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Identification of Exponentially Damped Sinusoidal Signals ⋆

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Abstract: A discrete-time internal model principle based adaptive algorithm for identifying signals composed of a sum of exponentially damped sinusoids is presented. The time varying state variables of an internal model principle controller in a feedback loop can provide estimates of the exponentially damped sinusoidal signal parameters, the damping factor and the frequency. By using additional integral controllers, the estimation errors can be eliminated. The convergence of the proposed algorithm is justified using discrete-time averaging theory. Simulation results demonstrate the performance of this algorithm for signal identification.

1. INTRODUCTION

In this paper, we consider signals composed of a sum of exponentially damped sinusoids with the following form,

$$s(k) = \sum_{i=1}^{N} s_i(k) = \sum_{i=1}^{N} a_i e^{-\sigma_i k} \sin(\omega_i k + \varphi_i) \tag{1}$$

where the uncertain σ_i and ω_i are the damping factor and the frequency, respectively. This form can also represent constant-amplitude sinusoids, and constant signals. The objective is to estimate the parameters σ_i and ω_i. There have been many techniques developed to deal with predictable signals, such as narrow-band or sinusoidal signals, since they appear in both signal processing and control applications, including active noise control, radar signals, rotating mechanical systems, computer hard disk drive *etc.*, Brown and Zhang [2003]. These techniques include linear quadratic regulator based modern control, higher harmonic control, Sievers and von Flotow [1992], adaptive notch filter, Regalia [1991], adaptive feedforward cancellation (AFC), Bodson and Douglas [1997], adaptive observer technique, Marino and Tomei [2002].

Another common approach for perfect cancellation of signals is based on a fundamental control principle, the *internal model principle*, (IMP) Francis and Wonham [1976]. This principle states that perfect disturbance rejection or reference tracking is achieved when a model of the dynamic structure of the disturbance or reference signal is incorporated in a stable feedback loop. The accuracy of regulation depends critically on the fidelity of the IMP controller. Errors of less than one percent in model coefficients can lead to unacceptable residual errors. Thus, the ability to adaptively tune the model parameters, which can be completely specified as damping factors and frequencies, is of great benefit. Then adaptive IMP controllers can provide exact reproduction of the predictable part of a signal, and when they do, they provide highly accurate estimates of the signal parameters. One application of the IMP to periodic disturbance rejection is repetitive control for time-lag systems and multi-link manipulators, Tsao et al. [2000]. Serrani et al. [2001] also presented a solution to a nonlinear output regulation problem based on the IMP.

An IMP based adaptive algorithm for canceling quasi-periodic, or narrow-band signals with uncertain frequencies is presented in Brown and Zhang [2003]. This approach begins with a state space implementation of the standard IMP controller with the best estimate of the frequency used. A simple mapping from the states of the controller to the error in the frequency estimate was developed and this "measurement" of the frequency error is used to update the parameters of the IMP controller. When this adaptive IMP controller is placed into a feedback loop, the resulting closed-loop system achieves perfect frequency estimation of the elements of a sum of sinusoidal signals.

In addition to sinusoidal signals, EDS signals are often used to model audio signals, such as speech or music, which contain relatively fast variations in amplitude. The conventional sinusoidal model is thus extended by allowing the amplitude to evolve exponentially as given by (1). A well-known approach to EDS signal parameter estimation is the polynomial or linear prediction method as in Kumaresan and Tufts [1982]. Traditionally, EDS signal model is associated with a high-resolution parameter estimation method, such as matrix pencil, ESPRIT or Kung's algorithm, Boyer and Rosier [2002]. Hua and Sarkar [1990] presented a matrix pencil method as an alternative approach which exploits the structure of a matrix pencil of the EDS signal $s_i(t)$, instead of the structure of prediction equations satisfied by $s_i(t)$. In Badeau et al. [2002], the EDS signal parameters, damping factor and frequency, are estimated using a subspace-based matrix pencil high resolution method. The tracking of the slow variation of the signal parameters is achieved using an adaptive least mean square algorithm.

Motivated by the IMP based adaptive algorithm in Brown and Zhang [2003], an extended adaptive algorithm for EDS disturbance cancellation was developed in Lu and Brown [2007]. The control law results in a system output error that decays exponentially fast with a decay rate independent of σ_i. This is equivalent to what is meant when integral control is said to provide perfect set-point tracking. This work was developed only in continuous-time framework and strictly as a control algorithm. Here we develop a discrete-time implementation of

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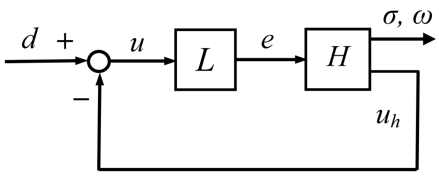

Fig. 1. Block diagram of an internal model control system

this algorithm, and convert the control algorithm into a signal processing algorithm.

