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Optimal estimation for linear singular systems using moving horizon estimation

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Abstract: In this paper, the moving horizon recursive state estimator for linear singular systems is derived from the minimum variance estimation problem. The proposed estimate of the state using the measured outputs samples on the recent finite time horizon is unbiased and independent of any *a priori* information of the state on the horizon. The convergence and stability of the filter are evoked. A numerical example is presented to prove the performance of the proposed filter.

1. INTRODUCTION

The study of estimation problem of singular systems (Descriptor systems or implicit systems) is presented in this work. This study is motivated by the fact that systems in descriptor formulation frequently arise naturally in economical systems, image modeling, and robotics, etc. Ishihara et al. [2005]. The problem of state estimation for singular systems has been treated by several authors Dai [1989], Darouach et al. [1992], Darouach et al. [1993], Darouach et al. [1995], Bassong-Onana and Darouach [1992], Nikoukhah et al. [1992], Nikoukhah et al. [1999], Zhang et al. [1998], Zhang et al. [1999], Ishihara et al. [2004] and Ishihara et al. [2005]. Most of these researchers present the generalized Kalman filter as a solution for recursive state estimation problem. The Kalman filtering problem through a deterministic approach is studied in Darouach et al. [1992], Darouach et al. [1993], Ishihara et al. [2004] and Ishihara et al. [2005].

Investing the success of receding horizon control in the estimation of dynamic states and parameters, for linear and nonlinear systems, recent attention has been concentrated on the moving horizon estimation (MHE). In the framework of MHE, only a fixed amount of measurement data is used to solve an optimization problem, so that the oldest measurement sample is discarded as a new sample becomes available. Among the reasons of successes of MHE approach are the possibility of incorporating the equality and inequality constraints and also the reduced size of data used for estimation.

In this paper, we derive the unbiased minimum-variance estimation. We consider singular systems in the most general case with no assumption regarding regularity and causality. Using the moving horizon estimation strategy (MHE), an optimal recursive filter is developed. Necessary and sufficient conditions for convergence and stability of the optimal filter are given.

The organisation of this paper is as follows. The problem is stated in Section 2. The optimal solution of the estimation problem for singular systems using moving horizon approach is derived in section 3. In section 4, the convergence and stability properties of the solution are presented and finally a numerical example is given in section 5 to illustrate this approach.

2. PROBLEM STATEMENT AND PRELIMINARIES

Let us consider the following discrete-time linear stochastic singular system described by :

$$Ex_{k+1} = Ax_k + Bu_k + w_k, \qquad k = 0, 1, 2, ... \quad (1a)$$

$$y_k = Cx_k + v_k \quad (1b)$$

where $x_k \in \Re^n$ is the state vector, $u_k \in \Re^l$ is the input, $y_k \in \Re^m$ is the measured output, $w_k \in \Re^n$ is a state noise vector, and $v_k \in \Re^m$ is a measurement noise vector. E, A, B and C are matrices with appropriate dimensions. Let us consider the initial condition $x_0 \in \Re^n$ be random variable having mean $\bar{x}_0$ and covariance P_0; the state noise w_j and the measurement noise v_j are assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matrices

$$\boldsymbol{E}\left\{\begin{bmatrix} w_i \\ v_j \end{bmatrix}\begin{bmatrix} w_i \\ v_j \end{bmatrix}^T\right\} = \begin{bmatrix} Q_w & 0 \\ 0 & V_v \end{bmatrix}\delta_{ij} > 0$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. We assume that x_0 is independent of v_j and w_j for all k.

