Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Observers Synthesis Method for a Class of Nonlinear Discrete-Time Systems with Extension to Observer-Based Control

Ali Zemouche * **Mohamed Boutayeb** **

* *LSIIT-UMR CNRS-ULP 7005, ENSPS, Bd. Sébastien Brant 67412 Illkirch Cedex France*
** *CRAN-CNRS UPRES-A 7039, University of Henri Poincaré, Nancy I 186, 54400 Cosnes et Romain, France*

Abstract: This note deals with a new observer synthesis method for a class of nonlinear discrete-time systems. Thanks to the use of the Differential Mean Value Theorem (DMVT), we have obtained easily an extension of the work established in Zemouche et al. (2006) and Fan and Arcak (2003) to the discrete-time case. Based on the Lyapunov stability, a new sufficient synthesis condition is proposed. This condition is expressed in term of Linear Matrix Inequality (LMI) and then it is easily tractable using standard convex optimization algorithms. An extension to observer-based control is also presented. The design of the observer-controller gains is given in two manners. Firstly, the gains are computed by solving a LMI condition under an equality constraint. Since this latter induce a conservatism for the approach, then a systematic algorithm, that avoids the equality constraint, is proposed to solve the problem of observer-based control in two steps.

Keywords: Nonlinear discrete-time systems, Observer design, Observer-based control, LMI approach, Lyapunov stability, the Differential Mean Value Theorem (DMVT), Linear Parameter Varying (LPV) systems.

1. INTRODUCTION

In the past, observer design problem for nonlinear continuous time systems has been widely investigated, and several state observer design techniques are proposed Keller (1987), Krener and Respondek (1985), Rajamani (1998), Raghavan and Hedrick (1994), Gauthier et al. (1992), Gauthier and Kupka (1994), Hou and Pugh (1999). Little attention has been paid toward discrete-time case. The only approach used in the past for the discrete time case is the famous Extended Kalman Filter (EKF) Boutayeb and Aubry (1999), Reif et al. (1999). However, it has suffered from the lack of guaranteed stability. Only local convergence is ensured with an additional drawback that is great sensitivity to initializations.

On the other hand, research in the state observation field was recently directed towards nonlinear discrete time systems. Many alternative observer design methods to the EKF are established in the recent literature, see Ibrir (2007a), Ibrir et al. (2005), Ibrir (2007b), Yaz et al. (2007), Sundarapandian (2004) and Sundarapandian (2006) just to mention some recent works.

In this paper, we present a new state observer design method for a class of nonlinear discrete time systems. The asymptotic convergence of the proposed observer is guaranteed globally and without any approximation of the parameters of the system. Thanks to the use of the DMVT, we have established easily an extension, to discrete time systems, of the result of Zemouche et al. (2006), which is also an improvement of works proposed in Fan and Arcak (2003) and Arcak and Kokotovic (2001). Based on the Lyapunov stability theory, a new design method is proposed. Using a new reformulation of the Lipschitz property for differentiable functions, a new sufficient synthesis condition is established. This condition, expressed in term of LMI, overcome the conservatism related to the approaches based directly on the classical Lipschitz condition, as shown in Zemouche et al. (2006) and Zemouche et al. (2007) for the continuous time case.

The proposed approach is also applied to solve the problem of observer-based control. Two methods are proposed to solve the problem. In the first one, the observer-controller gains are computed by solving a LMI condition under an equality constraint. Since this equality constraint is difficult or impossible to be solved for certain nonlinear systems with single input as shown in Ibrir et al. (2005), then a second alternative method is presented. This latter can be summarized in a systematic algorithm that solve the problem of observer-based control in two steps. The main advantage of this algorithm is that the equality constraint, that induce a conservatism for the approach, is eliminated. This algorithm consists to compute firstly the controller gain by solving a certain set of Linear Matrix Inequalities (LMIs). Using the resulting value of the controller gain, we determine the observer gains by solving a new LMI.

It should be noticed that the two approaches used to synthesize the observer-controller gains are different. Indeed, in the first step of the algorithm, it is required to proceed as in Zemouche et al. (2007) in order to lead to a LMI

978-1-1234-7890-2/08/$20.00 © 2008 IFAC

10.3182/20080706-5-KR-1001.3671

condition that provides easily the controller gain. In fact, the idea consists to use the DMVT in order to transform the stabilization problem of a Lipschitz nonlinear system to the stability of a certain LPV system.