This paper is organized as follows: In Section 2, the motivation, the continuous-time state-space representation of the IMP controller, is introduced. Then the derivation of the IMP based adaptive algorithm for EDS signal identification in discrete-time is presented. The convergence and stability property of the proposed adaptive algorithm is analyzed based on a discrete-time two time scale averaging theory in Section 3. Simulation results are demonstrated in Section 4, followed by a conclusion in Section 5.

2. INTERNAL MODEL PRINCIPLE BASED ADAPTIVE ALGORITHM

2.1 Adaptive Algorithm in Continuous-Time

The basic structure of the feedback system is shown in Fig. 1, where L is a tuning function that is properly designed to stabilize the system, H represents the IMP controller. d is an EDS signal, which is defined as follows,

$$d(t) = ae^{-\sigma_c t}\sin(\omega_c t+\varphi), \quad \sigma_c>0,\ \omega_c>0,\ a>0 \tag{2}$$

where σ_c, ω_c are the damping factor and frequency, respectively. e is the error signal. Thus, following IMP theory, a continuous-time state-space representation of an IMP controller is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\sigma_c & \omega_c\\ -\omega_c & -\sigma_c\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} e \tag{3}$$

$$u_h = [K_1\ \ K_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{4}$$

where K_1, K_2 are tuning gains.

If the initial conditions for x_1 and x_2 are given by $x_1(0) = \frac{a\cos\varphi}{\sqrt{K_1^2+K_2^2}}$, $x_2(0) = \frac{a\sin\varphi}{\sqrt{K_1^2+K_2^2}}$, then for all $t>0$, $e=0$ and

$$x_1(t) = \frac{a}{\sqrt{K_1^2+K_2^2}}e^{-\sigma_c t}\cos(\omega_c t+\varphi) \tag{5}$$

$$x_2(t) = \frac{a}{\sqrt{K_1^2+K_2^2}}e^{-\sigma_c t}\sin(\omega_c t+\varphi) \tag{6}$$

By letting $\mathbf{x} = |\mathbf{x}|\angle\theta = x_1(t)+\mathrm{j}x_2(t)$, we can get

$$|\mathbf{x}| = \frac{a}{\sqrt{K_1^2+K_2^2}}e^{-\sigma_c t} \tag{7}$$

$$\theta = \tan^{-1}\left(\frac{x_2(t)}{x_1(t)}\right) = \omega_c t+\varphi \tag{8}$$

where $\tan^{-1}(\cdot)$ is defined to have a range given by real numbers such that θ is continuous in t. Differentiating both sides of (7) and (8), we have

$$\sigma_c = -\frac{1}{|\mathbf{x}|}\frac{\mathrm{d}|\mathbf{x}|}{\mathrm{d}t} \tag{9}$$

and

$$\omega_c = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{10}$$

In practice, the model parameters in (3) and (4) are approximations, giving

$$\hat{\sigma}_c = -\frac{1}{|\mathbf{x}|}\frac{\mathrm{d}|\mathbf{x}|}{\mathrm{d}t} = -\frac{x_1\dot{x}_1+x_2\dot{x}_2}{x_1^2+x_2^2} = \sigma - \frac{ex_2}{x_1^2+x_2^2} \tag{11}$$

and

$$\hat{\omega}_c = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\tan^{-1}\left(\frac{x_2(t)}{x_1(t)}\right) = \frac{\dot{x}_1x_2-x_1\dot{x}_2}{x_1^2+x_2^2} = \omega - \frac{ex_1}{x_1^2+x_2^2} \tag{12}$$

Thus the error between $\hat{\sigma}_c$ and σ can be expressed as

$$\widetilde{\sigma} = -\frac{ex_2}{x_1^2+x_2^2} \tag{13}$$

Similarly, the error between $\hat{\omega}_c$ and ω is

$$\widetilde{\omega} = -\frac{ex_1}{x_1^2+x_2^2} \tag{14}$$

(11) and (12) can be used to estimate the damping factor and frequency of the EDS signal.

