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An Invariant Observer for Earth-Velocity-Aided Attitude Heading Reference Systems

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Abstract: In this paper we propose an invariant nonlinear observer (i.e. a "filter") for estimating the velocity vector and orientation of a flying rigid body, using measurements from low-cost Earth-fixed velocity, inertial and magnetic sensors. It has a nice geometric structure which respects meaningful physical symmetries of the system. It can be seen as an easier-to-tune and computationally much simpler alternative to an Extended Kalman Filter.

Keywords: observers; sensor fusion; nonlinear filters; strapdown systems; invariance; inertial navigation; extended Kalman filters.

1. INTRODUCTION

Aircraft, especially Unmanned Aerial Vehicles (UAV), commonly need to know their orientation and velocity to be operated, whether manually or with computer assistance. When cost or weight is an issue, using very accurate inertial sensors for "true" (i.e. based on the Schuler effect due to a non-flat rotating Earth) inertial navigation is excluded. Instead, low-cost systems –sometimes called velocity-aided Attitude Heading Reference Systems (AHRS)– rely on light and cheap "strapdown" gyroscopes, accelerometers and magnetometers "aided" by velocity sensors (provided for example in body-fixed coordinates by an air-data or Doppler radar system, or in Earth-fixed coordinates by a GPS engine). The various measurements are then "merged" according to the motion equations of the aircraft assuming a flat non-rotating Earth, usually with a linear complementary filter or an Extended Kalman Filter (EKF). For more details about avionics, various inertial navigation systems and sensor fusion, see for instance the books Collinson [2003], Kayton and Fried [1997], Grewal et al. [2007] and the references therein.

In this paper we propose as an alternative to an EKF a simple nonlinear observer. It has a nice geometric structure which respects meaningful physical symmetries of the system and is derived from the general method developed in Bonnabel et al. [2007]. It should be easier to tune and computationally much simpler than an EKF, with a similar local performance and an expected larger domain of convergence. Several nonlinear observers have recently been designed for various strapdown systems, though with different sensors, e.g. Thienel and Sanner [2003], Mahony et al. [2005], Hamel and Mahony [2006], Bonnabel et al. [2006], Cheviron et al. [2007], Baldwin et al. [2007], Martin and Salaün [2007].

The paper is organized as follows. Section 2 presents the model used to design the observer. Section 3 recaps the general method of Bonnabel et al. [2007], which is used in section 4 to derive an observer for the system under study. Section 5 is devoted to the choice of the observer tuning parameters. Finally, section 6 illustrates on simulations the good behavior of the observer while section 7 compares it on actual data to a commercial device.

2. PHYSICAL EQUATIONS AND MEASUREMENTS

2.1 Motion equations

The motion of a flying rigid body (assuming the Earth is flat and defines an inertial frame) is described by

$$\dot{q} = \frac{1}{2} q * \omega$$
$$\dot{V} = A + q * a * q^{-1},$$

where

- q is the quaternion representing the orientation of the body-fixed frame with respect to the Earth-fixed frame
- ω is the instantaneous angular velocity vector
- V is the velocity vector of the center of mass with respect to the Earth-fixed frame
- $A = (0\,0\,g)^T$ is the (constant) gravity vector in North-East-Down coordinates
- a is the specific acceleration vector, i.e. all the non-gravitational forces divided by the body mass.

The first equation describes the kinematics of the body, the second is Newton's force law. It is customary to use quaternions (also called Euler 4-parameters) instead of Euler angles since they provide a global parametrization of the body orientation, and are well-suited for calculations and computer simulations. For more details about this section, see any good textbook on aircraft modeling, for instance Stevens and Lewis [2003], and Martin and Salaün [2007] for useful formulas used in this paper.

2.2 Measurements

We use four triaxial sensors, yielding twelve scalar measurements: 3 gyros measure $\omega_m = \omega + \omega_b$, where ω_b is a constant vector bias; 3 accelerometers measure $a_m = a_s a$, where $a_s > 0$ is a constant scaling factor; 3 magnetometers measure $y_B = q^{-1} * B * q$, where $B = (B_1\ 0\ B_3)^T$ is the Earth magnetic field in NED coordinates; the velocity vector V is provided by the navigation solutions y_V of a GPS engine (the GPS velocity is obtained from the carrier phase and/or Doppler shift data, and not by differentiating the GPS position, hence is of rather good quality). All these measurements are of course also corrupted by noise.

