\in\mathbb{C}, \operatorname{rk}\left[\lambda I-A,B\right]=n.$$

Let $\Lambda(A)$ be the spectrum of A. Since $\operatorname{rk}[\lambda I-A]=n,\forall\lambda\notin\Lambda(A)$, then it is sufficient to study the controllability only for the spectrum of matrix A, so the PBH criterion becomes

$$\operatorname{rk}\left[B,\dots,A^{n-1}B\right]=n \Leftrightarrow \forall\lambda\in\Lambda(A), \operatorname{rk}\left[\lambda I-A,B\right]=n.$$

3.2 Network input-delayed controllability problem

Let $r\in\mathbb{N}$. Starting at an initial moment k_0, we denote by

$$F_i = F(k_0+i),\ 0\le i\le r.$$

Based on the state evolution equation (7), the length-r controllability matrix of system (6) is defined by

$$C_r(k_0) = \left[A^{r-1}F_0, A^{r-2}F_1,\dots,AF_{r-2},F_{r-1}\right]. \tag{8}$$

From relation (8), the following proposition may be easily deduced.

Proposition 1. A necessary and sufficient condition for the controllability of system (6) at time k_0 is the existence of $r\in\mathbb{N}$ such that

$$\operatorname{rk}\left[A^{r-1}F_0, A^{r-2}F_1,\dots,AF_{r-2},F_{r-1}\right]=n.$$

Suppose that non-delayed system (1a) is controllable. The input-delayed system is represented by (4) with the restriction (5). Since the input matrices of the input delayed system are obtained from the initial control matrix B, it is useful to study relationship between the controllability of non-delayed system (1a) and the absolute controllability of the input-delayed system (4) with restriction (5).

3.3 Impact of input delay evolution hypothesis

First of all, we present in Fig. 1 a time-delay diagram.

The diagonal lines select the blocks for building matrices F_0, F_1, F_2. From equalities (3) and (5), it results that every column of matrix B may be found in only one matrix $B_j(k)$ $(j\in\{0,\dots,M\})$ at a given fixed time moment k. For that reason, there exists a single cell in every row that contains the column. Let c be a generic column of matrix B corresponding to a given input channel (i.e. $c\in\{c_B^1,c_B^2,\dots,c_B^m\}$). Suppose for simplicity that $c\in B_0(0)$.

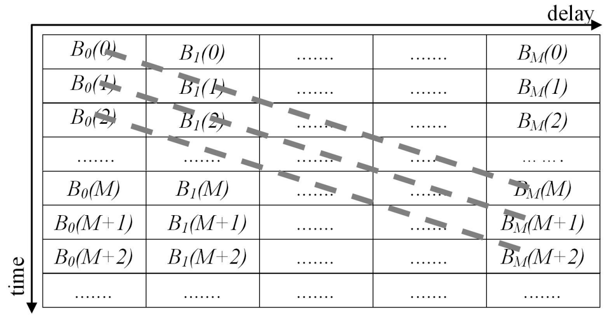

Fig. 1. Time-delay distribution in matrices $B_i(j)$

We represent in Fig 2 a possible time evolution of the corresponding input channel delay by marking with a point the column c placement. Shadowed cells signify that the

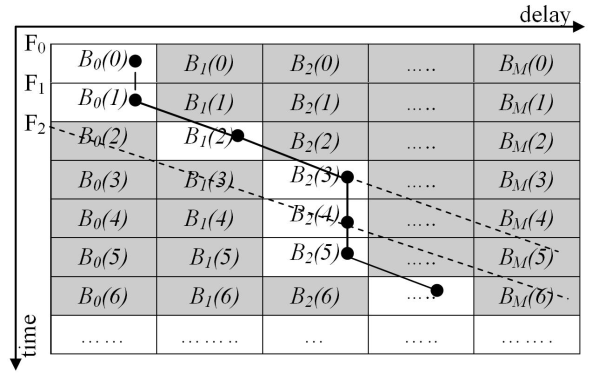

Fig. 2. A possible evolution of a time delay

column c cannot be found in corresponding matrices as a consequence of (5).

