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Positive invariant sets for fault tolerant multisensor control schemes

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Abstract: Positive invariance is a common analysis and control design tool for systems affected by constraints and disturbances. The present paper revisits the construction of ϵ-approximations of minimal robust positive invariant sets proposing contractive procedures in the cases of switching between different sets of disturbances and the inclusion of a predefined region of the state space. The results are used in multisensor control schemes which have to deal with specific problems originated by the switching between different estimators and by the presence of faults. Within this framework, global stability of the switching strategies can be assured if the invariant sets topology allows the exclusive selection of estimates obtained from healthy sensors.

1. INTRODUCTION

Multisensor schemes have originated substantial research on the aggregation of the information available from the plant in order to improve reliability and robustness. Sensor fusion has been one of the techniques traditionally employed in multisensor schemes where the construction of improved estimators is the main concern (Dasarathy, 1997; Luo et al., 2002; Sun and Deng, 2004). As is usually the case with low cost diversification and miniaturisation, components are predisposed to failures. For multisensor schemes, the presence of faults is manifested by the alteration of the estimations of the features of interest. The control strategy has to be equipped with fault detection capabilities in order to avoid the construction of the control action based upon erroneous feedback information.

Recently, multisensor switching feedback control strategies have provided interesting solutions with fault tolerance guarantees (Seron et al., 2008). The design procedure uses a switching strategy motivated by receding horizon optimal control principles. At each sampling time, the switching strategy implements the control action by selecting the sensor-estimator pair that provides the best predicted closed-loop performance according to a predefined criterion.

The present paper revisits the conditions for closed-loop stability for such multisensor switching control schemes and reduces the conservatism of the assumptions by refining the invariant sets for healthy and for faulty functioning. Indeed, positive invariance is a common analysis and control design tool for systems affected by constraints and/or disturbances. The switching between sensor-estimator pairs introduced by these schemes implies switching between systems affected by different sets of disturbances. In addition, the presence of failures implies structural modifications which have to consider the functioning regime previous to the fault. It will be shown how these issues can be embedded in the invariant sets definition and constructive solutions will be provided.

Ideally, the use of minimal robust positive invariant (mRPI) sets would provide the exact information about the closed-loop behavior under different operating conditions needed to perform (explicit or implicit) fault detection. From a practical point of view, exact mRPI sets can be obtained only for restricted classes of systems and, in general, ϵ-approximations have to be employed instead. Existing results on ϵ-approximations of robust positive invariant sets in Raković et al. (2005) are extended in the present paper along the following lines:

- The set confining the disturbance does not have to contain the origin;
- The iterative procedure constructs the approximations in a contractive manner;
- The result is extended to the construction of invariant sets containing a given region of the state space.

The remainder of the paper is organised as follows. Section 2 introduces the multisensor control structure. Section 3 details the invariant set computation and several refinements are provided. Section 4 describes the fault tolerant switching control scheme and discusses conditions for global stability. In Section 5 an application is presented and Section 6 draws some conclusions. Due to space limitations, we have not included the proofs of the theorems in this final version of the paper.

2. MULTISENSOR CONTROL SCHEME

2.1 System structure with multiple sensors and estimators

The multisensor control scheme assumes the existence of a linear discrete-time state space model Σ, for the plant considered. The state vector of the system, $x \in \mathbb{R}^n$, is not directly available. Instead, combinations of the states, given by $C_\Sigma x$, can be measured via N sensors. The output

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signal of each sensor, $y_i \in \mathbb{R}^{p_i}$, $i = 1, \ldots, N$, carries useful information for control purposes and its treatment has to take into account the internal dynamics of each sensor, described by the evolution of the internal state $\xi_i \in \mathbb{R}^{n_i}$. The first column of Table 1 contains the linear models for the plant and the sensors, as well as the dynamics of the estimators. A block description of the control scheme is depicted in Figure 1.

It is assumed that the sensor matrices A_{s_i} have all their eigenvalues strictly inside the unit circle and that the estimators are designed in order to exhibit a good dynamic behavior for the extended state estimate $\left[\hat{x}_i \ \hat{\xi}_i\right]^T$. This is achieved by adequate choice of matrices L_i, L_{s_i} such that the resulting matrices $A_{L_i}, i = 1, \ldots, N$ have all their eigenvalues strictly inside the unit circle.