The rest of this paper is organized as follows. In section 2, we introduce the problem formulation. the observer synthesis method is given in section 3. An extension of the proposed method to solve the problem of observer-based control is presented in section 4. Finally, we end this note by some conclusions in section 5.

Notations : Throughout this paper, we use the following notations :

- $(\star)$ is used for the blocks induced by symmetry;
- A^T represents the transposed matrix of A;
- I_r represents the identity matrix of dimension r;
- for a square matrix S, $S > 0$ ($S < 0$) means that this matrix is positive definite (negative definite);
- $e_s(i) = \big(\underbrace{0,...,0,\overbrace{1}^{i\text{ th}},0,...,0}_{s\text{ components}}\big)^T \in \mathbb{R}^s, s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$;
- The set $Co(x,y)$ is the convex hull of the set $\{x,y\}$, i.e.

$$Co(x,y) = \Big\{\lambda x + (1-\lambda)y, \lambda \in [0,1]\Big\}.$$

2. PROBLEM FORMULATION

Consider the class of nonlinear discrete-time systems described by the following equations :

$$x_{k+1} = A_x x_k + A_u u_k + Bf(x_k) \quad (1a)$$
$$y_k = Cx_k \quad (1b)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector and $y \in \mathbb{R}^p$ is the output vector. A_x, A_u, B and C are constant matrices of adequate dimensions. The function f: $\mathbb{R}^n \mapsto \mathbb{R}^q$ is differentiable with respect to x and without loss of generality, we assume that $f(0) = 0$. We know that there exists always a matrix $H_i \in \mathbb{R}^{s_i \times n}$ for all $i \in \{1,...,q\}$ so that :

$$f(x) = \begin{bmatrix} f_1(H_1 x) \\ . \\ . \\ . \\ f_q(H_q x) \end{bmatrix}. \quad (2)$$

Assumption 1. Assume that the function f satisfies the following condition :

$$a_{ij} \leq \frac{\partial f_i}{\partial \zeta^i_j}(\zeta) \leq b_{ij}, \quad \forall\, \zeta^i \in \mathbb{R}^{s_i} \quad (3)$$

The condition (3) implies that the differentiable function f is γ-Lipschitz where

$$\gamma = \sqrt{\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \max\Big(|a_{ij}|^2, |b_{ij}|^2\Big)}.$$

The reformulation of the Lipschitz condition for differentiable functions as in (3) plays an important role on the feasibility of the synthesis conditions and avoids high gain as shown in Zemouche et al. (2007). In addition, it is shown in Alessandri (2004) that the use of the classical Lipschitz property leads to restrictive synthesis conditions. Some new results to cope with this restriction are detailed in Zemouche and Boutayeb (2006) for discrete-time systems but remain conservative.

Remark 2. Without loss of generality we assume that the nonlinear function f satisfies (3) with $a_{ij} = 0$ for all $i = 1,...,q$ and $j = 1,...,s$, where $s = \max_{1\leq i\leq q}(s_i)$. Indeed, if there exist subsets $S_1 \subset \{1,...,q\}$ and $S_2 \subset \{1,...,s\}$ such that $a_{ij} \neq 0$ for all $(i,j) \in S_1 \times S_2$, we can consider a new function

$$\tilde{f}(x_k) = f(x_k) - \Big(\sum_{(i,j)\in S_1\times S_2} a_{ij} H_{ij} H_i\Big) x_k$$

where

$$H_{ij} = e_q(i) e^T_{s_i}(j).$$

Therefore, $\tilde{f}$ satisfies (3) with $\tilde{a}_{ij} = 0$ and $\tilde{b}_{ij} = b_{ij} - a_{ij}$, and then we rewrite (1a) as

$$x_{k+1} = \tilde{A} x_k + A_u u_k + B\tilde{f}(x_k)$$

with

$$\tilde{A} = A_x + B\sum_{(i,j)\in S_1\times S_2} a_{ij} H_{ij} H_i.$$

Now, we consider the following state observer, which is a generalization of that of Fan and Arcak (2003) to discrete-time systems :