Remark 1. It is important to note that a column position evolution (corresponding to an input-delay evolution) is possible in the vertical up-down direction (on the columns of the diagram) if the delay remain constant, or in the diagonal left-right direction (if input-delay increases with unitary increments at consecutive time moments). For example, consider the vertical column time evolution $B_2(3)$, $B_2(4)$ and $B_2(5)$ in the above figure. At instant $k = 4$, column c belongs only to $B_2(4)$. From the definition of $F(k)$, the matrix block F_2 will contain in the corresponding input position only the column $A^{-2}c$. If we consider a diagonal time evolution like $B_0(1)$, $B_1(2)$ and $B_2(3)$ in the above figure, column c belongs to $B_0(1)$, $B_1(2)$ and $B_2(3)$ and the matrix block $A^{-1}F_1$ will contain in the corresponding input position the column $\left(A^{-1} + A^{-2} + A^{-3}\right) c$.

The following lemma addresses the situation where a given input channel has a constant delay during n steps.

Lemma 1. If system (6) is non-controllable, then there exists a n-column vector q ($q \neq 0_{1\times n}$) such that for all time delay $d \in \{1, \dots, M\}$, discrete instant $h \in \mathbb{Z}$ verifying $h \geq d+k_0$, and column $c = c_B^\ell$ of B ($\ell \in \{1, \dots, m\}$) which is also column of matrices $B_d(h), ..., B_d(h+n+1)$, we have

$$\forall k \in \mathbb{Z}, qA^k c = 0.$$

Proof. The proof of the lemma relies on delay evolution assumptions, which were illustrated in the previous remarks. Without loss of generality, and to simplify the notation, assume that $k_0 = 0$. The system (6) is non-controllable. As a consequence,

$$\forall r \in \mathbb{N}, \mathrm{rk}\left[A^{r-1}F_0, A^{r-2}F_1, ..., AF_{r-2}, F_{r-1}\right] < n.$$

Since the delay of each input channel may only remain constant or increase with unitary increments, and is bounded by M, then there exists an instant $\mathcal{M}$ where the all the delays of the input channel remain indefinitely constant. Considering the sequence length $r = \mathcal{M}+n+1$, we obtain

$$\mathrm{rk}\left[A^{\mathcal{M}+n}F_0, ..., AF_{\mathcal{M}+n-1}, F_{\mathcal{M}+n}\right] < n.$$

In follows that there exists a n-column vector q such that $q \neq 0_{1\times n}$ and

$$q\left[A^{\mathcal{M}+n}F_0, ..., AF_{\mathcal{M}+n-1}, F_{\mathcal{M}+n}\right] = 0_{1\times(\mathcal{M}+n+1)m}.$$

Let $d \in \{1, \dots, M\}$, $h \in \mathbb{Z}$ verifying $h \geq d+k_0$, and $c = c_B^\ell$ be a column of B ($\ell \in \{1, \dots, m\}$) which is also a column of matrices $B_d(h), ..., B_d(h+n+1)$ Since A is invertible, the previous equality implies that

$$q\left[A^{h+n}F_0, \dots, A^dF_{h-d+n}\right] = 0_{1\times(h-d+n+1)m}.$$

This implies that for all $i \in \{1, \dots, n\}$,

$$q \cdot A^{d+n-i}F_{h-d+i} = 0_{1\times m}. \tag{9}$$

For all $i \in \{1, \dots, n\}$, $c = c_B^\ell$ is a column of $B_d(h+i-1)$, $B_d(h+i)$ and $B_d(h+i+1)$. Due to delay evolution assumptions 2 and 3, and as shown in the previous remarks, c_B^ℓ does not belong to any matrix $B_{d+j}(h+i+j)$ ($-d \leq j < 0$ and $0 < j \leq M-d$) and the ℓ^{th} columns of these matrices are zero. Recalling that

$$F_{h-d+i} = \sum_{j=-d}^{M-d} A^{-(d+j)}B_{d+j}(h+i+j),$$

it follows that the ℓ^{th} column of F_{h-d+i} is equal to $A^{-d}c_B^\ell$. Considering only the multiplication of qA^{d+n-i} and the ℓ^{th} column of F_{h-d+i}, equality (9) implies that

$$qA^{d+n-i}\left(A^{-d}c\right) = 0,$$

which is equivalent to

$$qA^i c = 0 \text{ for } i = 0, 1, 2, ..., n-1.$$

Using Cayley-Hamilton theorem, the proof is terminated. $\square$

3.4 Controllability problem solution

Using the equivalence between (4) and (6), the main result is given in the following theorem.

Theorem 2. The non-delayed system (1a) is controllable if and only if system (6) is controllable.