2.2 Control objective and exogenous signals

The control objective is to ensure that the state of the system tracks a reference signal x_{ref} which, in turn, follows the dynamics given in Table 1 for the reference model. The reference tracking of the global system requires a reference signal for each sensor state. The tracking errors are given by the difference between the state and the respective reference signal, as can be seen in the last column of Table 1.

In order to derive a control strategy, a description of the exogenous signals u_{ref}, w, η_i, is needed. The present study is dedicated to the case of bounded signals, with no other assumption on their properties. The disturbance and measurement noises are assumed to be contained in polyhedral sets $w \in W$ and $\eta_i \in N_i$. In addition, bounds on the input reference signal u_{ref} induce bounds on the state of the reference model:

$$x_{ref} \in X_{ref} \tag{1}$$

with X_{ref} a closed (polyhedral) set.

2.3 Sensor failure model

The faults considered in this paper are of the type of total sensor outage. Namely, it is assumed that during sensor failure the output of the sensor ceases to carry information about the sensor state (even though the sensor state continues to evolve with the same dynamics). The failure is thus equivalent to the following switching on the observation equation:

$$y_i = C_{s_i}\xi_i + \eta_i \quad \xrightarrow{FAULT} \quad y_i = 0 \cdot \xi_i + \eta_i^F \tag{2}$$

$$y_i = C_{s_i}\xi_i + \eta_i \xleftarrow{RECOVERY} y_i = \qquad \eta_i^F \tag{3}$$

The noise level during the fault, η_i^F, may in general be different from the noise during healthy operation, η_i. The bounds on the noise during the fault are denoted $\eta_i^F \in N_i^F$.

2.4 The missing link

The presence of faults implies, through the structural changes (2), a change in the input of the dynamic equation of the corresponding estimator. Indeed, a fault-recovery cycle will bring the system back to the operational framework (Table 1) but the reinitialisation of the estimator's state has to be carefully considered. The evolution of the estimation error, under healthy sensor operation, will verify:

$$\begin{bmatrix} \tilde{x}_i^+ \\ \tilde{\xi}_i^+ \end{bmatrix} = \begin{bmatrix} x^+ \\ \xi_i^+ \end{bmatrix} - \begin{bmatrix} \hat{x}_i^+ \\ \hat{\xi}_i^+ \end{bmatrix} = A_{L_i} \begin{bmatrix} \tilde{x}_i \\ \tilde{\xi}_i \end{bmatrix} + \begin{bmatrix} E_\Sigma \\ 0 \end{bmatrix} w - \begin{bmatrix} L_i \\ L_{s_i} \end{bmatrix} \eta_i \tag{4}$$

We assume that the pairs $\left(\begin{bmatrix} A_\Sigma & 0 \\ B_{s_i}C_\Sigma & A_{s_i} \end{bmatrix}, [0 \ \ C_{s_i}]\right)$ are detectable for $i = 1, \ldots, N$, and that the gains L_i, L_{s_i} are such that matrices A_{L_i} have all their eigenvalues strictly inside the unit circle (this is always possible by the detectability assumption).

The estimation error might be seen as the "missing link" (due to the fact that it is not directly measurable) between the estimator tracking error and the tracking error:

$$\underbrace{\begin{bmatrix} \hat{z}_i \\ \hat{\zeta}_i \end{bmatrix}}_{\substack{\text{Estimator} \\ \text{tracking error}}} = \underbrace{\begin{bmatrix} z \\ \zeta_i \end{bmatrix}}_{\substack{\text{Tracking} \\ \text{error}}} - \underbrace{\begin{bmatrix} \tilde{x}_i \\ \tilde{\xi}_i \end{bmatrix}}_{\substack{\text{Estimation} \\ \text{error}}} \tag{5}$$

3. INVARIANT SETS CONSTRUCTION

3.1 A contractive procedure

Consider a general discrete-time linear time-invariant systems subject to disturbance:

$$x^+ = Ax + B\delta \tag{6}$$

with A strictly stable and $\delta \in \Delta$ a polytopic set. The minimal robust positive invariant (mRPI) set, defined as the RPI set contained in any closed RPI set is known to be unique, compact and—in the case when Δ contains the origin—to contain the origin (Kolmanovsky and Gilbert, 1998). Its construction is dependent on the structure of A, B and the topology of Δ (the framework used to describe the disturbance set Δ will be the polytopic one).

