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AN INTERSECTION MODEL BASED ON THE GSOM MODEL

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Abstract:

In the field of the traffic modelling, among the many problems to be solved, the intersection modelling problem constitutes on of the most difficult. In particular, network modelling requires an intersection model which must be both simple and realistic in order to describe the behaviour of the flow or vehicles. In this paper, the intersection model based on the GSOM model class (generic second order macroscopic model) is presented and discussed. Based on several simulation runs, simulation results are described. According to the fixed boundary conditions, acceptable results are obtained and demonstrate the right information propagation in/out the node. *Copyright © 2008 IFAC*

Keywords: Traffic model, intersection, boundary conditions, Riemann problem.

1. INTRODUCTION: INTERSECTION MODELLING IN THE LWR MODEL

The motorway network is viewed as an oriented graph including arcs which represent the sections, and nodes which correspond to geometrical change of characteristics (modification of lane numbers, on/off ramps, capacity change etc) or actual intersections. To simulate the dynamic aspects of the traffic flow, we need to model the different elements constituting the network. In this paper, traffic dynamics on both arcs and nodes are considered.

The main difficulties with modelling traffic on sections and intersections lie with choosing the correct definition of the boundary conditions.
Let us first consider the case of the macroscopic first order LWR (Lighthill-Whitham-Richards) model (Lighthill-Whitham 1955, Lighthill 1956)

Let us consider the simple case of a road section limited by two nodes (a) at the entrance and (b) at the exit (see figure 1).

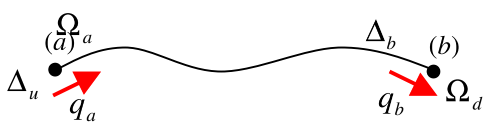

Figure 1. Boundary conditions of a section

The boundary condition of the in-link node (a) is given by the couple demand/supply: upstream demand Δ_u and link supply Ω_a. In Lebacque 1996, the in-flow at the node (a) of the network is shown to be:

$$q_a = Min[\Delta_u, \Omega_a] \qquad (1)$$

For the out-link node (b), the boundary conditions are defined in a symmetric way. In particular the out-flow at node (b) is shown to be:

$$q_b = Min[\Delta_b, \Omega_d] \qquad (2)$$

with Δ_b the link demand and Ω_d the downstream traffic supply.
However, if we consider the general case where the node admits several in/out links, the main difficulty consists in connecting the boundary conditions adjacent to the node in order to solve the node modelling problem.

During the last decade, the intersection modelling has become an active research area. In the literature of traffic modelling of a single intersection, several approaches have been developed and suggested.

In this paper, two point wise intersection types are referred to:

- Static point wise intersections without an internal state
- Dynamic point wise intersections with an internal node state

For both types the dimension of the intersection is neglected, when compared to the dimension of the motorway sections.
The emphasis of the paper will be on the internal state node model.

The modelling of the dynamic point wise intersection including an internal state has been introduced in (Lebacque and Khoshyaran, 2005). The modelling consists to consider the node as a section whose state is described by the number of vehicles present in the node. The traffic dynamics of the intersection result from the combination of the demand/supply couples for the node and the conservation of vehicles (Full details in section 3). In the static intersection model, the size and the internal state are neglected.

The node model of a point wise intersection without an internal state is characterized by the demands at the upstream sections δ_i, and the supplies of the downstream sections σ_j and the flows entering and exiting the intersection, i.e. q_i and r_j (see figure 2.).

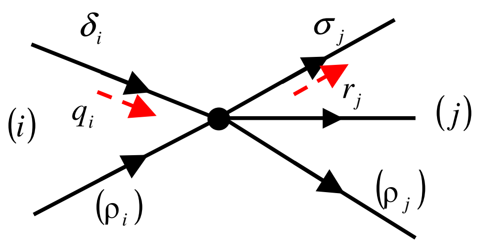

Figure 2. A point wise intersection without internal state

In general, intersection models are defined by a relation between the traffic conditions, i.e. the supplies and the demands, and the through-flows: $[q,r]=f(\delta,\sigma)$.
Lebacque and Khoshyaran, 2005 have established that this kind of models must satisfy the *invariance principle* in order to be consistent. Mathematical development of intersections modelling can be found in (Coclite and Piccoli, 2002; Lebacque and Khoshyaran, 2005).