Based on the measure z over time $(0, k)$ and the initial state x_0, the estimate of x_k, denoted $\hat{x}_{k|k}$ (the notation $*_{i|j}$ means that this is a discrete time variable at time i given information up to time j) is computed from the minimum variance estimation problem. In this case, we seek to find an optimal estimate which minimise the mean square error defined by

$$J = E\{\left(x_k - \hat{x}_{k|k}\right)\left(x_k - \hat{x}_{k|k}\right)^T\}. \quad (2)$$

The objective of this work is to apply the MHE approach to singular linear systems. The MHE attempt, to preserve old information by using a "information" window that slides over the measurements. The state is estimated,with the MHE approach, from the horizon of the most recent $N+1$ output measurements that moves forward at each sampling time when a new measurement is available. The old information is incorporated using a startup estimate $\bar{x}_{k-N}$ that is calculated from old filtered states and a specific weight $P_{k-N|k-N}$.

Throughout this paper the following assumption is taken to hold

Assumption 1. We suppose that $\begin{bmatrix} E \\ C \end{bmatrix}$ has full column rank (see Darouach et al. [1993, 1995], Nikoukhah et al. [1992], Ishihara et al. [2004]).

3. OPTIMAL ESTIMATION FOR SINGULAR SYSTEMS

In the Kalman filter recursion, the optimal state estimation at time k is determined recursively from the optimal state estimate and output measurements at time $k-1$. In the receding horizon formulation, the optimal state estimate at time k is determined recursively from the optimal state at time $k-N$ and the most recent $N+1$ output measurements using the minimum variance estimation formulation.

If the assumption 1 is verified, a non singular matrix $\bar{J}$ exist, such as

$$\underbrace{\begin{bmatrix} \bar{J}_1 & \bar{J}_2 \\ \bar{J}_3 & \bar{J}_4 \end{bmatrix}}_{\bar{J}} \begin{bmatrix} E \\ C \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \tag{3}$$

Firstly, let us present the current state x_k in the horizon $[k-N, K]$ using the initial state x_{k-N}.

Multiplying (1a) by $\bar{J}_1$, we find

$$\bar{J}_1 E x_k = \bar{J}_1 (A x_{k-1} + B u_{k-1} + w_{k-1}) \tag{4}$$

which is equivalent to

$$\begin{aligned} x_k &= \bar{J}_2 (y_k - v_k) + \bar{J}_1 (A x_{k-1} + B u_{k-1} + w_{k-1}) \\ &= F x_{k-1} + \bar{B} r_k + T_1 \eta_k \end{aligned} \tag{5}$$

with

$$\bar{B} = \left[\bar{J}_1 B \;\; \bar{J}_2 \right], T_1 = \left[\bar{J}_1 \;\; \bar{J}_2 \right], F = \bar{J}_1 A \tag{6a}$$

$$r_k = \begin{pmatrix} u_{k-1} \\ y_k \end{pmatrix}, \quad \eta_k = \begin{pmatrix} w_{k-1} \\ -v_k \end{pmatrix}. \tag{6b}$$

using the same idea for $k-1$ until $k-N+1$, the state x_k can be calculated recursively as

$$\begin{aligned} x_k &= F x_{k-1} + \bar{B} r_k + T_1 \eta_k \\ &= F^2 x_{k-2} + F \bar{B} r_{k-1} + \bar{B} r_k + F T_1 \eta_{k-1} + T_1 \eta_k \\ &= \ldots \\ &= F^N x_{k-N} + \left[F^{N-1} \bar{B} \;\; F^{N-2} \bar{B} \; \ldots \; \bar{B} \right] \bar{r}_k \\ &+ \left[F^{N-1} \;\; F^{N-2} \; \ldots \; I \right] \overline{W}_k \end{aligned} \tag{7}$$

with $\overline{w}_k = T_1 \eta_k$ and

$$\bar{r}_k = \left[r_{k-N+1}^T \; r_{k-N+2}^T \; \ldots \; r_k^T \right]^T \tag{8a}$$

$$\overline{W}_k = \left[\overline{w}_{k-N+1}^T \; \overline{w}_{k-N+2}^T \; \ldots \; \overline{w}_k^T \right]^T. \tag{8b}$$

Secondly, the measurement can be represented in a batch form in the interval $[k-N, k]$.