$$\hat{x}_{k+1} = A_x\hat{x}_k + A_u u_k + B\hat{f}(\hat{x}_k) + L\Big(y_k - C\hat{x}_k\Big) \quad (4a)$$
$$\hat{f}_i(\hat{x}_k) = f_i\Big(H_i\hat{x}_k + K_i\Big(y_k - C\hat{x}_k\Big)\Big) \quad (4b)$$

where $\hat{f}_i$ is the i^{th} component of $\hat{f}$.
Then, the aim is to find the gains $L \in \mathbb{R}^{n\times p}$ and $K_i \in \mathbb{R}^{s_i\times p}$ for $i = 1,...,q$, such that the estimation error

$$\varepsilon_k = x_k - \hat{x}_k \quad (5)$$

converges exponentially towards zero.
The dynamics of the estimation error is given by :

$$\varepsilon_{k+1} = \Big(A_x - LC\Big)\varepsilon_k + B\Big(f(x_k) - \hat{f}(\hat{x}_k)\Big). \quad (6)$$

Using the DMVT given firstly in Zemouche et al. (2007), there exist $z_i \in Co(v^i, w^i)$ for all $i = 1,...,q$ such that :

$$f(x_k) - \hat{f}(\hat{x}_k) = \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} h_{ij}(k) H_{ij}\chi_i \quad (7)$$

where

$$\chi_i = \Big(H_i - K_i C\Big)\varepsilon_k \quad (8)$$
$$h_{ij}(k) = \frac{\partial f_i}{\partial v^i_j}\Big(z_i(k)\Big) \quad (9)$$
$$v^i = H_i x_k, \quad w^i = H_i\hat{x}_k + K_i\Big(y_k - C\hat{x}_k\Big). \quad (10)$$

Then, the estimation error dynamics (6) becomes :

$$\varepsilon_{k+1} = \Big(A_x - LC\Big)\varepsilon_k + \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} h_{ij}(k) B H_{ij}\chi_i \quad (11)$$

3. OBSERVER SYNTHESIS METHOD

In this section, we introduce the main result of this paper, which consists in a new observer synthesis method for a class of nonlinear discrete-time systems. We give a new sufficient stability condition ensuring the exponential convergence of the estimation error towards zero. This condition is expressed in term of LMI easily tractable.

Theorem 3. The estimation error (5) converges asymptotically towards zero if there exist matrices $P = P^T > 0$, R, $K_i, i = 1, ..., q$ of adequate dimensions such that the following LMI is feasible :

$$\begin{bmatrix} -P & \mathbb{M}(K_1,...,K_q) & A_x^T P - C^T R \\ (\star) & -\Upsilon & \Sigma P \\ (\star) & (\star) & -P \end{bmatrix} < 0 \quad (12)$$

$$\mathbb{M}(K_1,...,K_q) = \Big[\mathbb{M}_1(K_1)\cdots\mathbb{M}_q(K_q)\Big], \quad (13)$$

$$\mathbb{M}_i(K_i) = \Big[\underbrace{(H_i - K_iC)^T ... (H_i - K_iC)^T}_{s_i \ \ times}\Big] \quad (14)$$

$$\Sigma = \Big(B\Big[H_{11}\cdots H_{1s_1}\ H_{21}\cdots H_{qs_q}\Big]\Big)^T, \quad (15)$$

$$\Upsilon = \mathrm{diag}\Big(\beta_{11}I_{s_1},...,\beta_{1s_1}I_{s_1},\beta_{21}I_{s_2},...,\beta_{qs_q}I_{s_q}\Big), \quad (16)$$

$$\beta_{ij} = \frac{2}{b_{ij}} \quad (17)$$

Then, the gain L is given by $L = P^{-1}R^T$ and the matrices K_i are free solutions of the LMI (12).

Proof. Choose the Lyapunov function candidate as follows :

$$V_k = \varepsilon_k^T P \varepsilon_k$$

Considering the difference $\Delta V = V_{k+1} - V_k$ along the system (1), we have

$$\begin{aligned} \Delta V = {} & \varepsilon^T\Big[\Big(A_x - LC\Big)^T P\Big(A_x - LC\Big) - P\Big]\varepsilon \\ & + 2\varepsilon^T\Big(A_x - LC\Big)^T P\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} BH_{ij}\zeta_{ij}\right) \\ & + \left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} BH_{ij}\zeta_{ij}\right)^T P\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} BH_{ij}\zeta_{ij}\right) \end{aligned} \quad (18)$$

where

$$\zeta_{ij} = h_{ij}(k)\chi_i. \quad (19)$$

By exploiting the inequalities (3) in assumption 1, we deduce that

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\zeta_{ij}^T\left(\frac{1}{h_{ij}} - \frac{1}{b_{ij}}\right)\zeta_{ij} \geq 0 \quad (20)$$