The exact computation of the mRPI set is assured only under restrictive assumptions of nilpotent system dynamics for the subsystem affected by the disturbances (Mayne and Schroeder, 1997). In Hirata and Ohta (2003) a recursive procedure is proposed to find an ϵ-outer approximation of the mRPI set. In Raković et al. (2005) an improved algorithm provides the maximal number of iterations for the obtention of the outer approximation for a given ϵ.

In the following we state the results in the general case where (although not necessarily) it is allowed for $0 \notin \Delta$. It will be shown that a certified ϵ-outer approximation can be obtained using a contractive procedure starting from an initial RPI set. This initial set can be obtained upon ultimate bounds, for example using the results provided in the next theorem. In the sequel, inequalities between vectors are to be interpreted componentwise.

Theorem 1. Consider the system (6), let $A = V\Lambda V^{-1}$ be the Jordan decomposition of A and consider a bounding box for the set Δ. If this bounding box is described by the vector $\bar{\delta}$ which satisfies $|\delta| \le \bar{\delta}$, $\forall \delta \in \Delta$ then the set:

$$\Phi_0 = \left\{ x \in \mathbb{R}^n : \left|V^{-1} x\right| \le (I - |\Lambda|)^{-1} \left|V^{-1}B\right| \bar{\delta} \right\}, \tag{7}$$

is robust positively invariant (RPI).

	Dynamics	Reference Model	Tracking Error
Plant Σ	$x^+ = A_\Sigma x + B_\Sigma u + E_\Sigma w$	$x^+_{ref} = A_\Sigma x_{ref} + B_\Sigma u_{ref}$	$\begin{aligned} z^+ &= x^+ - x^+_{ref} \\ &= A_\Sigma z + B_\Sigma \underbrace{(u - u_{ref})}_{v} + E_\Sigma w \end{aligned}$
Sensors $S_i (i = 1, \ldots, N)$	$\begin{aligned} \xi_i^+ &= A_{s_i}\xi_i + B_{s_i} C_\Sigma x, \\ y_i &= C_{s_i}\xi_i + \eta_i \end{aligned}$	$\xi^+_{i,ref} = A_{s_i}\xi_{i,ref} + B_{s_i} C_\Sigma x_{ref}$	$\begin{aligned} \zeta_i^+ &= \xi_i^+ - \xi^+_{i,ref} \\ &= A_{s_i}\zeta_i + B_{s_i} C_\Sigma z \end{aligned}$
Estimators $F_i (i = 1, \ldots, N)$	$\begin{bmatrix} \hat{x}_i^+ \\ \hat{\xi}_i^+ \end{bmatrix} = \underbrace{\begin{bmatrix} A_\Sigma & -L_i C_{s_i} \\ B_{s_i} C_\Sigma & A_{s_i} - L_{s_i} C_{s_i} \end{bmatrix}}_{A_{L_i}} \begin{bmatrix} \hat{x}_i \\ \hat{\xi}_i \end{bmatrix} + \begin{bmatrix} B_\Sigma \\ 0 \end{bmatrix} u + \begin{bmatrix} L_i \\ L_{s_i} \end{bmatrix} y_i$	$\begin{bmatrix} \hat{x}_{i,ref} \\ \hat{\xi}_{i,ref} \end{bmatrix} = \begin{bmatrix} x_{ref} \\ \xi_{i,ref} \end{bmatrix}$	$\begin{bmatrix} \hat{z}_i \\ \hat{\zeta}_i \end{bmatrix} = \begin{bmatrix} \hat{x}_i \\ \hat{\xi}_i \end{bmatrix} - \begin{bmatrix} \hat{x}_{i,ref} \\ \hat{\xi}_{i,ref} \end{bmatrix}$

Table 1. Plant, sensor and estimator models. The tracking error in each case is given with respect to the state of the corresponding reference model. ($^+$ *denotes the successor state*)