Proof. We have to prove that

$$\mathrm{rk}\left[\lambda I - A, B\right] = n, \forall\lambda \in \Lambda\left(A\right)$$

is equivalent to

$$\exists r > 0, \mathrm{rk}\left[\Phi\left(r,1\right)F_0, \Phi\left(r,2\right)F_1, ..., \Phi\left(r,r\right)F_{r-1}\right] = n.$$

($\Leftarrow$) Suppose first that (6) is controllable, i.e.

$$\exists r > 0, \mathrm{rk}\left[\Phi\left(r,1\right)F_0, \Phi\left(r,2\right)F_1, ..., \Phi\left(r,r\right)F_{r-1}\right] = n.$$

Suppose (by absurd) that (1a) is un-controllable, i.e.

$$\exists\lambda \in \Lambda\left(A\right), \mathrm{rk}\left[\lambda I - A, B\right] < n.$$

The previous relation is equivalent to the existence of a non zero n-column vector q such that

$$qA = \lambda q \text{ and } qB = 0_{1\times m}. \tag{10}$$

Relation (10) proves that q is a left eigenvector for matrix A and is orthogonal with every column of matrix B. Using repeatedly the first equality in (10) we obtain

$$\begin{aligned} qA &= \lambda q \Rightarrow q = \lambda q A^{-1} \Rightarrow qA^{-1} = \frac{1}{\lambda} q \\ qA^{-2} &= \frac{1}{\lambda} qA^{-1} = \frac{1}{\lambda^2} q \\ &\vdots \\ qA^{-g} &= \frac{1}{\lambda} qA^{-g+1} = \frac{1}{\lambda^g} q. \end{aligned} \tag{11}$$

From the second equality in (10) we obtain

$$qB = 0_{1\times m} \Leftrightarrow q\left[c_B^1, c_B^2, ..., c_B^m\right] = 0_{1\times m}.$$

Now, we study the value of

$$q\left[\Phi(r,1)F_0, \Phi(r,2)F_1, ..., \Phi(r,r-1)F_{r-2}, \Phi(r,r)F_{r-1}\right].$$

Let $i \in \{0, r-1\}$, and consider the value of

$$q \cdot \Phi(r, i+1) \cdot F_i = q \cdot A^{r-i-1} \cdot F_i.$$

From (11) $q \cdot A^h = \lambda^h \cdot q$ and the above relation will be

$$q \cdot \Phi(r, i+1) \cdot F_i = \lambda^{r-i-1} \cdot q \cdot F_i.$$

Replacing the expression of $F(k)$ in (11) we obtain

$$q \cdot \Phi(r, i+1) \cdot F_i = \lambda^{r-i-1} \cdot \sum_{j=0}^{M} \left(q \cdot A^{-j}\right) \cdot B_j(i+j).$$

Using (11) in the previous equality, we get

$$q \cdot \Phi(r, i+1) \cdot F_i = \lambda^{r-i-1} \cdot \sum_{j=0}^{M} \lambda^{-j} \cdot q \cdot B_j(i+j).$$

Using the second equality in (10) and the structure of $B_j(k)$, we get

$$q \cdot \Phi(r, i+1) \cdot F_i = 0_{1\times m}, \text{ for } i \in \{0, 1, ..., r-1\}.$$

and we conclude that

$$q\left[\Phi(r,1)F_0, \Phi(r,2)F_1, ..., \Phi(r,r)F_{r-1}\right] = 0_{1\times(r\times m)}.$$

This relation contradicts the starting controllability hypothesis of system (6), therefore we conclude that the initial system (1a) controllable.

($\Rightarrow$) Assume that $\operatorname{rk}[\lambda I - A, B] = n, \forall \lambda \in \Lambda(A)$. Suppose (by absurd) (6) is non-controllable. Since the input delay is bounded by M, for each input channel ℓ, there exists an instant h_ℓ such that the delay affecting every input channel ℓ remains contant (at a value $d_\ell \in \{1, \dots, M\}$) during the $n+1$ following instants and consequently, and each column c_B^ℓ of B will be also a column of $B_{d_\ell}(h_\ell), \dots, B_{d_\ell}(h_\ell + n + 1)$. Using Lemma 1, there exists a n-column vector q such that $q \neq 0_{1\times n}$ and for all column c of B,

$$\forall k \in \mathbb{Z}, qA^k c = 0.$$

This implies that

$$\forall k \in \mathbb{Z}, qA^k B = 0,$$

This relation contradicts the controllability assumption of system (1a), which completes the proof of the theorem. □

4. OBSERVABILITY PROBLEM STATEMENT

4.1 Network output delayed observability problem

In this section we consider the observability problem, when measured plant outputs are sent to the controller via a spatially distributed network, which induces different delays in each transmission channel (verifying assumptions 1, 2 and 3). To this end, we consider the following model.