Using (8) and (19), the inequality (20) becomes

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\varepsilon^T\Big(H_i - K_iC\Big)^T\zeta_{ij} - \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\frac{1}{b_{ij}}\zeta_{ij}^T\zeta_{ij} \geq 0 \quad (21)$$

Therefore,

$$\begin{aligned} \Delta V \leq {} & \varepsilon^T\Big[\Big(A_x - LC\Big)^T P\Big(A_x - LC\Big) - P\Big]\varepsilon \\ & + 2\varepsilon^T\Big(A_x - LC\Big)^T P\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} BH_{ij}\zeta_{ij}\right) \\ & + \left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} BH_{ij}\zeta_{ij}\right)^T P\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} BH_{ij}\zeta_{ij}\right) \\ & + 2\varepsilon^T\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\Big(H_i - K_iC\Big)^T\zeta_{ij} - \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\frac{2}{b_{ij}}\zeta_{ij}^T\zeta_{ij} \end{aligned} \quad (22)$$

The inequality (22) can be rewritten in the following simple form :

$$\Delta V \leq \begin{bmatrix}\varepsilon \\ \zeta\end{bmatrix}^T \begin{bmatrix}\Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Sigma^T P\Sigma - \Upsilon\end{bmatrix}\begin{bmatrix}\varepsilon \\ \zeta\end{bmatrix} \quad (23)$$

where

$$\Gamma_{11} = \Big(A_x - LC\Big)^T P\Big(A_x - LC\Big) - P \quad (24)$$

$$\Gamma_{12} = \mathbb{M}^T(K_1,...,K_q) + \Big(A_x - LC\Big)^T P\Sigma \quad (25)$$

$$\zeta = [\zeta_{11}^T,...,\zeta_{1s_1}^T,\zeta_{21}^T,...,\zeta_{qs_q}^T]^T \quad (26)$$

and $\mathbb{M}(K_1,...,K_q)$, Σ, Υ are defined in (13), (15) and (16) respectively.

Using the Schur Lemma and the notation $R = L^T P$, the inequality (12) is equivalent to

$$\begin{bmatrix}\Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Sigma^T P\Sigma - \Upsilon\end{bmatrix} < 0. \quad (27)$$

Consequently, we deduce that under the condition (12), the estimation error converges asymptotically towards zero. This ends the proof of theorem 3.

4. OBSERVER-BASED CONTROL

In this section, we present an extension of the previous result to observer-based control. The aim is to investigate the stabilization problem of the class of nonlinear systems defined in (1)-(2)-(3). We shall design an observer of the form (4) so that the system (1) under the linear feedback

$$u_k = -F\hat{x}_k$$

is globally asymptotically stable. We have

$$\begin{cases} x_{k+1} = \Big(A_x - A_uF\Big)x_k + A_uF\varepsilon_k + Bf(x_k) \\ \varepsilon_{k+1} = \Big(A_x - LC\Big)\varepsilon_k + B\Big(f(x_k) - \hat{f}(\hat{x}_k)\Big) \end{cases} \quad (28)$$

We can write (28) as follows :

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ \varepsilon_{k+1}\end{bmatrix} = {} & \begin{bmatrix}\Big(A_x - A_uF\Big) & A_uF \\ 0 & \Big(A_x - LC\Big)\end{bmatrix}\begin{bmatrix}x_k \\ \varepsilon_k\end{bmatrix} \\ & + \begin{bmatrix}B & 0 \\ 0 & B\end{bmatrix}\begin{bmatrix} f(x_k) \\ \Big(f(x_k) - \hat{f}(\hat{x}_k)\Big)\end{bmatrix} \end{aligned} \quad (29)$$

From (7), we have

$$f(x_k) - \hat{f}(\hat{x}_k) = \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} h_{ij}(k)H_{ij}\Big(H_i - K_iC\Big)\varepsilon_k.$$