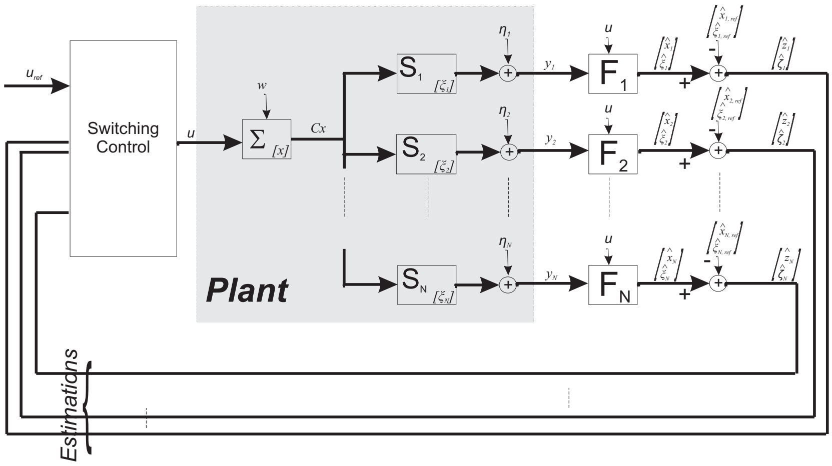

Fig. 1. Configuration of the multisensor scheme with the plant Σ, sensors S_i $(i = 1, \ldots, N)$, estimators F_i $(i = 1, \ldots, N)$ and the switching control block (fault tolerant selection of the sensor-estimator pairs).

In order to refine this invariant set, a sequence of sets can be recursively built by considering the Minkowski sum between the image of an RPI set through the linear transformation A and the polyhedral set $B\Delta$:

$$\Phi_{k+1} = A\Phi_k \oplus B\Delta \tag{8}$$

The definition (8) of the sequence Φ_k preserves the invariance properties.

Theorem 2. Let Φ_0 be as defined in Theorem 1. Then the sequence $\Phi_k, \forall k \in \mathbb{N}$, satisfies $\Phi_{k+1} \subset \Phi_k$, and Φ_k is convex, compact and an RPI set with respect to (6).

Remark 3. The initial set Φ_0 in the set recursion (8) can be in fact any RPI set for the dynamics (6). Theorem 1 provides a simple and direct way of obtaining the initial condition using ultimate bounds (7), in view of an algorithmic implementation.

Our next theorem shows that the limiting sequence obtained from (8) will lead to an approximation of the mRPI. Indeed, let Ω be the mRPI set, defined as the limit of all the possible trajectories of (6). Equivalently (see Raković et al. (2005)) the mRPI set can be described as $\Omega = \lim_{k\to\infty} \Omega_k$ with

$$\Omega_k = \bigoplus_{i=0}^{k} A^i B\Delta \tag{9}$$

Theorem 4. $\Phi_k \to \Omega$ for $k \to \infty$.

The following theorem uses the set recursion (8) to obtain outer ϵ-approximations of the mRPI set Ω

Theorem 5. For all $\epsilon > 0$ there exists an $s \in \mathbb{N}^+$ such that the following RPI outer ϵ-approximation exists:

$$\Omega \subset \Phi_{s+1} \subset \Omega \oplus \mathbb{B}_p^n(\epsilon) \tag{10}$$

The previous theorems show that iterating the set construction in (8) a contractive refinement of the invariant set obtained using ultimate bounds, Φ_0, can be found. Moreover, the theorems show that the set sequence obtained with (8) converges to the minimal positive invariant set Ω and that the maximal number of iterations needed to find an ϵ-approximation can be computed *a priori*, thus providing an effective stopping criterion.

Remark 6. Observing that the intersection of RPI sets is an RPI set, an improved ϵ-approximation can be obtained by working in parallel with a contractive and an expansive algorithm. For the case $0 \in int(\Delta)$ the presented technique and the one in Raković et al. (2005) can be used to provide, in a fix number of steps, a better ϵ-approximation.

The construction of the approximation of the mRPI set is summarised in the following algorithm.

Algorithm 1: Approximation of the mRPI set
Input arguments: The pair (A, B), the disturbance set Δ and the scalar $\epsilon > 0$.
Output: The RPI ϵ-approximation of the mRPI set.

1) Compute the Jordan decomposition of A;
2) Build the initial RPI set Φ_0 using ultimate bounds (7);

3) Find s, such that $A^{s+1}\Phi_0 \subset \mathbb{B}_p^n(\epsilon/2)$;
4) For $k = 1$ to $k = s + 1$

compute the set Φ_k using (8). ■

3.2 Switching between sets of disturbances

In the framework of the multisensor fault tolerant control systems that will be explained below, in Section 4, the case of arbitrary switches between N different sets of disturbances has to be considered, namely:

$$\begin{array}{c} x^+ = Ax + B_l\delta_l \\ \delta_l \in \Delta_l, l \in \{1, \ldots, N\} \end{array} \tag{11}$$

The mRPI set is, in this case, in general nonconvex. In order to obtain a convex RPI approximation, a certain degree of conservativeness has to be introduced by considering the convex hull of the sets of disturbances. This leads to a linear model similar to (6), that is:

$$\begin{array}{l} x^+ = Ax + \nu, \quad \nu \in \Delta \\ \Delta = Conv.Hull\{B_1\Delta_1, \ldots, B_N\Delta_N\} \end{array} \tag{12}$$

The construction of a refined RPI set can follow the lines presented in the previous subsection.