From (1b) and (3), we have

$$\begin{aligned} -\bar{J}_4 y_{k-N+1} = \bar{J}_3 A x_{k-N} + \bar{J}_3 B u_{k-N} + \bar{J}_3 w_{k-N} \\ - \bar{J}_4 v_{k-N+1} \end{aligned} \tag{9}$$

equation (9) can be written as

$$z_{k-N+1} = \underline{C} x_{k-N} + \overline{v}_{k-N+1} \tag{10}$$

with

$$\begin{aligned} z_{k-N+1} = -D r_{k-N+1}, \quad D = \left[\bar{J}_3 B \;\; \bar{J}_4 \right], \quad \underline{C} = \bar{J}_3 A, \\ T_2 = \left[\bar{J}_3 \;\; \bar{J}_4 \right], \quad \overline{v}_{k-N+1} = T_2 \eta_{k-N+1} \end{aligned} \tag{11}$$

on the horizon $[k-N, k]$, using (5), the measurements can be expressed as follows

$$\begin{aligned} z_{k-N+1} &= \underline{C} x_{k-N} + \overline{v}_{k-N+1} \\ z_{k-N+2} &= \underline{C} F x_{k-N} + \underline{C} \bar{B} r_{k-N+1} + \underline{C} \overline{w}_{k-N+1} \\ &\quad + \overline{v}_{k-N+2} \\ z_{k-N+3} &= \underline{C} F^2 x_{k-N} + \underline{C} F \bar{B} r_{k-N+1} + \underline{C} \bar{B} r_{k-N+2} \\ &\quad + \underline{C} F \overline{w}_{k-N+1} + \underline{C} \overline{w}_{k-N+2} + \overline{v}_{k-N+3} \\ &\vdots \\ z_k &= \underline{C} F^{N-1} x_{k-N} + \underline{C} F^{N-2} \bar{B} r_{k-N+1} \\ &\quad + \ldots + \underline{C} \bar{B} r_{k-1} + \underline{C} F^{N-2} \overline{w}_{k-N+1} \\ &\quad + \ldots + \underline{C} \overline{w}_{k-1} + \overline{v}_k \end{aligned}$$

The finite number of measurements is given by

$$Z_k = \widetilde{C}_N x_{k-N} + \widetilde{B}_N \bar{r}_k + \widetilde{G}_N \overline{W}_k + \overline{V}_k \tag{12}$$

with

$$\begin{aligned} Z_k &= \left[z_{k-N+1}^T \; z_{k-N+2}^T \; \cdots \; z_k^T \right]^T \\ \overline{V}_k &= \left[\overline{v}_{k-N+1}^T \; \overline{v}_{k-N+2}^T \; \cdots \; \overline{v}_k^T \right]^T \\ \widetilde{C}_N &= \begin{bmatrix} \underline{C} \\ \underline{C} F \\ \vdots \\ \underline{C} F^{N-1} \end{bmatrix} \\ \widetilde{B}_N &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \underline{C} \bar{B} & 0 & \cdots & 0 & 0 \\ \underline{C} F \bar{B} & \underline{C} \bar{B} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \underline{C} F^{N-2} \bar{B} & \underline{C} F^{N-3} \bar{B} & \cdots & \underline{C} \bar{B} & 0 \end{bmatrix} \\ \widetilde{G}_N &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \underline{C} & 0 & \cdots & 0 & 0 \\ \underline{C} F & \underline{C} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \underline{C} F^{N-2} & \underline{C} F^{N-3} & \cdots & \underline{C} & 0 \end{bmatrix}. \end{aligned}$$