Also, from the DMVT, there exist $\bar{z}_i(k) \in Co(0, x_k)$ such that

$$f(x_k) = \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \bar{h}_{ij}(k) H_{ij} H_i x_k \tag{30}$$

where

$$\bar{h}_{ij}(k) = \frac{\partial f_i}{\partial v_j^i}\Big(\bar{z}_i(k)\Big)$$

Therefore, (29) becomes

$$\begin{cases} \begin{bmatrix} x_{k+1} \\ \varepsilon_{k+1} \end{bmatrix} = \begin{bmatrix} \Big(A_x - A_u F\Big) & A_u F \\ 0 & \Big(A_x - LC\Big) \end{bmatrix} \begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix} \\ \qquad + \displaystyle\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} H_{ij} & 0 \\ 0 & H_{ij} \end{bmatrix} \eta_{ij} \\ \eta_{ij} = \begin{bmatrix} \bar{h}_{ij} & 0 \\ 0 & h_{ij} \end{bmatrix} \begin{bmatrix} H_i & 0 \\ 0 & \Big(H_i - K_i C\Big) \end{bmatrix} \begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix}. \end{cases} \tag{31}$$

The objective consists to determine the matrices F, L and K_i, $i = 1, ..., q$ such that the system (31) is globally asymptotically stable. Sufficient synthesis conditions are given in the following theorem :

Theorem 4. If there exist matrices $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, a full rank matrix $\bar{P}_1$, matrices X, Y and $K_i, i = 1, ..., q$ of adequate dimensions such that :

- the following LMI is feasible

$$\Lambda\Big(P_1, P_2, X, Y, K_1, ..., K_q\Big) < 0 \tag{32}$$

  where Λ is defined in (34);
- the equality constraint holds

$$P_1 A_u = A_u \bar{P}_1 \tag{33}$$

then, the system (31) is globally asymptotically stable under the action of the observer-based linear static feedback

$$F = \bar{P}_1^{-1} X$$

with

$$L = P_2^{-1} Y^T$$

and matrices K_i, $i = 1, ..., q$ free solutions of (32).

Proof. To prove this theorem, we use the Lyapunov function candidate

$$V_k = \begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix}$$

Now, we calculate the difference $\Delta V = V_{k+1} - V_k$.

$$\begin{aligned} \Delta V = \begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix}^T \Bigg(& \begin{bmatrix} \Big(A_x - A_u F\Big) & A_u F \\ 0 & \Big(A_x - LC\Big) \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \times \\ & \begin{bmatrix} \Big(A_x - A_u F\Big) & A_u F \\ 0 & \Big(A_x - LC\Big) \end{bmatrix} - \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \Bigg) \begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix} \\ & + 2 \begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix}^T \begin{bmatrix} \Big(A_x - A_u F\Big) & A_u F \\ 0 & \Big(A_x - LC\Big) \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \times \\ & \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} H_{ij} & 0 \\ 0 & H_{ij} \end{bmatrix} \eta_{ij} \right) \\ & + \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} H_{ij} & 0 \\ 0 & H_{ij} \end{bmatrix} \eta_{ij} \right)^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \times \\ & \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} H_{ij} & 0 \\ 0 & H_{ij} \end{bmatrix} \eta_{ij} \right). \end{aligned} \tag{39}$$

Using inequalities (3), we deduce that

$$\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \eta_{ij}^T \begin{bmatrix} \dfrac{1}{\bar{\bar{h}}_{ij}} - \dfrac{1}{b_{ij}} & 0 \\ 0 & \dfrac{1}{h_{ij}} - \dfrac{1}{b_{ij}} \end{bmatrix} \eta_{ij} \geq 0$$

which is equivalent to

$$\begin{aligned} & \begin{bmatrix} x_k \\ \varepsilon_k \end{bmatrix}^T \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \begin{bmatrix} H_i & 0 \\ 0 & \Big(H_i - K_i C\Big) \end{bmatrix}^T \eta_{ij} \\ & - \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \eta_{ij}^T \begin{bmatrix} \dfrac{1}{b_{ij}} & 0 \\ 0 & \dfrac{1}{b_{ij}} \end{bmatrix} \eta_{ij} \geq 0. \end{aligned} \tag{40}$$

Consequently, using the notations (35), (36), (37) and (38), we have

$$\Delta V \leq \begin{bmatrix} x_k \\ \varepsilon_k \\ \eta_k \end{bmatrix}^T \Pi \begin{bmatrix} x_k \\ \varepsilon_k \\ \eta_k \end{bmatrix} \tag{41}$$

where

$$\eta_k = [\eta_{11}^T, ..., \eta_{1s_1}^T, \eta_{21}^T, ..., \eta_{qs_q}^T]^T$$

and Π is given in (42). Using the equality constraint (33) and the notations $Y = L^T P_2, X = \bar{P}_1 F$, the inequality $\Pi < 0$ is equivalent to (32) by Schur lemma. This ends the proof of theorem 4.