In this paper, the developed model is based on the resolution of the dynamic point wise intersection, in the context of the GSOM model introduced in Lebacque, Mammar, Haj-Salem 2007. The next section is dedicated to the description of this model, which extends considerably the LWR model.

2. THE GSOM MODELLING APPROACH

Following the pioneering LWR model, a great number of models reproducing the dynamics of traffic in a realistic way have been developed in the literature. However, during the last decade, the researches are focused on the higher order dynamic traffic modelling aspect. The recent models of Aw-Rascle 2000 and Zhang 2002 (see also Garavello and Piccoli, 2006) have solved the defects of the previous generation of models such as the Payne model (Payne, 1971) or the (Zhang, 1998) model for instance. The (Aw-Rascle 2000) and (Zhang 2002) models (called ARZ model) have recently been investigated and analytical and numerical solutions are proposed for different traffic conditions: in the homogeneous (Mammar *et al*, 2005) and heterogeneous (Lebacque *et al*, 2005) case.

The proposed intersection model is based on the EARZ (Extended Aw-Rascle Zhang) second order model recently introduced by Lebacque (Lebacque *et al*, 2007), now extended as the GSOM model class (Khoshyaran Lebacque 2007). The GSOM exhibits the particularity that it permits an adequate adaptation of the simulation model to field data measurements. This is achieved by using the invariant denoted *I*. The GSOM model is defined by the following equations:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho I) + \partial_x(\rho v I) = 0 \\ v = \Im(\rho, I) \end{cases} \quad (3)$$

The system (3) is conservative. The conserved variables and flux vector are $U = (\rho, y = \rho I)^t$ and $F(U) = \left(\rho v, p \overset{def}{=} \rho v I\right)^t$ respectively.
where p is the *relative traffic pressure*.

(Mammar *et al*, 2006), (Lebacque *et al*, 2007) have shown that an adequate choice of invariant *I,* makes it possible to derive several traffic models already developed in the literature. For instance, with $\Im(\rho, I) = V_e(\rho) + I$, where $v = V_e(\rho)$ denotes the equilibrium speed-density relationship (fundamental diagram), the derived model corresponds to the ARZ model.

The analytic resolution of the EARZ / GSOM model is based on the properties of invariant *I*:

- According to equation (3), the invariant *I* is constant along the vehicular trajectories:

$$\dot{I} = \partial_t I + v\partial_x I = 0 \quad (4)$$

- The discontinuities of invariant I are propagated with the traffic speed.

The invariant I is a *driver attribute*.
Let us consider the Riemann problem depicted in (Figure 3). The Riemann problem consists in finding the solution at the discontinuity point of two traffic conditions: upstream and downstream of the discontinuity point defined by the traffic conditions, (ρ_l, I_l) and (ρ_r, I_r) respectively.

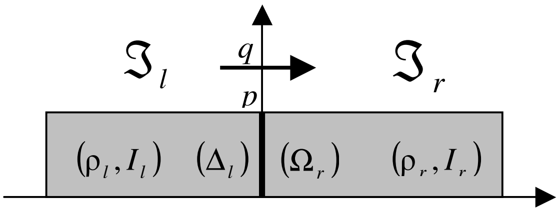

Figure 3. A Solution for the Riemann problem

Under the initial condition and the assumption that the variable I is piecewise constant, Lebacque (Lebacque *et al*, 2007) has demonstrated that the analytic resolution of the GSOM is equivalent to the piecewise resolution of the LWR model.