The vectors $\widetilde{G}_N \overline{W}_k$ and $\overline{V}_k$ are strongly correlated but still white. To remove the correlation, equation (10) can be transformed as follows

$$\begin{aligned} z_{k-N+2} &= \underline{C} F x_{k-N} + \underline{C} \bar{B} r_{k-N+1} + \underline{C} \overline{w}_{k-N+1} \\ &\quad + \overline{v}_{k-N+2} + \underline{C} S R^{-1} (z_{k+1} - z_{k+1}) \\ &= \underline{C} F^s x_{k-N} + \underline{C} \bar{B}^s r_{k-N+1} + \underline{C} \overline{w}_{k-N+1}^s \\ &\quad + \overline{v}_{k-N+2} \end{aligned}$$

and

$$x_k = F^{sN} x_{k-N} + L_{\bar{r},N} \bar{r}_k + L_{\overline{W}^s,N} \overline{W}_k^s \tag{13}$$

with

$$S = T_1 \begin{bmatrix} Q_w & 0 \\ 0 & V_v \end{bmatrix} T_2^T, R = T_2 \begin{bmatrix} Q_w & 0 \\ 0 & V_v \end{bmatrix} T_2^T$$
$$Q = T_1 \begin{bmatrix} Q_w & 0 \\ 0 & V_v \end{bmatrix} T_1^T, \quad Fs = F - SR^{-1}\underline{C}$$
$$\bar{B}^s = \bar{B} - SR^{-1}D, \quad T_1^s = T_1 - SR^{-1}T_2$$
$$Q^s = Q - SR^{-1}S^T, R^s = R, \overline{w}_{k+1}^s = T_1^s \eta_{k+1}.$$

here Q and R is the covariance of $\overline{w}_{k+1}$ and $\overline{v}_{k+1}$ respectively. It can be checked that Q and R are positive definite.

Equation (12) can be written as

$$Z_k = \widetilde{C}_N^s x_{k-N} + \widetilde{B}_N^s \bar{r}_k + \widetilde{G}_N^s \overline{W}_k^s + \overline{V}_k \tag{14}$$

where the matrices $\widetilde{C}_N^s$, $\widetilde{B}_N^s$ and $\widetilde{G}_N^s$ are the matrices $\widetilde{C}_N$, $\widetilde{B}_N$ and $\widetilde{G}_N$ respectively while replacing the matrices F and $\bar{B}$ by F^s and $\bar{B}^s$ respectively.

Equation (7) is also transformed as follows

$$x_k = F^{sN} x_{k-N} + L_{\bar{r},N} \bar{r}_k + L_{\overline{W}^s,N} \overline{W}_k^s \tag{15}$$

with

$$L_{\bar{r},N} = \left[F^{sN-1}\bar{B}^s \ F^{sN-2}\bar{B}^s \ \ldots \ \bar{B}^s \right]$$
$$L_{\overline{W}^s,N} = \left[F^{sN-1} \ F^{sN-2} \ \ldots \ I \right]$$

The MHE estimator is presented as

$$\hat{x}_{k|k} = M\bar{x}_{k-N} + HZ_k + L\bar{r}_k \tag{16a}$$

with

$$H = [H_N \ H_{N-1} \ \ldots \ H1] \tag{17a}$$
$$L = [L_N \ L_{N-1} \ \ldots \ L1] \tag{17b}$$
$$M = M_1 M_2 \ldots M_N \tag{17c}$$

the startup value $\bar{x}_{k-N}$ is determined from the filtered optimal estimate computed $N+1$ time intervals in the past and is named $\hat{x}_{k-N|k-N}$ which is the estimate of initial state x_{k-N}.

The estimation error can be written as

$$\begin{aligned} e_k &= x_k - \hat{x}_{k|k} \\ &= F^{sN} x_{k-N} + L_{\bar{r},N}\bar{r}_k + L_{\overline{W}^s,N}\overline{W}_k^s - M\hat{x}_{k-N|k-N} \\ &\quad -HZ_k - L\bar{r}_k \end{aligned}$$

by replacing Z_k by their expression, we botain

$$\begin{aligned} e_k &= F^{sN} x_{k-N} + L_{\bar{r},N}\bar{r}_k + L_{\overline{W}^s,N}\overline{W}_k^s - M\hat{x}_{k-N|k-N} \\ &\quad -L\bar{r}_k - H\left(\widetilde{C}_N^s x_{k-N} + \widetilde{B}_N^s \bar{r}_k + \widetilde{G}_N^s \overline{W}_k^s + \overline{V}_k\right) \\ &= \left(F^{sN} x_{k-N} - H\widetilde{C}_N^s\right) x_{k-N} - M\hat{x}_{k-N|k-N} + H\overline{V}_k \\ &\quad + \left(L_{\bar{r},N} - H\widetilde{B}_N^s - L\right)\bar{r}_k + \left(L_{\overline{W}^s,N} - H\widetilde{G}_N^s\right)\overline{W}_k^s \end{aligned}$$