Note that the design of the gains L and F may be difficult or impossible for some nonlinear systems because of the required equality constraint (33). This is, for example, the case of nonlinear systems with single input, where the equality constraint $P_1 A_u = A_u \bar{P}_1$ is reduced to $P_1 A_u = \alpha A_u$ with $\alpha \in \mathbb{R}$.

In order to eliminate the equality constraint (33) from theorem 4, we must choose *a priori* the controller gain F and then we determine the observer gains L and $K_i, i = 1, ..., q$, which ensure $\Delta V < 0$, by proceeding as follows :

Let $A_F = A_x - A_u F$ and $\bar{A} = A_u F$. Therefore, starting from inequality (41), we have $\Delta V < 0$ if $\Pi < 0$ or equivalently

$$\Lambda\Big(P_1, P_2, X, Y, K_1, ..., K_q\Big) = \begin{bmatrix} -\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} & \Delta(K_1, ..., K_q) & \begin{bmatrix} A_x^T P_1 - X^T A_u^T & 0 \\ X^T A_u^T & A_x^T P_2 - C^T Y \end{bmatrix} \\ (\star) & -\Phi & \Omega \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ (\star) & (\star) & -\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \end{bmatrix} \quad (34)$$

$$\Delta(K_1, ..., K_q) = \Big[\Delta_1(K_1) \cdots \Delta_q(K_q)\Big], \quad (35)$$

$$\Delta_i(K_i) = \left[\underbrace{\begin{bmatrix} H_i & 0 \\ 0 & (H_i - K_iC)^T \end{bmatrix} \cdots \begin{bmatrix} H_i^T & 0 \\ 0 & (H_i - K_iC)^T \end{bmatrix}}_{s_i \ times} \right] \quad (36)$$

$$\Omega = \left(\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \left[\begin{bmatrix} H_{11} & 0 \\ 0 & H_{11} \end{bmatrix} \cdots \begin{bmatrix} H_{1s_1} & 0 \\ 0 & H_{1s_1} \end{bmatrix} \ \begin{bmatrix} H_{21} & 0 \\ 0 & H_{21} \end{bmatrix} \cdots \begin{bmatrix} H_{qs_q} & 0 \\ 0 & H_{qs_q} \end{bmatrix} \right] \right)^T, \quad (37)$$

$$\Phi = \mathrm{diag}\left(\begin{bmatrix} \beta_{11} I_{s_1} & 0 \\ 0 & \beta_{11} I_{s_1} \end{bmatrix}, ..., \begin{bmatrix} \beta_{1s_1} I_{s_1} & 0 \\ 0 & \beta_{1s_1} I_{s_1} \end{bmatrix}, \begin{bmatrix} \beta_{21} I_{s_2} & 0 \\ 0 & \beta_{21} I_{s_2} \end{bmatrix}, ..., \begin{bmatrix} \beta_{qs_q} I_{s_q} & 0 \\ 0 & \beta_{qs_q} I_{s_q} \end{bmatrix} \right). \quad (38)$$

$$\Pi = \begin{bmatrix} \Psi^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \Psi - \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} & \Delta(K_1, ..., K_q) + \Psi^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \Omega \\ (\star) & \Omega \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \Omega^T - \Phi \end{bmatrix} \quad (42)$$

$$\Psi = \begin{bmatrix} \Big(A_x - A_u F\Big) & A_u F \\ 0 & \Big(A_x - LC\Big) \end{bmatrix} \quad (43)$$

$$\begin{bmatrix} -\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} & \Delta(K_1, ..., K_q) & \begin{bmatrix} A_F^T P_1 & 0 \\ \bar{A}^T P_1 & A_x^T P_2 - C^T Y \end{bmatrix} \\ (\star) & -\Phi & \Omega \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ (\star) & (\star) & -\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \end{bmatrix} < 0 \quad (44)$$

with

$$Y = L^T P_2. \quad (45)$$

Then, the gain $L = P_2^{-1} Y^T$ and the free solutions $K_{i,i=1,...,q}$ ensure the global asymptotic stability of the system (31).