2.2 Derivation of the Adaptive Algorithm in Discrete-Time

The discrete-time state-space equation of the IMP controller can be converted from its continuous-time counterpart (3) and (4) as

$$\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix} = e^{-\sigma}\begin{bmatrix}\cos\omega & \sin\omega\\ -\sin\omega & \cos\omega\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}e(k) \tag{15}$$

$$u_h(k) = [K_1\ \ K_2]\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix} \tag{16}$$

Therefore at sampling instant $t = kT_s$, continuous-time estimation errors (13) and (14) are equal to the following discrete-time estimation errors

$$\widetilde{\sigma}(k) = -\frac{e(k)x_2(k)}{x_1^2(k)+x_2^2(k)} \tag{17}$$

$$\widetilde{\omega}(k) = -\frac{e(k)x_1(k)}{x_1^2(k)+x_2^2(k)} \tag{18}$$

and the estimates of the damping factor and frequency can be updated by using two integral controllers

$$\begin{aligned}\sigma(k+1) &= \sigma(k)+\varepsilon\widetilde{\sigma}(k)\\ &= \sigma(k)-\varepsilon\frac{e(k)x_2(k)}{x_1^2(k)+x_2^2(k)}\end{aligned} \tag{19}$$

$$\begin{aligned}\omega(k+1) &= \omega(k)+\varepsilon K_b\widetilde{\omega}(k)\\ &= \omega(k)-\varepsilon K_b\frac{e(k)x_1(k)}{x_1^2(k)+x_2^2(k)}\end{aligned} \tag{20}$$

where ε is a small adaptation gain, K_b is a constant, and both are positive.

From (19) and (20), it can be seen that if $x_1(k)=0$ and $x_2(k)=0$, the integral update laws are undefined as both the numerators and denominators are 0. This problem can be avoided by adding a small constant C in the denominators of both equations, or setting $\varepsilon = 0$ when $|\mathbf{x}|$ is less than a constant.

Fig. 2 and Fig. 3 show the structure of the IMP based adaptive feedback system. The function $F_k(\mathbf{x},e)$ is what have been derived in (17) and (18). Due to the structure of the IMP controller, a controller can only achieve identification of a

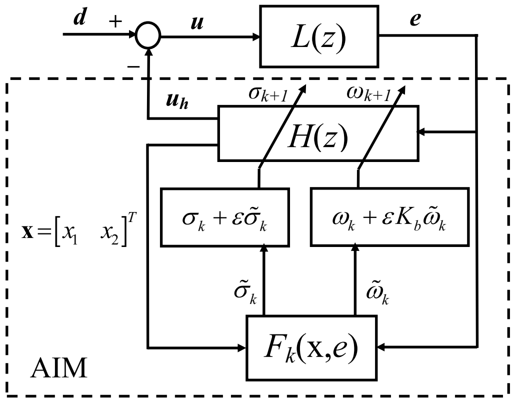

Fig. 2. Block diagram of the adaptive IMP control system

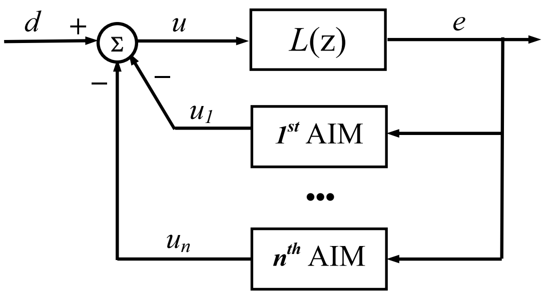

Fig. 3. Block diagram of an adaptive IMP feedback control system for multi-EDS signals

single EDS mode. If the signal $d(k)$ is composed of multiple EDS modes, multiple adaptive IMP controllers are placed in parallel.

By incorporating the adaptive update laws (19)-(20), the IMP controller parameters, σ and ω, converge to the true values σ_c and ω_c of the EDS signal. If σ_c or ω_c changes with time, e is not 0 and the adaptive algorithm will estimate and track the changing parameters.

3. CONVERGENCE AND STABILITY ANALYSIS

The stability analysis of the proposed adaptive algorithm in continuous-time is presented in Lu and Brown [2007]. Singular perturbation theorem, Khalil [2002] and averaging theorem, Sastry and Bodson [1989] are used for the analysis. Bai *et al.* adapted these two theorems in a combined discrete-time version, *Theorem 2.2.4. Exponential Stability Theorem for Two-Time Scale System*, in Bai et al. [1988].