$$\begin{aligned} &\partial_t \rho + \partial_x \Re(\rho, I, x) = 0 \\ &\text{with} \quad \Re(\rho, I, x) = \rho \Im(\rho, I_l, x) \end{aligned} \tag{5}$$

Note that the function $\Im$ admits a discontinuity with respect to position: $\Im(\rho, I, x) = \Im_l(\rho, I)$ if $x < 0$ and $\Im(\rho, I, x) = \Im_r(\rho, I)$ if $x > 0$.
The solution of the Riemann problem is based on the application of the shifted supply and demand by a quantity I as depicted in Figure 4.

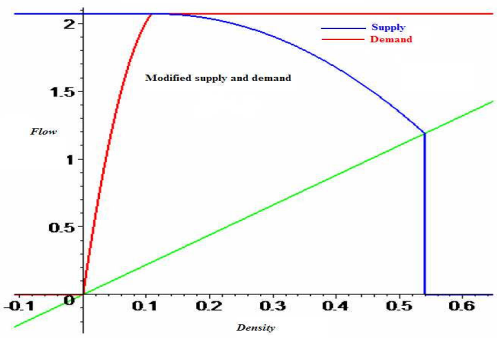

Figure 4. The translated supply/demand by I >0

Recall that the supply function (resp. demand) corresponds to the maximum number of vehicles that can enter (resp. wish to exit) a cell during one time slice. The functions supply and demand are expressed as follows:

$$\begin{aligned} &\Delta_i(\rho, I) = Max_{0 \le r \le \rho} \Re_i(\rho, I) \quad \forall i = l, r \\ &\Omega_i(\rho, I) = Max_{r \ge \rho} \Re_i(\rho, I) \quad \forall i = l, r \end{aligned} \tag{6}$$

Flow q and pressure p are given by the usual formula of Min:

$$\begin{cases} q = Min[\sigma_r, \delta_l] \\ p = q.I_l \end{cases} \tag{7}$$

with $\delta_l \overset{def}{=} \Delta_l(\rho_l, I_l)$ the upstream demand and with

$$\begin{aligned} \sigma_r &= \Omega_r\left(\Im^{-1}_{r,\rho}(v_r, I_l), I_l\right) \\ \sigma_r &= \Omega_r\left(\Im^{-1}_{r,\rho}(\Im_r(\rho_r, I_r), I_l), I_l\right) \overset{def}{=} \Xi_r(\rho_r, I_r, I_l) \end{aligned} \tag{8}$$

the downstream supply (v_r denotes the downstream speed and $\Im^{-1}_{r,\rho}$ the inverse of the downstream equilibrium speed function with respect to density).

Note that the density $\rho_m \overset{def}{=} \Im^{-1}_{r,\rho}(v_r, I_l)$ is such that $v_r = \Im_r(\rho_m, I_l)$.
The reader is referred to (Lebacque *et al* 2005) and (Lebacque *et al* 2007) for the derivation of the above expression (8) of the downstream supply.
Note that the upstream demand and downstream supply as defined in (7) and (8) can be used to define proper boundary conditions, (Lebacque et al 2007), thus generalizing the LWR boundary conditions (Lebacque 1996).

The supply downstream to the section depends on the upstream invariant I of this section. These dependencies constitute the main problem of the intersection modelling to be solved. This is the aim of the next section.

3. GSOM BASED INTERSECTION MODEL

In this section, the intersection model based on the GSOM is described. In order to help readers understand the difficulty of modelling intersections, we consider an intersection composed of node (*J*), upstream (ingoing) arcs (*i*) and downstream (outgoing) arcs (*j*) as depicted in Figure 1. To each arc (ℓ) of the intersection, a density ρ_ℓ and an invariant I_ℓ are associated.