On the other hand, the arbitrary choice of the matrix F is not desirable. To be rigorous or to obtain a systematic method to synthesize the controller gain, we propose to find F that stabilizes the system

$$x_{k+1} = \Big(A_x - A_u F\Big)x_k + Bf(x_k). \quad (46)$$

For this, we proceed as in Zemouche et al. (2007). From (31), the system (46) has the following Linear Parameter Varying (LPV) form :

$$x_{k+1} = \Big(\Theta(\rho) - A_u F\Big)x_k \quad (47)$$

where

$$\Theta(\rho) = A_x + B \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \bar{h}_{ij}(k) H_{ij} H_i \quad (48)$$

and

$$\rho = \Big(\bar{h}_{11}, ..., \bar{h}_{qs_q}\Big). \quad (49)$$

From assumption 1 (inequality (3)), the parameter vector ρ belongs to a bounded convex domain $\mathcal{H}_q$ of which the set of vertices is defined as follows :

$$\mathcal{V}_{\mathcal{H}_q} = \Big\{\alpha = (\alpha_{11}, ..., \alpha_{1s_1}, ..., \alpha_{qs_q}) \ : \ \alpha_{ij} \in \{a_{ij}, b_{ij}\}\Big\}. \quad (50)$$

By considering as Lyapunov function $W_k = x_k^T S^{-1} x_k$ with $S = S^T > 0$, we have $W_{k+1} - W_k < 0$ if

$$\Big(\Theta(\rho) - A_u F\Big)^T S^{-1} \Big(\Theta(\rho) - A_u F\Big) - S^{-1} < 0.$$

Pre- and post multiplying the last inequality by S, we obtain $W_{k+1} - W_k < 0$ if

$$S\Big(\Theta(\rho) - A_u F\Big)^T S^{-1} \Big(\Theta(\rho) - A_u F\Big)S - S < 0$$

or equivalently

$$\begin{bmatrix} -S & S\Theta^T(\rho) - SF^T A_u^T \\ (\star) & -S \end{bmatrix} < 0. \quad (51)$$

Using the convexity principle Boyd and Vandenberghe (2001) and the notation $X = SF^T$, the inequality (51) holds if the following linear matrix inequalities (LMIs) are feasible :

$$\begin{bmatrix} -S & S\Theta^T(\alpha) - XA_u^T \\ (\star) & -S \end{bmatrix} < 0, \quad \forall\, \alpha \in \mathcal{V}_{\mathcal{H}_q}. \qquad (52)$$

Now, we can state the following theorem :

Theorem 5. If there exist a symmetric positive definite matrix S and a matrix X of adequate dimensions so that the LMIs (52) are feasible, then the nonlinear system (46) is globally asymptotically stable for $F = X^T S^{-1}$.

To summarize, a procedure that allows to solve the observer-based control problem or to design the observer-controller gains, without any equality constraint, is given in the following algorithm :

Algorithm : *The observer-based control problem* (28) *is solved in two steps. The advantage of this algorithm is that the equality constraint of theorem 4, which induce a conservatism for the approach, is omitted.*

(i) *Solve the LMIs* (52) *with respect to* S, X *and set the controller gain*

$$F = X^T S^{-1}$$

if a solution exists;

(ii) *For the same value of* F *given in* (i), *solve the LMI* (44) *with respect to* P_1, P_2, Y *and* $K_i, i = 1, ..., q$. *If a solution exists, then the observer gains are given by*

$$L = P_2^{-1} Y^T$$

and $K_i, i = 1, ..., q$ *are free solutions of* (44).

Remark 6. Note that in section 3, we cannot proceed as in Zemouche et al. (2007). Indeed, if it is the case, there will be couplings between the Lyapunov matrix P and the gains K_i, which leads to nonlinear synthesis condition.