There is some freedom in the modeling of the sensors imperfections. A simple first-order observability analysis reveals that up to six unknown constants can be estimated. The two extra constants could be used to model two imperfections on y_B, or one on y_B and one more on a_m. Nevertheless, it is not possible to use them to model three imperfections on a_m: in particular if we write $a_m = a + a_b$, with a_b a constant vector bias, only two components of a_b are observable; moreover, only one imperfection on a_m can be estimated without relying on the possibly disturbed magnetic measurement y_B. The choice adopted here, 3 biases on the gyros and one scaling factor on the acceleros, ensures that in level flight the estimated velocity equals the measured velocity (see §5.2).

2.3 The considered system

To design our observers we therefore consider the system

$$\dot{q} = \frac{1}{2} q * (\omega_m - \omega_b) \tag{1}$$

$$\dot{V} = A + \frac{1}{a_s} q * a_m * q^{-1} \tag{2}$$

$$\dot{\omega}_b = 0 \tag{3}$$

$$\dot{a}_s = 0, \tag{4}$$

where ω_m and a_m are seen as known inputs, together with the output

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V \\ q^{-1} * B * q \end{pmatrix}. \tag{5}$$

This system is observable provided $B \times (q * a_m * q^{-1}) \neq 0$ since all the state variables can be recovered from the known quantities ω_m, a_m, y_V, y_B and their derivatives. Indeed from (2), $a_s = \frac{\|a_m\|}{\|\dot{y}_V - A\|}$ and $\frac{a_m}{\|a_m\|} = q^{-1} * \frac{\dot{y}_V - A}{\|\dot{y}_V - A\|} * q$. We thus know the action of q on the two known vectors B and $\dot{y}_V - A$, which are independent by the above assumption; this completely defines q in function of $y_B, \dot{y}_B, a_m$. Finally $\omega_b = \omega_m - 2q^{-1}\dot{q}$ is determined from (1).

3. THEORY OF INVARIANT OBSERVERS

We briefly recall here the main ideas of Bonnabel et al. [2007]. The theory is constructive and is directly applicable to the system considered in this paper.

3.1 Invariant systems and compatible outputs

Definition 1. Let G be a Lie Group with identity e and Σ an open set (or more generally a manifold). A *transformation group* $(\phi_g)_{g \in G}$ on Σ is a smooth map

$$(g, \xi) \in G \times \Sigma \mapsto \phi_g(\xi) \in \Sigma$$

such that:

- $\phi_e(\xi) = \xi$ for all ξ
- $\phi_{g_2} \circ \phi_{g_1}(\xi) = \phi_{g_2 g_1}(\xi)$ for all g_1, g_2, ξ.

By construction ϕ_g is a diffeomorphism on Σ for all g. The transformation group is *local* if $\phi_g(\xi)$ is defined only for g around e. In this case the transformation law $\phi_{g_2} \circ \phi_{g_1}(\xi) = \phi_{g_2 g_1}(\xi)$ is imposed only when it makes sense. We consider in the sequel only local transformation groups. "For all g" thus means "for all g around e, and "for all ξ" means "for all ξ in some neighborhood".

Consider now the smooth output system

$$\dot{x} = f(x, u) \tag{6}$$

$$y = h(x, u) \tag{7}$$

where x belongs to an open subset $\mathcal{X} \subset \mathbb{R}^n$, u to an open subset $\mathcal{U} \subset \mathbb{R}^m$ and y to an open subset $\mathcal{Y} \subset \mathbb{R}^p$, $p \leq n$.

We assume the signals $u(t), y(t)$ known (y is measured, and u is measured or known as a control input).

Consider also the local group of transformations on $\mathcal{X} \times \mathcal{U}$ defined by $(X, U) = \big(\varphi_g(x), \psi_g(u)\big)$, where φ_g and ψ_g are local diffeomorphisms.