3.3 mRPI with inclusion preserving

Consider again the discrete-time LTI systems subject to disturbance:

$$x^+ = Ax + B\delta \tag{13}$$

with A strictly stable and $\delta \in \Delta$ a polytopic set. Also consider a given bounded set $P \subset \mathbb{R}^n$ which can be interpreted as the region where the state evolution is initiated. Following the ideas of the previous subsections, the construction of an RPI approximation for the minimal RPI set which assures the inclusion of P will be sketched.

In order to use a recursive procedure, an initial RPI set Ψ_0 with the desired property $\Psi_0 \supset P$ has to be devised. The set Φ_0 constructed using (7) does not necessarily satisfy the inclusion but it has the auxiliary property that $0 \in int(\Phi_0)$. Using this property and the scaling factor:

$$\mu^* = \mu^*(\Phi_0, P) = \min_{\substack{\mu \geq 1 \\ P \subset \mu\Phi_0}} \mu \tag{14}$$

the next result is available.

Proposition 7. $\Psi_0 = \mu^*\Phi_0$ is a robust positive invariant set for (13) and $P \subset \Psi_0$.

In order to refine recursively this RPI set while preserving the inclusion, the following sequence is considered:

$$\Psi_{k+1} = Conv.Hull\{P,\, A\Psi_k \oplus B\Delta\} \tag{15}$$

Theorem 8. $\Psi_{k+1} \subset \Psi_k$ and $P \subset \Psi_k$, $\forall k \in \mathbb{N}$. If Φ_0 is bounded then Ψ_k is convex, compact and an RPI set for (13).

Denoting Ω_P the minimal RPI set that preserves the inclusion $P \subset \Omega_P$, and using the fact that the mRPI set Ω is an attractor for the trajectories of (13), it follows that Ω_P can be described explicitly as the union of all the trajectories starting inside P and leading to Ω:

$$\Omega_P = P\bigcup_{i=0}^{\infty}\left\{A^{i+1}P\bigoplus_{j=0}^{i}A^jB\Delta\right\} \cup \Omega \tag{16}$$

As it can be seen from (16), Ω_P may in general be nonconvex. Consider now the set:

$$\bar{\Omega}_P = Conv.Hull\left\{P\bigcup_{i=0}^{\infty}\left\{A^{i+1}P\bigoplus_{j=0}^{i}A^jB\Delta\right\} \cup \Omega\right\}, \tag{17}$$

then, it can be easily shown (from the robust positive invariance of Ω_P and the convexity of $\bar{\Omega}_P = Conv.Hull\{\Omega_P\}$) that $\bar{\Omega}_P$ is an RPI set. In fact, $\bar{\Omega}_P$ is the minimal convex RPI set that preserves the inclusion $P \subset \bar{\Omega}_P$.

Theorem 9. $\Psi_k \to \bar{\Omega}_P$ for $k \to \infty$.

The following result proves that an ϵ-approximation for $\bar{\Omega}_P$ can be found in a finite number of iterations.

Theorem 10. For all $\epsilon > 0$ there exists an $s \in \mathbb{N}^+$ such that the following RPI outer ϵ-approximation exists:

$$\bar{\Omega}_P \subset \Psi_{s+1} \subset \bar{\Omega}_P \oplus \mathbb{B}_p^n(\epsilon) \tag{18}$$

4. SWITCHING CONTROL STRATEGY

In this section we use the invariant sets constructed in the preceding sections to define a switching control strategy with fault-tolerance guarantees for the multisensor scheme described in Section 2.

4.1 Invariant sets for estimation errors and sensor reference signals

The polyhedral sets that describe the disturbances and the measurement noises, W, N_i and N_i^F, are assumed to contain the origin in their interiors. These sets can be confined inside symmetric bounding boxes:

$$|w| \leq \bar{w};\ |\eta_i| \leq \bar{\eta}_i;\ |\eta_i^F| \leq \bar{\eta}_i^F; \tag{19}$$

where $\bar{w}$, etc., are vectors of bounds with positive elements.