5. CONCLUSION

In this paper, we proposed a new observer design method for a class of Lipschitz discrete-time systems. Thanks to the use of the DMVT, a reformulation of the Lipschitz property is introduced. The objective of this reformulation is to reduce the conservatism of the techniques based on the classical Lipschitz property. The stability analysis is performed using a quadratic Lyapunov function. This latter provided a less restrictive synthesis condition expressed in term of LMI. Two methods are proposed to extend the proposed approach to solve the problem of observer-based control. In the first one, the observer-controller gains are computed by solving a LMI condition under an equality constraint. Since this equality constraint is difficult or impossible to be solved for certain nonlinear systems, then an alternative method, that avoids solving the equality constraint, is presented. This latter is summarized in an algorithm that solve the problem in two steps. Because of the lack of space here, no numerical example is given.

REFERENCES

A. Alessandri. Design of observers for Lipschitz nonlinear systems using LMI. *NOLCOS, IFAC Symposium on Nonlinear Control Systems, Stuttgart, Germany*, 2004.

M Arcak and P. Kokotovic. Observer-based control of systems with slope-restricted nonlinearities. *IEEE Trans. on Automatic Control*, 46(7):1146–1150, 2001.

M. Boutayeb and D. Aubry. A strong tracking extended Kalman observer for nonlinear discrete-time systems. *IEEE Trans. on Automatic Control*, 44(8), 1999.

S. Boyd and L. Vandenberghe. Convex optimization with engineering applications. *Lecture Notes, Stanford University, Stanford*, 2001.

X. Fan and M Arcak. Observer design for systems with multivariable monotone nonlinearities. *Systems and Control Letters*, 50:319–330, 2003.

J. P. Gauthier and I. A. K. Kupka. Observability and observers for nonlinear systems. *SIAM J. Control and Optimization*, 32(4):975–994, 1994.

J. P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems. Applications to bioreactors. *IEEE Trans. on Automatic Control*, 37(6):875–880, 1992.

M. Hou and A.C. Pugh. Observer with linear error dynamics for nonlinear multi output systems. *" Systems and Control Letters*, 37(1):1–9, 1999.

S. Ibrir. Nonlinear circle-criterion observers in discrete-time. In *2007 American Control Conference ACC'07*, New York City, USA, July 2007a.

S. Ibrir. Circle-criterion approach to discrete-time nonlinear observer design. *Automatica*, 43, 2007b.

S. Ibrir, Wen Fang Xie, and Chun-Yi Su. Observer-based control of discrete-time Lipschitzian non-linear systems : application to one-link flexible joint robot. *International Journal of Control*, 78(6):385–395, 2005.

H. Keller. Nonlinear observer design by transformation into a generalized observer canonical form. *Int. J. of Control*, 46(6):1915–1930, 1987.

A. J. Krener and W. Respondek. Nonlinear observer with linearizable error dynamics. *SIAM J. Control and Optimization*, 23(2):197–216, 1985.

S. Raghavan and J. K. Hedrick. Observer design for a class of nonlinear systems. *Int. J. of Control*, 59(2):515–528, 1994.

A Rajamani. Observers for Lipschitz nonlinear systems. *IEEE Trans. Automatic Control*, 43(3), 1998.

K. Reif, F. Sonnemann, and R. Unbehauen. Nonlinear state observation using H_∞-filtring riccati design. *IEEE Trans. Automatic Control*, 44(1):203–208, 1999.

V. Sundarapandian. New results on general observers for discrete-time nonlinear systems. *Applied Mathematics Letters*, 17, 2004.

V. Sundarapandian. Reduced order observer design for discrete-time nonlinear systems. *Applied Mathematics Letters*, 19, 2006.

E. E. Yaz, C. S. Jeong, A. Bahakeem, and Y. I. Yaz. Discrete-time nonlinear observer design with general criteria. *Journal of The Franklin Institute*, 344, 2007.

A. Zemouche and M. Boutayeb. Observer design for Lipschitz nonlinear systems. The discrete-time case. *IEEE Trans. Circuits Syst. II*, 53(8):777–781, 2006.

A. Zemouche, M. Boutayeb, and G. I. Bara. Observers design for a class of nonlinear systems. In *45th IEEE Conference on Decision and Control CDC 2006*, San Diego, California, USA, December 2006.

A. Zemouche, M. Boutayeb, and G. I. Bara. Observers for a class of Lipschitz systems with extension to $\mathcal{H}_\infty$ performance analysis. *Systems & Control Letters*, 57(1): 18–27, 2007.