Definition 2. The system $\dot{x} = f(x, u)$ is *G-invariant* if $f\big(\varphi_g(x), \psi_g(u)\big) = D\varphi_g(x) \cdot f(x, u)$ for all g, x, u.

The property also reads $\dot{X} = f(X, U)$, i.e., the system is left unchanged by the transformation.

Definition 3. The output $y = h(x, u)$ is *G-compatible* if there exists a transformation group $(\varrho_g)_{g \in G}$ on $\mathcal{Y}$ such that $h\big(\varphi_g(x), \psi_g(u)\big) = \rho_g\big(h(x, u)\big)$ for all g, x, u.

With $(X, U) = \big(\varphi_g(x), \psi_g(u)\big)$ and $Y = \varrho_g(y)$, the definition reads $Y = h(X, U)$.

3.2 Invariant preobservers

Definition 4. (Preobserver). The system $\dot{\hat{x}} = F(\hat{x}, u, y)$ is a *preobserver* of (6)-(7) if $F\big(x, u, h(x)\big) = f(x, u)$ for all x, u.

An observer is then a preobserver such that $\hat{x}(t) \to x(t)$ (possibly only locally).

Definition 5. The preobserver $\dot{\hat{x}} = F(\hat{x}, u, y)$ is *G-invariant* if $F\big(\varphi_g(\hat{x}), \psi_g(u), \rho_g(y)\big) = D\varphi_g(\hat{x}) \cdot F(\hat{x}, u, y)$ for all $g, \hat{x}, u, y$.

The property also reads $\dot{\hat{X}} = F(\hat{X}, U, Y)$, i.e., the system is left unchanged by the transformation.

The key idea to build an invariant observer is to use an invariant output error.

Definition 6. The smooth map $(\hat{x}, u, y) \mapsto E(\hat{x}, u, y) \in \mathcal{Y}$ is an *invariant output error* if

- the map $y \mapsto E(\hat{x}, u, y)$ is invertible for all $\hat{x}, u$

- $E\big(\hat{x}, u, h(\hat{x}, u)\big) = 0$ for all $\hat{x}, u$
- $E\big(\varphi_g(\hat{x}), \psi_g(u), \rho_g(y)\big) = E(\hat{x}, u, y)$ for all $\hat{x}, u, y$

The first and second properties mean E is an "output error", i.e. it is zero if and only if $h(\hat{x}, u) = y$; the third property, which also reads $E(\hat{X}, U, Y) = E(\hat{x}, u, y)$, defines invariance.

Similarly, the key idea to study the convergence of an invariant observer is to use an invariant state error.

Definition 7. The smooth map $(\hat{x}, x) \mapsto \eta(\hat{x}, x) \in \mathcal{X}$ is an *invariant state error* if

- it is a diffeomorphism on $\mathcal{X} \times \mathcal{X}$
- $\eta(x, x) = 0$ for all x
- $\eta\big(\varphi_g(\hat{x}), \varphi_g(x)\big) = \eta(\hat{x}, x)$ for all $\hat{x}, x$.

We now state the two main results –based on the Cartan moving frame method– in the special case where $g \mapsto \varphi_g(x)$ is invertible (i.e. when G is of dimension n), see Bonnabel et al. [2007] for the general case. The *moving frame* $x \mapsto \gamma(x)$ is obtained by solving for g the so-called *normalization equation* $\varphi_g(x) = c$ for some arbitrary constant c; in other words $\varphi_{\gamma(x)}(x) = c$.

Theorem 8. The general invariant preobserver reads

$$F(\hat{x}, u, y) = f(\hat{x}, u) + \sum_{i=1}^{n} \big(L_i(E, I) \cdot E\big) w_i(\hat{x}),$$

where:

- $w_i, i = 1, \ldots, n$, is the invariant vector field defined by
$$w_i(\hat{x}) = \big[D\varphi_{\gamma(\hat{x})}(\hat{x})\big]^{-1} \cdot \frac{\partial}{\partial x_i},$$
with $\frac{\partial}{\partial x_i}$ the i^{th} canonical vector field on $\mathcal{X}$
- E is the invariant error defined by
$$E(\hat{x}, u, y) = \rho_{\gamma(\hat{x})}\big(h(\hat{x}, u)\big) - \rho_{\gamma(\hat{x})}(y)$$
- I is the (complete) invariant defined by
$$I(\hat{x}, u) = \psi_{\gamma(\hat{x})}(u)$$
- L_i, $i = 1, \ldots, n$, is a $1 \times p$ matrix with entries possibly depending on E and I, and can be freely chosen.