The intersection model consists in determining, at each moment, the in- and out-flows $q_i(t)$ and $r_j(t)$ respectively by taking into account the upstream and downstream boundary conditions of the intersection. The boundary conditions are the upstream demands $\delta_i = \Delta_i(\rho_i(t), I_i)$ and the downstream supplies

$$\sigma_j = \Xi_j\left(\rho_j(t), I_j(t), \tilde{I}_j(t)\right)$$

which depend on the attribute $\tilde{I}_j$ of the drivers entering the link (j). We recall that, following (8), the expression of the downstream supplies is:

$$\sigma_j = \Omega_j\left(\Im^{-1}_{j,\rho}(v_j, \tilde{I}_j), \tilde{I}_j\right) = \Omega_j\left(\Im^{-1}_{j,\rho}(\Im_j(\rho_j, I_j), \tilde{I}_j), \tilde{I}_j\right)$$

(with v_j the downstream speed and $\Im^{-1}_{j,\rho}$ the inverse of the equilibrium speed function with respect to density)
As we can notice, the difficulty of modelling an intersection lies in determining the attribute value $\tilde{I}_j$ of the drivers entering the links (j). The value of these driver attributes also depend on the driver attributes of the upstream links of the intersection following some relationship (Mammar *et al* 2006):

$$\tilde{I}_j = f_j(I_1, I_2, ...I_i)$$

In order to overcome this difficulty, we consider that, at each time, the intersection node is characterised by an internal state. This state is defined by:

- The total number of vehicles present in the node: $N(t)$.
- The number of vehicles $N_j(t)$ leaving the node to the out-link (j) where $N(t) = \sum_j N_j(t) \quad \forall t$
- A driver attribute $\tilde{I}$

Consequently, the supply $\Omega\left(N(t), \tilde{I}(t)\right)$ and the global demand $\Delta\left(N(t), \tilde{I}(t)\right)$ of the intersection depend on the total number of vehicles $N(t)$ and on the invariant $\tilde{I}(t)$ within the intersection.

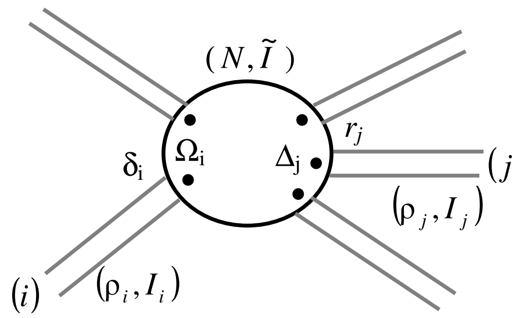

Figure 5. A punctual intersection with an internal state.

The calculation of the partial supplies at the level of each upstream section is given by a linear split of the global supply:

$$\Omega_i = \beta_i \Omega\left(N(t), \tilde{I}\right) \tag{9}$$

where the β_i are the *split coefficients*, which are proportional to the number of lanes of the upstream links.

Similarly, for the upstream section (j), we define the partial demand of the intersection using a FIFO model (demand proportional to composition):

$$\Delta_j(t) = \frac{N_j(t)}{N(t)} \Delta\left(N(t), \tilde{I}(t)\right) \tag{10}$$

To calculate the variation of the number of vehicles $N_j(t)$ at each time, the in/out vehicle flows must be calculated according to the following equations:

$$\begin{aligned} q_i(t) &= \text{Min}\left[\delta_i(t), \Omega_i(t)\right] \\ r_j(t) &= \text{Min}\left[\Delta_j(t), \sigma_j(t)\right] \\ \dot{N}_j(t) &= -r_j(t) + \sum_i \gamma_{ij} q_i(t) \\ \frac{d}{dt}(NI) &= \sum_i q_i I_i - \sum_j r_j I \end{aligned} \tag{11}$$

where γ_{ij} denotes the turning rate of vehicles exiting from section (*i*) and choosing section (*j*) at time *t*.

4. SIMULATION STUDY

The numeric resolution of the GSOM model is based on the principle of the Godunov scheme. This principle consists in decomposing the time and sections into time steps of duration Δt and cells of length Δx respectively (See

Figure 6).