we have

$$\hat{x}_{k-N|k-N} = x_{k-N} - e_{k-N} \tag{18a}$$

then

$$\begin{aligned} e_k &= \left(F^{sN} - H\widetilde{C}_N^s\right) x_{k-N} - M\left(x_{k-N} - e_{k-N}\right) + H\overline{V}_k \\ &\quad + \left(L_{\bar{r},N} - H\widetilde{B}_N^s - L\right)\bar{r}_k + \left(L_{\overline{W}^s,N} - H\widetilde{G}_N^s\right)\overline{W}_k^s \\ &= \left(F^{sN} - H\widetilde{C}_N^s - M\right) x_{k-N} + Me_{k-N} + H\overline{V}_k \\ &\quad + \left(L_{\bar{r},N} - H\widetilde{B}_N^s - L\right)\bar{r}_k + \left(L_{\overline{W}^s,N} - H\widetilde{G}_N^s\right)\overline{W}_k^s \end{aligned}$$

to satisfy the unbiased condition, i.e,, $E(e_k) = 0$, the following constraint should be satisfied

$$M = F^{sN} - H\widetilde{C}_N^s \tag{19a}$$
$$L = L_{\bar{r},N} - H\widetilde{B}_N^s. \tag{19b}$$

substituting M and L in e_k, we have

$$\begin{aligned} e_k &= \left(F^{sN} - H\widetilde{C}_N^s\right) e_{k-N} + \left(L_{\overline{W}^s,N} - H\widetilde{G}_N^s\right)\overline{W}_k^s \\ &\quad + H\overline{V}_k \end{aligned} \tag{20a}$$

Then the problem is reduced to determine matrix H such that the trace of the error covariance is minimum, which is given by

$$P_{k|k} = E\{e_k e_k^T\} \tag{21a}$$
$$\begin{aligned} &= \left(F^{sN} - H\widetilde{C}_N^s\right) P_{k-N|k-N} \left(F^{sN} - H\widetilde{C}_N^s\right)^T \\ &\quad + \left(L_{\overline{W}^s,N} - H\widetilde{G}_N^s\right) Q_k^s \left(L_{\overline{W}^s,N} - H\widetilde{G}_N^s\right)^T \\ &\quad + HR_k^s H^T \end{aligned} \tag{21b}$$

the minimum of trace $P_{k|k}$ is given by

$$\begin{aligned} H &= \left(F^{sN} P_{k-N|k-N} (\widetilde{C}_N^s)^T + L_{\overline{W}^s,N} Q_k^s (\widetilde{G}_N^s)^T\right) \\ &\quad \left(\widetilde{C}_N^s P_{k-N|k-N} (\widetilde{C}_N^s)^T + \widetilde{G}_N^s Q_k^s (\widetilde{G}_N^s)^T + R_k^s\right)^{-1}, \end{aligned} \tag{22}$$

the condition for the existence of H is that

$$\left(\widetilde{C}_N^s P_{k-N|k-N} (\widetilde{C}_N^s)^T + \widetilde{G}_N^s Q_k^s (\widetilde{G}_N^s)^T + R_k^s\right) > 0$$

substituting (22) in (21b), we obtain

$$P_{k|k} = \left[F^{sN} \ L_{\overline{w}^s,N} \right] \begin{bmatrix} W_{1,1} & W_{1,2} \\ W_{1,2}^T & W_{2,2} \end{bmatrix}^{-1} \begin{bmatrix} \left(F^{sN}\right)^T \\ L_{\overline{w}^s,N}^T \end{bmatrix}$$