The feedback system is now formulated as a two time scale model. Due to the limitation of space, we will address only the case where the signal is composed of a single EDS. The techniques for extending the proof to the multi-EDS case are shown in Brown and Zhang [2003], Lyndon J. Brown and Qing Zhang [2004]. The primary change in the proof is that the equilibria x_{10} and x_{20} defined in equations (32) and (33) will have terms corresponding to each mode, and the calculation of the averaging function will be far more complicated. However, as simple sinusoids, these extra terms will ultimately contribute nothing to the average, beyond possibly requiring longer averaging times. Detailed calculations of the averaged function has also been omitted for space reasons. The state-space equations for the adaptive feedback system in Fig. 2 are as follows:

$$x_p(k+1) = A_p x_p(k) + B_p u(k) \tag{21}$$

$$e(k) = C_p x_p(k) \tag{22}$$

$$u(k) = -K_1 x_1(k) - K_2 x_2(k) + d(k) \tag{23}$$

$$d(k) = a e^{-\sigma_c k}\sin(\omega_c k) + c_1 \tag{24}$$

$$x_1(k+1) = (e^{-\sigma}\cos\omega)x_1(k) + (e^{-\sigma}\sin\omega)x_2(k) \tag{25}$$

$$x_2(k+1) = -(e^{-\sigma}\sin\omega)x_1(k) + (e^{-\sigma}\cos\omega)x_2(k) + e(k) \tag{26}$$

$$\sigma(k+1) = \sigma(k) - \varepsilon\frac{e(k)x_2(k)}{x_1^2(k)+x_2^2(k)} \tag{27}$$

$$\omega(k+1) = \omega(k) - \varepsilon K_b\frac{e(k)x_1(k)}{x_1^2(k)+x_2^2(k)} \tag{28}$$

where (21) and (22) are state-space representation for $L(z)$. The presence of the bias c_1 in the exogenous signal (24) does not affect the convergence property of the algorithm if $L(1) = 0$.

Perturbation analysis proceeds by fixing $\sigma(k) = \sigma$, and $\omega(k) = \omega$. Under these conditions, the transfer function of the IMP controller is

$$H(z) = \frac{K_2 z + K_1 e^{-\sigma}\sin\omega - K_2 e^{-\sigma}\cos\omega}{z^2 - 2ze^{-\sigma}\cos\omega + e^{-2\sigma}} := \frac{N(z)}{D(z)}.$$

The tuning function can also be expressed as a rational polynomial form as $L(z) = \frac{B(z)}{A(z)}$. Thus the transfer function from $d(k)$ to $e(k)$ can be given by

$$G_{ed}(z) = \frac{B(z)D(z)}{A(z)D(z)+B(z)N(z)} = (z^2 - 2ze^{-\sigma}\cos\omega + e^{-2\sigma})Q(z)$$

where $Q(z) = \frac{B(z)}{A(z)D(z)+B(z)N(z)}$. The transfer functions from $d(k)$ to $x_1(k)$ and $x_2(k)$ can also be derived as

$$G_{x_1 d}(z) = (e^{-\sigma}\sin\omega)Q(z),$$
$$G_{x_2 d}(z) = (z - e^{-\sigma}\cos\omega)Q(z).$$

Now in order to formulate the two time scale model, we introduce new state variables, σ_e, ω_e, x_{pe}, x_{1e}, x_{2e}, as follows:

$$\sigma_e(k) = \sigma(k) - \sigma_c \tag{29}$$

$$\omega_e(k) = \omega(k) - \omega_c \tag{30}$$

$$x_{pe}(k) = x_p(k) - x_{p0}(k) \tag{31}$$

$$x_{1e}(k) = x_1(k) - a|Q|e^{-\sigma_c k - \sigma(k)}\sin\omega(k)\sin(\omega_c k + \angle Q) := x_1(k) - x_{10}(k) \tag{32}$$

$$x_{2e}(k) = x_2(k) - a|Q|e^{-\sigma_c k - \sigma(k)}\left[e^{-\sigma_c}\sin\omega_c\cos(\omega_c k + \angle Q) + (e^{-\sigma_c}\cos\omega_c - e^{-\sigma(k)}\cos\omega(k))\sin(\omega_c k + \angle Q)\right] := x_2(k) - x_{20}(k) \tag{33}$$

where x_{p0}, x_{10}, x_{20} are steady state solutions when the slow states are fixed at σ and ω. $|Q|$ and $\angle Q$ denote the magnitude and angle of $Q(z)$ evaluated at $z = -\sigma_c + j\omega_c$ respectively, and $\sigma = \sigma(k)$, $\omega = \omega(k)$. By letting

$$\mathbf{x}(k) = \begin{bmatrix}\sigma_e(k)\\ \omega_e(k)\end{bmatrix}, \quad \mathbf{y}(k) = \begin{bmatrix}x_{pe}(k)\\ x_{1e}(k)\\ x_{2e}(k)\end{bmatrix},$$

and substituting these new state variables into the system state equations, we can derive a two time scale model form