Theorem 9. The error system reads $\dot{\eta} = \Upsilon(\eta, I)$ where η is the invariant state error defined by

$$\eta(\hat{x}, x) = \varphi_{\gamma(x)}(\hat{x}) - \varphi_{\gamma(x)}(x).$$

This result greatly simplifies the convergence analysis of the preobserver, since the error equation is autonomous but for the "free" known invariant I. For a general nonlinear (not invariant) observer the error equation depends on the trajectory $t \mapsto \big(x(t), u(t)\big)$ of the system, hence is in fact of dimension $2n$.

4. CONSTRUCTION OF THE OBSERVER

4.1 Invariance of the system equations

The physical system is obviously unaffected by a constant velocity translation in the Earth-fixed frame and a constant rotation of the body-fixed frame. It is natural to expect a similar behavior from an observer. We therefore consider the following transformation group generated by rotations, translations and scaling

$$\varphi_{(q_0,V_0,\omega_0,a_0)}\begin{pmatrix} q \\ V \\ \omega_b \\ a_s \end{pmatrix} = \begin{pmatrix} q * q_0 \\ V + V_0 \\ q_0^{-1} * \omega_b * q_0 + \omega_0 \\ a_s a_0 \end{pmatrix}$$
$$\psi_{(q_0,V_0,\omega_0,a_0)}\begin{pmatrix} \omega_m \\ a_m \end{pmatrix} = \begin{pmatrix} q_0^{-1} * \omega_m * q_0 + \omega_0 \\ a_0 q_0^{-1} * a_m * q_0 \end{pmatrix}$$
$$\rho_{(q_0,V_0,\omega_0,a_0)}\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} y_V + V_0 \\ q_0^{-1} * y_B * q_0 \end{pmatrix}.$$

There are $3 + 2 * 3 + 1 = 10$ parameters: the unit quaternion q_0, the $\mathbb{R}^3$-vectors V_0, ω_0 and the positive scalar a_0. The group law $\diamond$ is given by

$$\begin{pmatrix} q_1 \\ V_1 \\ \omega_1 \\ a_1 \end{pmatrix} \diamond \begin{pmatrix} q_0 \\ V_0 \\ \omega_0 \\ a_0 \end{pmatrix} = \begin{pmatrix} q_0 * q_1 \\ V_0 + V_1 \\ q_1^{-1} * \omega_0 * q_1 + \omega_1 \\ a_0 a_1 \end{pmatrix}.$$

The system (1)–(4) is of course invariant by the transformation group since

$$\begin{aligned}
\dot{\overbrace{q * q_0}} &= \dot{q} * q_0 \\
&= \frac{1}{2}(q * q_0) * \big((q_0^{-1} * \omega_m * q_0 + \omega_0) \\
&\quad - (q_0^{-1} * \omega_b * q_0 + \omega_0)\big) \\
\dot{\overbrace{V + V_0}} &= \dot{V} \\
&= A + \frac{1}{a_s a_0}(q * q_0) * (a_s q_0^{-1} * a_m * q_0) * (q * q_0)^{-1} \\
\dot{\overbrace{q_0^{-1} * \omega_b * q_0 + \omega_0}} &= q_0^{-1} * \dot{\omega}_b * q_0 = 0 \\
\dot{\overbrace{a_s a_0}} &= \dot{a}_s a_0 = 0,
\end{aligned}$$

whereas the output (5) is compatible since

$$\begin{pmatrix} V + V_0 \\ (q * q_0)^{-1} * B * (q * q_0) \end{pmatrix} = \rho_{(q_0,V_0,\omega_0,a_0)} \begin{pmatrix} V \\ q^{-1} * B * q \end{pmatrix}.$$