with

$$\begin{aligned} W_{1,1} &= (\widetilde{C}_N^s)^T (R_N^s)^{-1} \widetilde{C}_N^s \\ W_{1,2} &= (\widetilde{C}_N^s)^T (R_N^s)^{-1} \widetilde{G}_N^s \\ W_{2,2} &= (\widetilde{G}_N^s)^T (R_N^s)^{-1} \widetilde{G}_N^s + (Q_N^s)^{-1} \\ R_N^s &= \mathrm{diag}(R^s, \ldots, R^s) \\ Q_N^s &= \mathrm{diag}(Q^s, \ldots, Q^s) \end{aligned}$$

Note that the matrix H can be written as follows

$$H = \left[F^{sN} \ L_{\overline{w}^s,N} \right] \begin{bmatrix} W_{1,1} + P_{k-N|k-N}^{-1} & W_{1,2} \\ W_{1,2}^T & W_{2,2} \end{bmatrix}^{-1} \begin{bmatrix} (\widetilde{C}_N^s)^T \\ (\widetilde{G}_N^s)^T \end{bmatrix} (R_N^s)^{-1} \tag{23}$$

These results are summarized in the following theorem.

Theorem 1. Let us suppose that the assumption 1 is verified, then the MHE estimate of the state vector at time k and the matrix of error covariance are computed as

$$\begin{aligned} \hat{x}_k &= F^{sN} \hat{x}_{k-N|k-N} + \left[F^{sN} \ L_{\overline{w}^s,N} \right] \\ &\quad \begin{bmatrix} W_{1,1} + P_{k-N|k-N}^{-1} & W_{1,2} \\ W_{1,2}^T & W_{2,2} \end{bmatrix}^{-1} \begin{bmatrix} (\widetilde{C}_N^s)^T \\ (\widetilde{G}_N^s)^T \end{bmatrix} (R_N^s)^{-1} \\ &\quad \times \left(Z_k - \widetilde{C}_N^s \hat{x}_{k-N|k-N} - \widetilde{B}_N^s \bar{r}_k\right) + L_{\bar{r},N}\bar{r}_k. \end{aligned}$$

$$P_{k|k} = \left[F^{sN} \ L_{\overline{w}^s,N} \right] \begin{bmatrix} W_{1,1} + P_{k-N|k-N}^{-1} & W_{1,2} \\ W_{1,2}^T & W_{2,2} \end{bmatrix}^{-1} \left[F^{sN} \ L_{\overline{w}^s,N} \right]^T \tag{24}$$

The solution given by the theorem 1 is equivalent to the solution proposed by lemma 2 who can be to check easily.

Lemma 2. The MHE estimate of system (1) given by theorem can be calculated as

$$\hat{x}_k = \left[F^{sN} \;\; L_{\overline{w}^s,N} \right] \begin{bmatrix} W_{1,1} + P^{-1}_{k-N|k-N} & W_{1,2} \\ W^T_{1,2} & W_{2,2} \end{bmatrix}^{-1} \times \left(\begin{bmatrix} P^{-1}_{k-N|k-N} \\ 0 \end{bmatrix} \hat{x}_{k-N|k-N} + \begin{bmatrix} (\widetilde{C}^s_N)^T \\ (\widetilde{G}^s_N)^T \end{bmatrix} (R^s_N)^{-1} \left(Z_k - \widetilde{B}^s_N \bar{r}_k \right) \right) + L_{\bar{r},N} \bar{r}_k.$$

$$P_{k|k} = \left[F^{sN} \;\; L_{\overline{w}^s,N} \right] \begin{bmatrix} W_{1,1} + P^{-1}_{k-N|k-N} & W_{1,2} \\ W^T_{1,2} & W_{2,2} \end{bmatrix}^{-1} \left[F^{sN} \;\; L_{\overline{w}^s,N} \right]^T \quad (25)$$

4. CONVERGENCE AND STABILITY

Let us giving the following lemma

Lemma 3. The MHE estimate of system 1 given by Lemma 2 is equivalent to the Kalman filter.