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \varepsilon\begin{bmatrix}f_1(k,\mathbf{x},\mathbf{y})\\ f_2(k,\mathbf{x},\mathbf{y})\end{bmatrix} \tag{34}$$

$$\mathbf{y}(k+1) = A(\mathbf{x}(k))\mathbf{y}(k) \tag{35}$$

where

$$f_1 = -\frac{C_p\left(x_{pe}(k)+x_{p0}(k)\right)\left(x_{2e}(k)+x_{20}(k)\right)}{\left(x_{1e}(k)+x_{10}(k)\right)^2+\left(x_{2e}(k)+x_{20}(k)\right)^2} \tag{36}$$

$$f_2 = -\frac{K_bC_p\left(x_{pe}(k)+x_{p0}(k)\right)\left(x_{1e}(k)+x_{10}(k)\right)}{\left(x_{1e}(k)+x_{10}(k)\right)^2+\left(x_{2e}(k)+x_{20}(k)\right)^2} \tag{37}$$

and

$$A = \begin{bmatrix} A_p & -K_1B_p & -K_2B_p \\ 0 & e^{-\sigma_e-\sigma_c}\cos(\omega_e+\omega_c) & e^{-\sigma_e-\sigma_c}\sin(\omega_e+\omega_c) \\ C_p & -e^{-\sigma_e-\sigma_c}\sin(\omega_e+\omega_c) & e^{-\sigma_e-\sigma_c}\cos(\omega_e+\omega_c) \end{bmatrix} \tag{38}$$

For the adaptive feedback system (34) and (35), we have the following stability and convergence theorem.

Theorem 1. Consider the dynamic system (34)-(35), with input signal given by (24). If the following assumptions are satisfied

(1) The tuning function $L(z)$ is not equal to zero when $z = e^{-\sigma_c+j\omega_c}$, and $a \neq 0$;
(2) For all fixed σ and ω, matrix A has eigenvalues less than one, i.e., the system of Fig. 2 with $H(z)$ given by (15), (16), has poles strictly within the unit circle;
(3) $L(1) = 0$,

then there exists ε^*, such that for all $0 < \varepsilon < \varepsilon^*$, the origin of (34) and (35) is locally exponentially stable. Note assumption (3) is not required if $d(k)$ is zero mean, i.e. $c_1 = 0$.

Proof of this theorem results from direct application of Theorem 2.2.4. in Bai et al. [1988]. This has two main requirements. It requires the fast system (35) to be stable, which is satisfied by assumption (2), and the average of the slow system (34) is also required to be stable. In Lu and Brown [2007], it is shown that

$$\frac{\omega_c}{2\pi}\int_0^{2\pi/\omega_c} f_1 = \omega-\omega_c+\frac{\omega_c}{\pi}\times\ln\frac{(\sigma-\sigma_c)^2+\omega^2}{\omega_c^2}$$

and

$$\frac{\omega_c}{2\pi}\int_0^{2\pi/\omega_c} f_2 = \sigma-\sigma_c+\frac{2}{\pi}\omega_c\tan^{-1}\frac{\sigma-\sigma_c}{\omega}$$

when x_{1e}, x_{2e} and x_{pe} are zero and time index k has been replaced by a continuous time variable. Stability of the resulting average system is easily shown by Lyapunov's first method. The details showing that the average calculated by summation is equivalent is more complicated and has been omitted. Other technical conditions of Theorem 2.2.4 can be easily verified.

The two time scale requirement of the theory leads to the idea that ε^* will be significantly less than the magnitude of the smallest eigenvalue of $(I-A)$. Practically, for exponential stability of $\mathbf{x}(k)$ to require convergence of σ and ω to their true values in the presence of noise, ε must be significantly greater than σ_c. Otherwise, $\mathbf{x}(k)$ will go to zero simply as a result of $d(k)$ going to zero. Thus selection of $L(z)$ and ε must be done to ensure that a three time scale system is generated in order for the algorithm to function.