Recall from (4) that the estimation error dynamics during healthy sensor operation satisfy:

$$\begin{bmatrix} \tilde{x}_i^+ \\ \tilde{\xi}_i^+ \end{bmatrix} = A_{L_i}\begin{bmatrix} \tilde{x}_i \\ \tilde{\xi}_i \end{bmatrix} + \begin{bmatrix} E_\Sigma & -L_i \\ 0 & -L_{s_i} \end{bmatrix}\begin{bmatrix} w \\ \eta_i \end{bmatrix} \tag{20}$$

with the matrices A_{L_i} strictly stable and with input $[w\ \ \eta_i]'$ bounded as in (19).

We begin by computing, using (7), the initial invariant set that corresponds to the dynamics (20) and bounds (19). Then, iterating (8) for system (20) and sets W, N_i according to Algorithm 1, we can obtain, for a given $\epsilon > 0$:

$\tilde{S}_i$ = *RPI ϵ-approximation of the mRPI set for the i-th estimation error under healthy sensor operation.*

The sensor reference signals $\xi_{i,ref}$ (see Table 1) satisfy:

$$\xi_{i,ref}^+ = A_{s_i}\xi_{i,ref} + B_{s_i}C_\Sigma x_{ref} \tag{21}$$

where, as explained in Section 2.2, the state of the reference model is contained in a set $X_{ref} \subset \mathbb{R}^n$ determined by the constraints on the reference input u_{ref} (notice that it is possible that $0 \notin X_{ref}$, and that this is allowed in Algorithm 1). Confining X_{ref} inside a symmetric box we can write:

$$|x_{ref}| \leq \bar{x}_{ref} \tag{22}$$

The invariant set construction procedure can be initialized using (7) for the dynamics (21) and bounds (22). Then, iterating (8) for system (21) and set X_{ref} we obtain, using Algorithm 1:

$S_{i,ref} =$ *RPI ϵ-approximation of the mRPI set for the i-th sensor reference signal.*

The sets $\tilde{S}_i$ and $S_{i,ref}$ defined above will be used in the following subsection to compute refined invariant sets for the tracking error dynamics under healthy and faulty operation.

4.2 Optimal control upon healthy estimations

Let us now consider an optimal control problem for the tracking error system (A_Σ, B_Σ) (see Table 1), with $Q > 0$ and $R > 0$ as the weighting parameters for the tracking error states and control effort respectively. By solving the associated Riccati equation one can obtain the optimal linear gain K and cost function matrix P as follows:

$$K = (R + B'_\Sigma P B_\Sigma)^{-1} B'_\Sigma P A_\Sigma \tag{23}$$

$$P = A'_\Sigma P A_\Sigma + Q - K'(R + B'_\Sigma P B_\Sigma) K \tag{24}$$

Consider the control law:

$$u = u_{ref} - K \hat{z}_l^{UP} \tag{25}$$

where $\hat{z}_l^{UP}$ is the estimation "update" for the l-th sensor, supposed healthy:

$$\hat{z}_l^{UP} = \hat{x}_l^{UP} - x_{ref} = \hat{x}_l + M_l(y_l - C_{s_l}\hat{\xi}_l) - x_{ref} \tag{26}$$

The update matrix M_l is obtained from:

$$\begin{bmatrix} A_\Sigma & 0 \\ B_{s_l}C & A_{s_l} \end{bmatrix} \begin{bmatrix} M_l \\ M_{s_l} \end{bmatrix} = \begin{bmatrix} L_l \\ L_{s_l} \end{bmatrix} \tag{27}$$

Assuming that the control law (25) can be based on information from any sensor $l \in \{1, \ldots, N\}$ which has been functioning without failure for sufficiently long time, we have the following closed-loop dynamics for the plant tracking error:

$$z^+ = \underbrace{(A_\Sigma - B_\Sigma K)}_{A_z} z + \underbrace{\begin{bmatrix} E_\Sigma & B_\Sigma K & -B_\Sigma K M_l C_{s_l} & -B_\Sigma K M_l \end{bmatrix}}_{B_z} \begin{bmatrix} w \\ \tilde{x} \\ \tilde{\xi}_l \\ \eta_l \end{bmatrix} \tag{28}$$