Proof. See Kwon and Han [2005].

Using the result of lemma 3, the MHE estimate can be written as follows

$$\begin{aligned} \hat{x}_{k|k} &= \left(F^s - K^s_{k-1} \underline{C} \right) \hat{x}_{k-1|k-1} + \left(\bar{B}^s - K^s_{k-1} D \right) r_k \\ P_{k|k} &= F^s P_{k-1|k-1} F^{sT} + Q^s - K^s_{k-1} \underline{C} P_{k-1|k-1} F^{sT} \end{aligned} \quad (26)$$

with

$$K^s_{k-1} = F^s P_{k-1|k-1} \underline{C}^T \left(\underline{C} P_{k-1|k-1} \underline{C}^T + R^s \right)^{-1},$$

and $P_{0|0} = P_0$.

The Riccati algebraic equation (RAE) associate to the Riccati difference equation (RDE) (26) is given by

$$P = F^s P F^{sT} + Q^s - K^s \underline{C} P F^{sT} \quad (27)$$

with

$$K^s = F^s P \underline{C}^T \left(\underline{C} P \underline{C}^T + R^s \right)^{-1}.$$

The necessary and sufficient conditions for the convergence of the sequence $\{P_{k|k}, k \geq 0\}$ to the strong or the stabilization solution of the ARE (when this solution exist)are given by theorem 4.

Theorem 4. Bassong-Onana and Darouach [1992]

- Assume that P is the unique strong solution of the ARE, and that the initial condition $P_{0|0}$ satisfies $P_{0|0} - P \geq 0$. Then, the sequence $\{P_{k|k}, k \geq 0\}$, generated by by the RDE, converges exponentially to P, if and only if $(\underline{C}, F^s)$ is detectable.

- Assume that P is the unique stabilization solution of the ARE and that $P_{0|0} \geq 0$. Then, the sequence $\{P_{k|k}, k \geq 0\}$, generated by by the RDE, converges exponentially to P, if and only if $(\underline{C}, F^s)$ is detectable and $(F^s), D^s$ (D^s is any square root matrix of Q^s) has no unreachable mode inside the unit circle.

Proof. See de Souza et al. [1986].

5. NUMERICAL EXAMPLE

Consider the singular discrete-time system described by the following equations

$$\begin{aligned} E x_{k+1} &= A x_k + B u_k + w_k, \qquad k = 0, 1, 2, ... \\ y_k &= C x_k + v_k \end{aligned}$$

with

$$E = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 0 & 0.59 \\ 0 & -1 & 0 & 0.5 \\ 1 & 0 & 1 & 0.09 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 0.6 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.7 \end{bmatrix}, \quad W = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}, \text{ and}$$

$$V = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$$

we take the initial state $\bar{x}_0 = [3\ 3\ 3\ 3]^T$, $N = 10$ and $k = 65$. The input u_k is given by figure (1). Note that matrix $\begin{bmatrix} E \\ C \end{bmatrix}$ is full column rank, then assumption 1 is verified.

The matrices $\bar{J}_1$,$\bar{J}_2$, $\bar{J}_3$ and $\bar{J}_4$ are given by

$$\bar{J}_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \quad \bar{J}_2 = \begin{bmatrix} 0 & 2 & 3 \\ -2 & -2 & -1 \\ 2 & 0 & -2 \\ -3 & 0 & 5 \end{bmatrix},$$

$$\bar{J}_3 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix}, \quad \bar{J}_4 = \begin{bmatrix} -1 & 2 & 4 \\ -2 & -2 & 0 \end{bmatrix}$$

The simulation results based on the filter given by theorem 1 is shown in the figures $(2) - (5)$. The true state $x(k)$ is plotted by the solid line and the MHE estimate is presented by the dashed line.

6. CONCLUSION

Using minimum-variance unbiased estimation and the moving horizon estimation approach, a recursive filter for discrete-time linear stochastic singular systems is derived. The necessary and sufficient condition for convergence and stability are presented. To show the obtained results, a numerical example has been presented.