4.2 Construction of the general invariant preobserver

We solve for $(q_0, V_0, \omega_0, a_0)$ the normalization equations

$$\begin{aligned} q * q_0 &= 1 \\ V + V_0 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} q_0^{-1} * \omega_b * q_0 + \omega_0 &= 0 \\ a_s a_0 &= 1 \end{aligned}$$

to find the moving frame

$$\gamma(q, V, \omega_b, a_s) = \begin{pmatrix} q^{-1} \\ -V \\ -q * \omega_b * q^{-1} \\ 1/a_s \end{pmatrix}.$$

We then get the 6-dimensional invariant error

$$\begin{aligned}
\begin{pmatrix} E_V \\ E_B \end{pmatrix} &= \rho_{\gamma(\hat{q},\hat{V},\hat{\omega}_b,\hat{a}_s)} \begin{pmatrix} \hat{y}_V \\ \hat{y}_B \end{pmatrix} - \rho_{\gamma(\hat{q},\hat{V},\hat{\omega}_b,\hat{a}_s)} \begin{pmatrix} y_V \\ y_B \end{pmatrix} \\
&= \begin{pmatrix} \hat{y}_V - y_V \\ B - \hat{q} * q^{-1} * B * q * \hat{q}^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \hat{y}_V - y_V \\ B - \hat{q} * y_B * \hat{q}^{-1} \end{pmatrix}
\end{aligned}$$

and the 6-dimensional complete invariant

$$\begin{pmatrix} I_\omega \\ I_a \end{pmatrix} = \psi_{\gamma(\hat{q},\hat{V},\hat{\omega}_b,\hat{a}_s)} \begin{pmatrix} \omega_m \\ a_m \end{pmatrix} = \begin{pmatrix} \hat{q} * (\omega_m - \hat{\omega}_b) * \hat{q}^{-1} \\ \frac{1}{\hat{a}_s}\hat{q} * a_m * \hat{q}^{-1} \end{pmatrix}.$$

Notice that I_ω,I_a,E_V and E_B are functions of the estimates and the measurements. Hence they are known quantities which can be used in the construction of the preobserver. It is straightforward to check they are indeed invariant. For instance,

$$\begin{aligned}E_B(\hat{q}*q_0,\hat{V}+V_0,q_0^{-1}*\hat{\omega}_b*q_0+\omega_0,\hat{a}_s a_0,q_0^{-1}*y_B*q_0)\\ &= B-(\hat{q}*q_0)*(q_0^{-1}*y_B*q_0)*(\hat{q}*q_0)^{-1}\\ &= B-\hat{q}*y_B*\hat{q}^{-1}\\ &= E_B(\hat{q},\hat{V},\hat{\omega}_b,\hat{a}_s,y_B).\end{aligned}$$

To find invariant vector fields, we solve for $w(q,V,\omega_b,a_s)$ the 10 vector equations

$$\left[D\varphi_{\gamma(q,V,\omega_b,a_s)}\begin{pmatrix}q\\V\\\omega_b\\a_s\end{pmatrix}\right]\cdot w(q,V,\omega_b,a_s) = \begin{pmatrix}e_i\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\e_i\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\e_i\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\e_{10}\end{pmatrix},\qquad i=1,2,3,$$

where the e_i's are the canonical basis of $\mathbb{R}^3$ (we have identified the tangent space of the unit norm quaternions space to $\mathbb{R}^3$). Since

$$\left[D\varphi_{(q_0,V_0,\omega_0,a_0)}\begin{pmatrix}q\\V\\\omega_b\\a_s\end{pmatrix}\right]\cdot\begin{pmatrix}\delta q\\\delta V\\\delta\omega_b\\\delta a_s\end{pmatrix}=\begin{pmatrix}\delta q*q_0\\\delta V\\q_0^{-1}*\delta\omega_b*q_0\\a_0\delta a_s\end{pmatrix},$$

this yields the 10 independent invariant vector fields

$$\begin{pmatrix}e_i*q\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\e_i\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\q^{-1}*e_i*q\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\a_s e_{10}\end{pmatrix},\qquad i=1,2,3.$$

These vector fields are invariant. Indeed for instance,

$$\left[D\varphi_{(q_0,V_0,\omega_0,a_0)}\begin{pmatrix}q\\V\\\omega_b\\a_s\end{pmatrix}\right]\cdot\begin{pmatrix}e_i*q\\0\\0\\0\end{pmatrix}=\begin{pmatrix}(e_i*q)*q_0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}e_i*(q*q_0)\\0\\0\\0\end{pmatrix}.$$