$$x_{k+1} = Ax_k + Bu_k \tag{12a}$$

$$y_k = Cx_k \tag{12b}$$

$$y'_k = y_{k-H(k)} \tag{12c}$$

where

$$\begin{aligned} y_{k-H} &= \left[\, y^1_{k-h_1(k)} \; y^2_{k-h_2(k)} \; \dots \; y^p_{k-h_p(k)} \,\right]^T \\ &= \left[\, Cx^1_{k-h_1(k)} \; Cx^2_{k-h_2(k)} \; \dots \; Cx^p_{k-h_p(k)} \,\right]^T \end{aligned}$$

The previous relation shows that the non-uniformly delayed output may be written as a combination of previous states of the non-delayed system. Since every state at moment k may be represented using the past states, we will consider the maximum time-delay over all communication channels:

$$H = \sup_{k\in\mathbb{Z}} \left\{ \max_{i=1,2,\dots p} \{h_i\} \right\}$$

Consider all possible states with bounded delay:

$$\{x_k, x_{k-1}, ..., x_{k-H}\}.$$

The delayed output of system (12) has the following equivalent representation:

$$y'_k = \sum_{j=0}^{H} C_j(k)\, x_{k-j}.$$

In the above representation, matrix $C_j(k)$ contains at moment k exactly the columns of output matrix C that multiply the components of the output vector having a delay of j steps. The other columns are zero. As a consequence, we have the equality

$$\sum_{j=0}^{H} C_j(k) = C$$

Every state at the moment $k - h_i(k)$ may be represented as a final state of state evolution starting at initial moment $k - H$.

$$\begin{aligned} x_{k-H} &= I x_{k-H} \\ x_{k-H+1} &= Ax_{k-H} + Bu_{k-H} \\ &\vdots \\ x_{k-H+d} &= A^d x_{k-H} + \sum_{i=1}^{d} A^{d-i} B u_{k-H-1+i} \end{aligned}$$

Defining

$$\bar{U} = \left[\, u^T_{k-H} \; u^T_{k-H+1} \; \dots \; u^T_{k-2} \; u^T_{k-1} \,\right]^T,$$

$$\bar{B} = [\, B \; 0 \; 0 \dots \; 0 \,],$$

$$\bar{C}(k) = \sum_{j=0}^{H} C_j\,(k)\,A^{H-j},$$

$$\bar{D}(k) = \text{line}\left(C_j\,(k)\sum_{i=1}^{H-j} A^{H-j-i}B\right), j = 0, 1, ..., H.$$

we obtain the model

$$x_{k-h+1} = Ax_{k-h} + \bar{B}\bar{U} \tag{13a}$$

$$y_k' = \bar{C}(k)x_{k-h} + \bar{D}\,(k)\,\bar{U}. \tag{13b}$$

4.2 Observability problem solution

Despite the fact that controllability and observability are dual concepts, we give the observability problem solution because we use a different system representation for observability. As presented in (Marinescu and Bourlès (2000)), this representation is more adequate for prediction because it realizes a clear decoupling between the state to be estimated and the deterministic variables (commands). If we know the measured outputs, we can estimate the state at moment $k - H$ if the linear system in (13) is observable. The observability property of system (13) has a slightly modified interpretation comparing to observability of non-delayed system: we may say that output-delayed system (12) is observable if and only if system (13) is observable. Both systems are equivalent, but system (12) has a "classical" input-state-output representation using the state . As a consequence, the observability is referred with respect to state not in the same manner as for a non-delayed system.

Proposition 2. The system (12) is observable if and only if there exists $r > 0$ such that

$$\text{rk}\left[\left(A^T\right)^{r-1}\bar{C}(0)^T, \left(A^T\right)^{r-2}\bar{C}(1)^T, \ldots, \bar{C}(r-1)\right] = n.$$

The main result is represented in next theorem.

Theorem 3. The non-delayed system (1b) is observable if and only if the network-delayed system (13) is observable.

Proof. The proof is simliar to that of theorem 1.

5. CONCLUSION

In the present paper we have studied the structural properties of systems with variable network-induced delays in command (input) and measure (output), resulting from a communication network overload. The main results are synthesized in two theorems, which state that, if the non-delayed system is controllable and/or observable, the same system is controllable and/or observable using a networked control architecture causing a delay that can either remain constant or increases with unitary increments.

Future work will focus on the more general case of linear time-varying systems with different variable input/output network induced delays.

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