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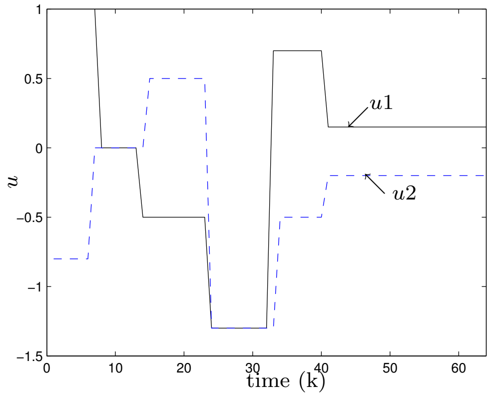

Fig. 1. The input $u_1(k)$ (solid line) and The input $u_2(k)$ (dashed line).

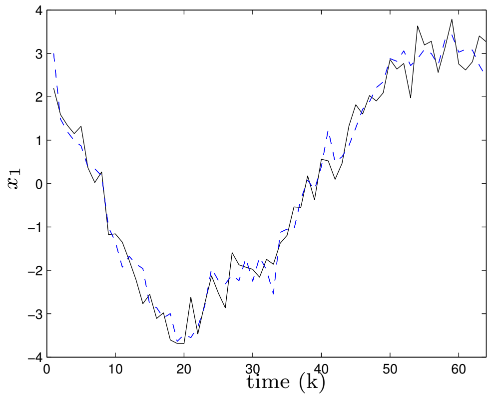

Fig. 2. The true value of state $x_1(k)$ (solid line) and the MHE estimate (dashed line).

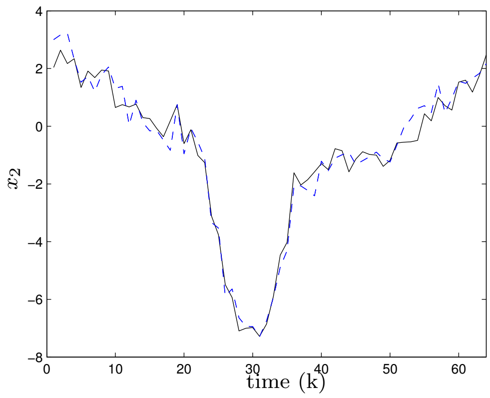

Fig. 3. The true value of state $x_2(k)$ (solid line) and the MHE estimate (dashed line).

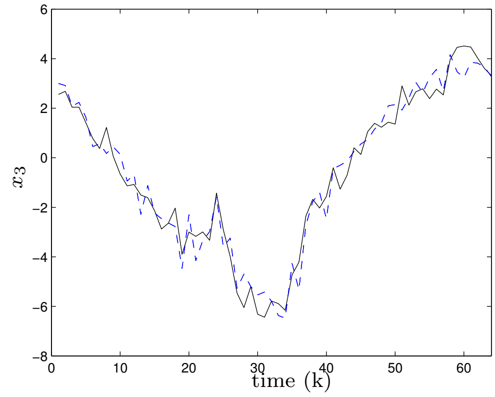

Fig. 4. The true value of state $x_3(k)$ (solid line) and the MHE estimate (dashed line).

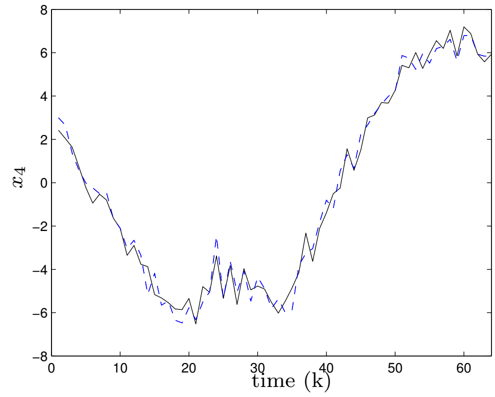

Fig. 5. The true value of state $x_4(k)$ (solid line) and the MHE estimate (dashed line).

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