By considering the traffic state of each cell approximated by a function which is piecewise constant, the flow of traffic between two consecutive cells during a step time *t* is obtained by solving a Riemann problem.

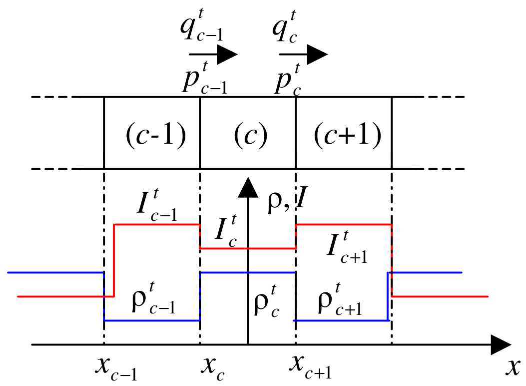

Figure 6. The principle of the Godunov scheme.

Assuming that the invariant *I* is bounded, we obtain the following discritized equations of the GSOM model:

$$\begin{aligned}
\rho_c^{t+1} &= \rho_c^t + \frac{\Delta t}{\Delta x}\left(q_{c-1}^t - q_c^t\right) \\
y_c^{t+1} &= y_c^t + \frac{\Delta t}{\Delta x}\left(p_{c-1}^t - p_c^t\right) \\
q_c^t &= \mathrm{Min}\left[\Delta_c\left(\rho_c^t, I_c^t\right), \Omega_c\left(\rho_{c+1}^t, I_c^t\right)\right] \\
p_c^t &= q_c^t I_c^t
\end{aligned} \qquad (12)$$

The stability of the numeric schema is guaranteed by the satisfiability of the following CFL condition:

$$\Delta x = \Delta t.\mathrm{Max}_{\rho \geq 0} \Im(\rho, I) \qquad (13)$$

The CFL condition depends on the values of *I*. In order to use (13), it is necessary to assume that the attribute *I* is bounded, an assumption which is easily satisfied by measurement data. By applying the same approach at the node level with a small temporal discretization step $\delta t = \frac{\Delta t}{\alpha}, \alpha \in \mathrm{N}$, we obtain the following discritized equations:

$$N_j(t+\delta t) = N_j(t) - r_j(t)\delta t + \sum_i \gamma_{ij}(t) q_i(t)\delta t$$

$$N(t+\delta t) = \sum_j N_j(t+\delta t)$$

$$\tilde{I}(t+\delta t) = \tilde{I}(t) + \frac{2}{N(t)+N(t+\delta t)} \cdot \left(\sum_i q_i(t) I_i(t) - r(t)\tilde{I}(t)\right)\delta t$$

with $r(t) \overset{def}{=} \sum_j r_j(t)$

To demonstrate the consistency of the numeric scheme, we consider the geometric configuration of the intersection given by Figure 5. For this study, we consider an invariant *I* equal to the differential between the current speed and equilibrium speed (the ARZ Aw-Rascle Zhang model).

In order to study the vehicles behaviour inside the intersection according to attribute *I*, we consider the network entries fed with a variable demand which depends on invariant *I* at the network entry. This invariant varies according to the time as shown in Figure 7.

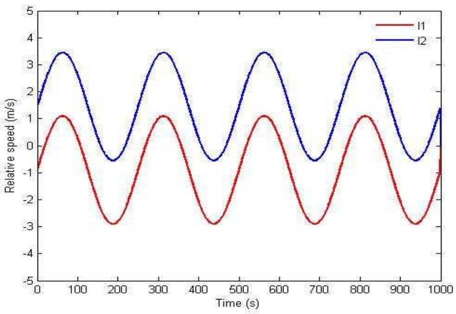

Figure 7. Relative speed (function of time)

The split coefficients are constant and equal to $\beta_1 = 0.75$ and $\beta_2 = 0.25$. Similarly, for each section, the numeric values of the proportions of the turning movements are equal to: $\gamma_{13} = 0.25$, $\gamma_{14} = 0.20$, $\gamma_{15} = 0.55$, $\gamma_{23} = 0.3$, $\gamma_{24} = 0.6$, $\gamma_{25} = 0.1$.