4. SIMULATIONS

In this section, the performance of the proposed adaptive algorithm is examined via simulations. All simulations are created in MATLAB and Simulink environment using discrete solver with time units normalized such that $T_s = 1$ unit time. In this case, the frequency unit rad/s means radians per sample. Three signals are used to conduct the simulations: (1) A single EDS signal with step changes on both parameters plus a constant offset and Gaussian noise; (2) A single EDS signal with time varying damping factor plus a constant offset and Gaussian noise; (3) A multi-EDS modes signal plus a constant offset and Gaussian noise. The tuning function is chosen as $L(z) = (z^2-z)/(z^2-0.75z+0.01)$, so that the closed-loop feedback system is stable. Note that for pure sinusoidal signals, Zhang and Brown [2006] present a performance analysis for the IMP based adaptive algorithm for uncertain frequency identification. In this article, the tuning function $L(s)$ is chosen as a function of ω such that the closed-loop system is a bandpass filter with a notch of width $\mathbf{W}$. By incorporating this bandpass filter in $L(s)$, formulae are derived for calculating the variance for the estimated frequency. With a signal to noise ration given by SNR the variance of ω was found to be $\varepsilon^2 * \mathbf{W} * \omega * \mathrm{SNR}$ where ε is the adaptation gain as in this work. This analysis can be extended to EDS signals that we are interested in.

For the first simulation, the EDS signal is given by

$$d(k) = 3e^{-0.005k}\sin(0.5k)+1+n(k), \quad 0 \le k \le 149.$$

At 150 sample point, its damping factor changes from 0.005 to 0.01 and its frequency has a step change from 0.5 rad/s to 0.6 rad/s. In order to avoid the discontinuities in the signal magnitude and phase, the signal is given by

$$d(k) = 3e^{-0.01k+0.75}\sin(0.6k-15)+1+n(k), \quad 150 \le k \le 300.$$

The additive Gaussian noise $n(k)$ has zero mean and variance 0.0001. The initial conditions are $\sigma(0) = 0.008$, $\omega(0) = 0.2$ rad/s. The tuning parameters are $K_1 = 0.5$, $K_2 = 0.3$, $\varepsilon = 0.05$, $K_b = 2$. The magnitude of the dominant eigenvalue of matrix A is 0.9062. The algorithm presented here, with an integral action contained in the tuning function $L(z)$, is not affected by the presence of constant offsets, unlike other algorithms in the literature. Fig. 4 shows the signal and error response plots. The error converges to zero at 40 samples with a decay rate significantly greater than the EDS signal's damping factor. Fig. 5 shows that the IMP controller's parameters converge to the true values of the EDS signal at about 40 samples. After the step changes in both parameters, it takes about 50 samples for σ and 25 samples for ω to converge to their new values. In order to evaluate the performance of the algorithm, we measure the variances of estimated parameters in a steady-state time period. From sample point 50 to sample point 150, the measurements are $\mathrm{var}(\sigma) = 7.7\times10^{-8}$, $\mathrm{var}(\omega) = 1.8\times10^{-7}$.

For the second simulation, the EDS signal is

$$d(k) = 3\exp\left(\sum_{i=0}^{k}0.02\sin(0.03i)\right)\sin(0.5k)+1+n(k),$$

with its damping factor defined as $\sigma_c(k) = 0.02\sin(0.03k)$. The additive Gaussian noise has zero mean and variance 0.0001. The initial conditions are $\sigma(0) = 0.05$, $\omega(0) = 0.2$ rad/s. The tuning parameters are $K_1 = 0.5$, $K_2 = 0.3$, $\varepsilon = 0.12$, $K_b = 0.83$. Since the damping factor is time varying, in order to minimize the tracking delay, the integral gain for σ has been increased while keeping the integral gain for ω unchanged. Fig. 6 shows the EDS signal with time varying damping factor and the error response of the adaptive feedback system. It can be observed that the error decays to zero at a rate independent of the EDS signal's damping factor. The top plot in Fig. 7 demonstrates the tracking performance of the adaptive algorithm. The estimated damping factor tracks the true value after 40 samples with 8.5 samples delay. The estimated constant frequency is illustrated in the bottom plot. From sample point 50 to sample point 260, the variances for estimated parameters are calcu-