The above corresponds to a system of the type (12), which switches between different sets of disturbances. Using the arguments in Subsection 3.2 and adapting Algorithm 1, we can construct, for the dynamics (28) and the sets $W, N_i, \tilde{S}_l$ with $l \in \{1, \ldots, N\}$, the set:

$S_z =$ *RPI ϵ-approximation of the mRPI set for the plant tracking error.*

Subsequently using the set S_z for each sensor dynamics as described in Table 1:

$$\zeta_i^+ = A_{s_i}\zeta_i + B_{s_i} C z \tag{29}$$

we obtain by direct application of Algorithm 1:

$S_{\zeta_i} =$ *RPI ϵ-approximation of the mRPI set for the i-th sensor tracking error.*

For the estimated tracking error corresponding to healthy sensors, and assuming that only healthy sensors are used to implement the control law, the closed-loop dynamics can be written explicitly as:

$$\begin{bmatrix} \hat{z}_i^+ \\ \hat{\zeta}_i^+ \end{bmatrix} = A_{L_i} \begin{bmatrix} \hat{z}_i \\ \hat{\zeta}_i \end{bmatrix} + B_{li}\nu_{li} \tag{30}$$

with

$$B_{li} = \begin{bmatrix} -B_\Sigma K & B_\Sigma K & -B_\Sigma K M_l C_{s_l} & -B_\Sigma K M_l & L_i C_{s_i} & L_i \\ 0 & 0 & 0 & 0 & L_{s_i} C_{s_i} & L_{s_i} \end{bmatrix}$$

$$\nu_{li} = \begin{bmatrix} z' & \tilde{x}'_l & \tilde{\xi}'_l & \eta'_l & \zeta'_i & \eta'_i \end{bmatrix}' \tag{31}$$

Considering (30) and combining all the intermediate invariant sets $S_z, \tilde{S}_l, N_l, S_{\zeta_i}, N_i$ we can obtain by means of Algorithm 1 (using as input argument the convex hull of the possible sets of disturbances):

$\hat{S}_i =$ *RPI ϵ-approximation of the mRPI set for the estimated tracking error of the i-th healthy sensor.*

In the case of a fault as described in (2), and assuming that only healthy sensors are used to implement the control law, the closed-loop dynamics of the estimated tracking error for the j-th failed sensor can be shown to become:

$$\begin{bmatrix} \hat{z}_j^+ \\ \hat{\zeta}_j^+ \end{bmatrix} = A_{L_j} \begin{bmatrix} \hat{z}_j \\ \hat{\zeta}_j \end{bmatrix} + B_{lj}\nu_{lj}^F \tag{32}$$

with

$$\nu_{lj}^F = \begin{bmatrix} z' & \tilde{x}'_l & \tilde{\xi}'_l & \eta'_l & -\xi_{j,ref}^{F\,\prime} & \eta_j^{F\prime} \end{bmatrix}' \tag{33}$$

and B_{lj} as in (30).

Proceeding as for the healthy sensors' tracking error, by replacing the sets S_{ζ_i}, N_i by $-S_{j,ref}$ and N_j^F, we can construct the set:

$\hat{S}_j^F =$ *RPI ϵ-approximation of the mRPI set for the estimated tracking error of the j-th faulty sensor*,

upon an algorithm which preserves the inclusion $\hat{S}_j^F \supset \hat{S}_j$, as explained in Subection 3.3.

4.3 Switching strategy and fault tolerance

The optimal control law (25) was based on the assumption that the estimation update is provided by a *healthy sensor.* The way of switching among these estimation updates corresponding to healthy sensors characterises the stability and fault tolerance properties of the multisensor control scheme. The following result establishes these properties.

Proposition 11. Suppose the following assumptions are fulfilled:

(1) At any time instant, at least one of the following two situations is true:
- **a)** all sensors are healthy;
- **b)** at least one sensor is healthy; in addition, all healthy sensors have estimation errors inside the invariant set $\tilde{S}_i$ and at least one healthy lth sensor has the states of the corresponding estimated tracking error in the invariant set $\hat{S}_l$.

(2) $(\hat{z}_i^{UP})' P \hat{z}_i^{UP} < (\hat{z}_j^{UP})' P \hat{z}_j^{UP}$ for all $i, j = 1, \ldots, N, i \neq j$ and

$$\hat{z}_i^{UP} \in \begin{bmatrix} I & -M_i C_{s_i} \end{bmatrix} \hat{S}_i \oplus M_i N_i \oplus M_i C_{s_i} S_{\zeta_i} \tag{34}$$

$$\hat{z}_j^{UP} \in \begin{bmatrix} I & -M_i C_{s_i} \end{bmatrix} \hat{S}_i^F \oplus M_i N_i^F \oplus M_i C_{s_i} S_{j,ref} \tag{35}$$

where P is the cost function matrix defined in (24).