Figure 8 and Figure 9 show the evolution of the traffic state in term of relative speed but also in term of number of vehicles inside the intersection which is function of the time.

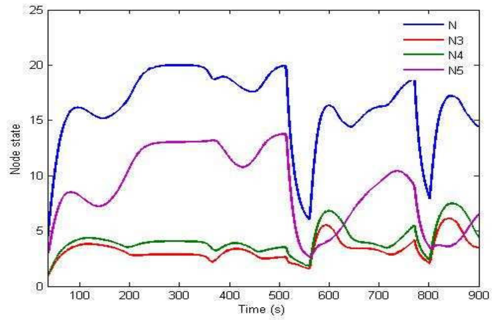

Figure 8 Time evolution of the number of vehicles inside the node

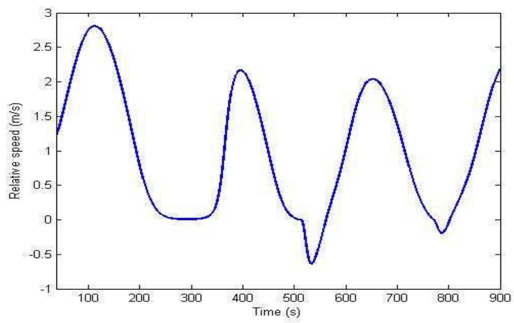

Figure 9. Time evolution of the relative speed inside the node

Figure 10 represents the evolution of the density at each intersection entry. As we can remark, the congestion generated at the second entry is more important to the one of the first in-link. This phenomenon is explained by the fact that the split coefficient of the first in-link is higher. Consequently, the traffic flow is more important.

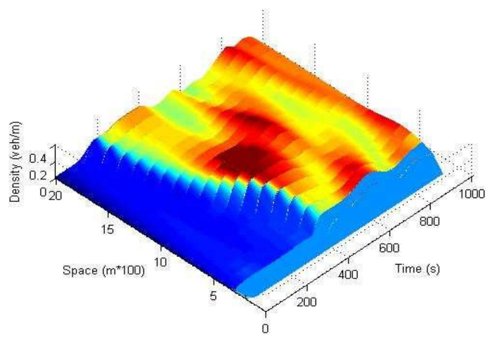

Figure 10. Time-space density evolution at the ingoing level.

Figure 11 represents the evolution of density on each outgoing link. As we can notice, at the outgoing level, the density varies accordingly to the value of the attribute *I* inside the node.

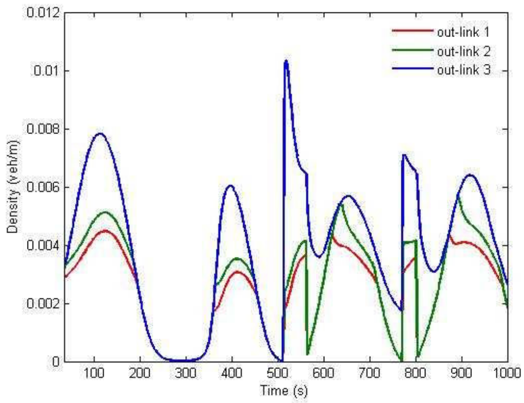

Figure 11. Time-space evolution of the density on outgoing links.

4. CONCLUSION

The approach used to model intersections is based on the GSOM model and the supply/demand notions which allow an adequate expression of the boundary conditions. It is fully compatible with the Godunov discretization of sections. The first results we have obtained in simulation are promising and completely satisfactory. This is why we think that the developed model is worth validating on real data.

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