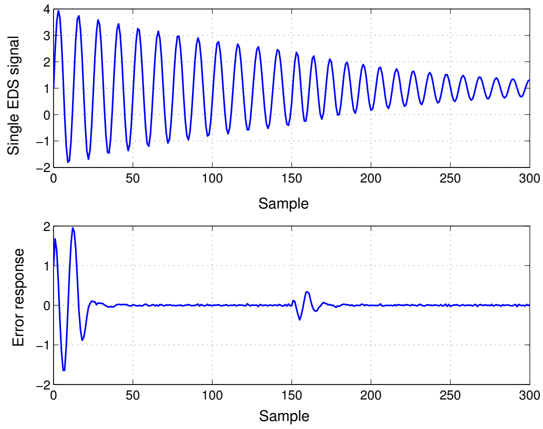

Fig. 4. Single EDS signal with a parameter step change and error response

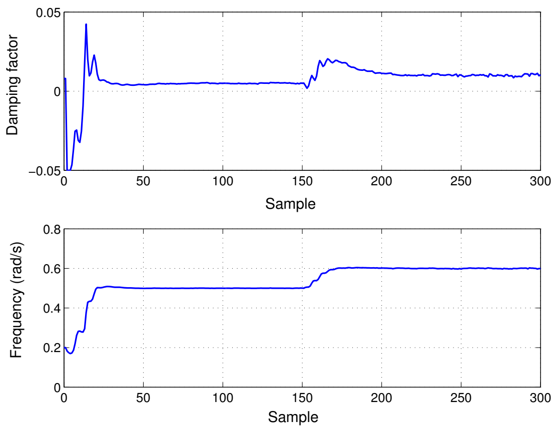

Fig. 5. Parameter estimation of a single EDS signal with a parameter step change

lated as $\mathrm{var}(\omega) = 1.2 \times 10^{-6}$ and the variance of σ defined as $\mathrm{var}(\sigma(i) - 0.02\sin(0.03(i-8.5)))$ equals 9.3×10^{-7}.

The signal for the third simulation is

$$d(k) = d_1(k) + d_2(k) + 2 + n(k),$$

where

$$d_1(k) = 2e^{-0.007k}\sin(0.3k),$$

$$d_2(k) = 3e^{-0.012k}\sin(0.5k)$$

with initial conditions

$$\sigma_1(0) = 0.005,\ \omega_1(0) = 0.2\ rad/s,$$

$$\sigma_2(0) = 0.015,\ \omega_2(0) = 0.65\ rad/s.$$

The additive Gaussian noise has zero mean and variance 0.0001. The tuning parameters are the same for each IMP controller as $K_1 = 0.1,\ K_2 = 0.1,\ \varepsilon = 0.03,\ K_b = 1$. The magnitude of the dominant eigenvalue of matrix A is 0.9504. Fig. 8 shows the multi-EDS signal and the error response of the feedback system. As ε has been reduced, we see slower convergence in Fig. 9 and 10. This has been seen to be required in practice and can be inferred from the averaging proof as the averaging period is now calculated for a sum of sinusoids. From sample point 200 to sample point 300, the variances for estimated parameters are $\mathrm{var}(\sigma_1) = 3.8 \times 10^{-7}$, $\mathrm{var}(\omega_1) = 1.1 \times 10^{-7}$, $\mathrm{var}(\sigma_2) = 4.6 \times 10^{-7}$, $\mathrm{var}(\omega_2) = 1.5 \times 10^{-7}$.

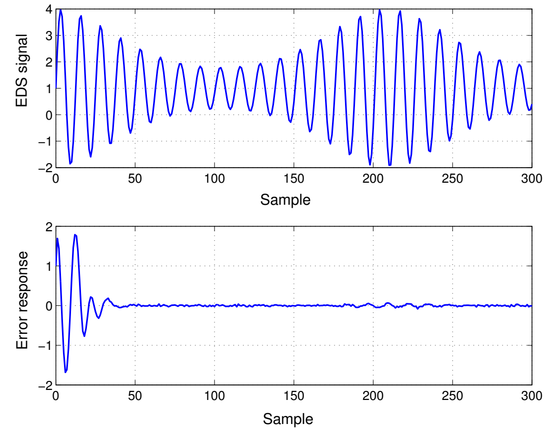

Fig. 6. Single EDS signal with time varying damping factor and error response

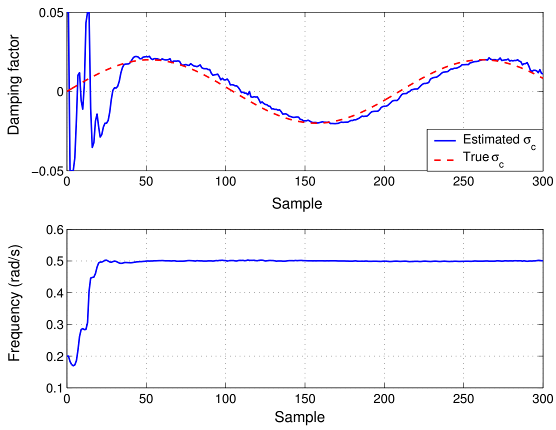

Fig. 7. Parameter estimation of a single EDS signal with time varying damping factor