(3) $\hat{S}_i^F \supset \hat{S}_i$ for all $i = 1, \ldots, N$.

Then the closed-loop system with:

$$u = u_{ref} - K \arg\min_{z_l^{UP}} (\hat{z}_l^{UP})' P \hat{z}_l^{UP} \qquad (36)$$

is stable and fault tolerant. ■

5. EXAMPLE

Consider the longitudinal control problem for a car following scenario as described in Martinez and Canudas de Wit (2004) and revisited in Seron et al. (2008). The interdistance dynamics are represented by a discretised double integrator model:

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; E = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}; C = [\,0\ \ 1\,] \qquad (37)$$

We consider two sensors with dynamics given by the linear models in Table 1 with

$$A_{s_1} = 0.6065, B_{s_1} = 0.5, C_{s_1} = 0.7869 \qquad (38)$$

$$A_{s_2} = 0.8187, B_{s_2} = 0.5, C_{s_2} = 0.3625 \qquad (39)$$

The disturbances and measurement noises are bounded as

$$|w| \leq 0.02, |\eta_i| \leq 0.1, |\eta_i^F| \leq 0.1,\ i = 1, 2 \qquad (40)$$

In Seron et al. (2008) it was shown that a switching scheme as described in Section 2 and based on an optimal control law designed with weights

$$Q = \begin{bmatrix} 0.1007 & 0 \\ 0 & 6.3187 \end{bmatrix}; R = 7.2598 \qquad (41)$$

in (23)–(24), can stabilise the system and provide fault tolerance guarantees for reference signals elementwise bounded as

$$\begin{bmatrix} 33 \\ -10 \end{bmatrix} \leq x_{ref} \leq \begin{bmatrix} 75 \\ 0.08 \end{bmatrix} \qquad (42)$$

The fault tolerance conditions given in item (2) of Proposition 11 can be interpreted geometrically as a separation in the $\hat{z}_i^{UP}$ space between the healthy and faulty invariant sets S_i^H and S_j^F. Note, however, that when the range of the reference signals is increased to:

$$\begin{bmatrix} 24 \\ -20 \end{bmatrix} \leq x_{ref} \leq \begin{bmatrix} 100 \\ 5 \end{bmatrix} \qquad (43)$$

the separation using the ultimate bound invariant sets does not hold anymore, as it can be observed in Figure 2.

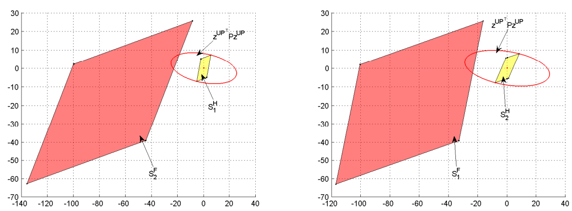

Fig. 2. Invariant sets based on ultimate bounds which fail to assure separation between the healthy and the faulty behaviour.

On the other hand, using the refined invariant set construction presented in this paper, where arbitrarily precise ϵ-approximations of the mRPI sets can be computed, it can be shown that the stability under fault of the switching scheme is preserved for the enlarged range of reference signals (43). The separation using these sets is illustrated in Figure 3.

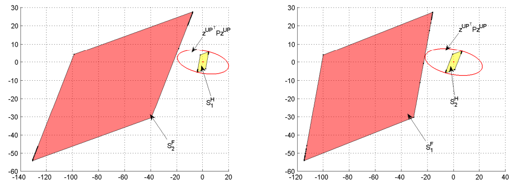

Fig. 3. Refined invariant sets assuring the separation between the healthy and the faulty behaviour.

6. CONCLUSIONS

This paper has explored a multisensor switching control scheme, focusing on a deterministic construction of invariant sets in order to obtain guarantees of fault tolerant functioning. Unfortunately, the construction of the minimal robust positive invariant sets is not finitely determined, not even for linear systems with regular disturbance sets. New contractive procedures for the construction of ϵ-approximations of these sets were proposed here with several adaptations for the multisensor control scheme.

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