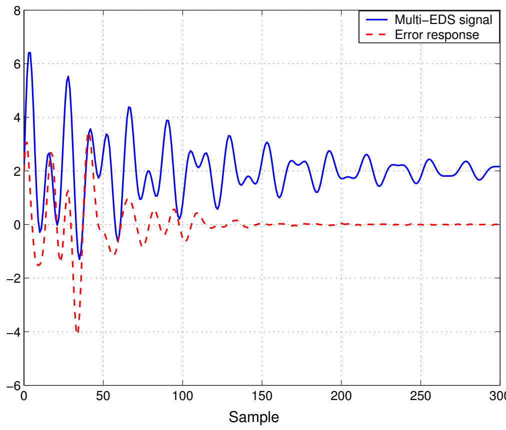

Fig. 8. Multi-EDS signal and error response

5. CONCLUSION

An IMP based adaptive algorithm is developed in discrete-time for identifying exponentially damped sinusoidal signals. Both the damping factor and frequency of the signal can be estimated using the time varying state variables of the IMP controller. This adaptive algorithm can not only identify constant parameters, but also track slowly time varying parameters. By constructing a series of IMP controllers in parallel, the adaptive

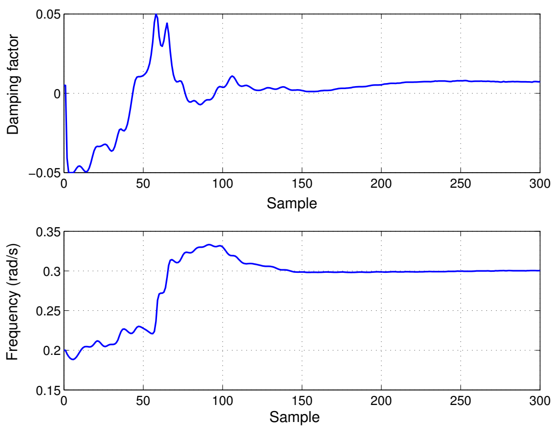

Fig. 9. Parameter estimation for EDS mode $d_1(k)$

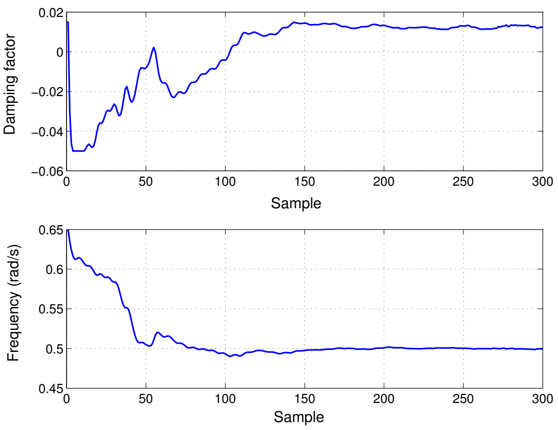

Fig. 10. Parameter estimation for EDS mode $d_2(k)$

feedback system can identify a signal composed of a sum of EDS components, with each IMP controller corresponding to one EDS component.

In the first simulation, the slow system has almost the same speed as the fast system. In the third simulation, the fast system is faster by a factor of 2 than the slow system. From these simulations, our algorithm has shown its functionality despite the limitation that the slow system shall be slower than the fast system. In order to speed up our algorithm, we can place the closed-loop poles closer to the origin by tuning the function $L(z)$. Also note that the variance measurements are zero for noise free simulation cases.

The proposed algorithm is shown to be locally exponentially stable, with convergence rates given by the design parameters, independent of the signal strength and almost independent of the signal parameters.(By almost we refer to the natural restrictions that convergence cannot be faster than $1/\omega_c$, and must be faster than σ_c.) Because of the local nature of the stability result, initial choice for σ and ω can be critical.

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