The general invariant preobserver then reads

$$\begin{aligned}\dot{\hat{q}} &= \frac{1}{2}\hat{q}*(\omega_m-\hat{\omega}_b)+\sum_{i=1}^{3}(L_{Vi}E_V+L_{Bi}E_B)e_i*\hat{q}\\ \dot{\hat{V}} &= A+\frac{1}{\hat{a}_s}\hat{q}*a_m*\hat{q}^{-1}+\sum_{i=1}^{3}(M_{Vi}E_V+L_{Bi}E_B)e_i\\ \dot{\hat{\omega}}_b &= \sum_{i=1}^{3}\hat{q}^{-1}*(N_{Vi}E_V+N_{Bi}E_B)e_i*\hat{q}\\ \dot{\hat{a}}_s &= \hat{a}_s(O_VE_V+O_BE_B),\end{aligned}$$

where the L_{Vi}, L_{Bi}, M_{Vi}, M_{Bi}, N_{Vi}, N_{Bi}, O_V, O_B are arbitrary 1×3 matrices with entries possibly depending on E_V, E_B, I_ω and I_a. Noticing

$$\sum_{i=1}^{3}(L_{Vi}E_V)e_i=\begin{pmatrix}L_{V1}\\L_{V2}\\L_{V3}\end{pmatrix}E_V=L_VE_V,$$

where L_V is the 3×3 matrix whose rows are the L_{Vi}'s, and defining L_B, M_V, M_B, N_V and N_B in the same way, we can rewrite the preobserver as

$$\dot{\hat{q}} = \frac{1}{2}\hat{q}*(\omega_m-\hat{\omega}_b)+(L_VE_V+L_BE_B)*\hat{q} \tag{8}$$

$$\dot{\hat{V}} = \frac{1}{\hat{a}_s}\hat{q}*a_m*\hat{q}^{-1}+A+(M_VE_V+M_BE_B) \tag{9}$$

$$\dot{\hat{\omega}}_b = \hat{q}^{-1}*(N_VE_V+N_BE_B)*\hat{q} \tag{10}$$

$$\dot{\hat{a}}_s = \hat{a}_s(O_VE_V+O_BE_B). \tag{11}$$

As a by-product of its geometric structure, the preobserver automatically has a desirable feature: the norm of $\hat{q}$ is left unchanged by (8), since $L_VE_V+L_BE_B$ is a vector of $\mathbb{R}^3$ (see Martin and Salaün [2007]); moreover $\hat{a}_s$ remains positive.

4.3 Error equations

The invariant state error is given by

$$\begin{pmatrix}\eta\\\nu\\\beta\\\alpha\end{pmatrix}=\varphi_{\gamma(q,V,\omega_b,a_s)}\begin{pmatrix}\hat{q}\\\hat{V}\\\hat{\omega}_b\\\hat{a}_s\end{pmatrix}-\varphi_{\gamma(q,V,\omega_b,a_s)}\begin{pmatrix}q\\V\\\omega_b\\a_s\end{pmatrix}=\begin{pmatrix}\hat{q}*q^{-1}-1\\\hat{V}-V\\q*(\hat{\omega}_b-\omega_b)*q^{-1}\\\dfrac{\hat{a}_s}{a_s}-1\end{pmatrix}.$$

It is in fact more natural –though completely equivalent– to take $\eta=\hat{q}*q^{-1}$ (rather than $\eta=\hat{q}*q^{-1}-1$), so that $\eta(x,x)=1$, the unit element of the group of quaternions. In the same way, we take $\alpha=\frac{\hat{a}_s}{a_s}$ to keep α in $\mathbb{R}^+$. Hence,

$$\begin{aligned}\dot{\eta} &= \dot{\hat{q}}*q^{-1}+\hat{q}*(-q^{-1}*\dot{q}*q^{-1})\\ &= -\frac{1}{2}\eta*\beta+(L_AE_A+L_CE_C)*\eta\\ \dot{\nu} &= \dot{\hat{V}}-\dot{V}\\ &= I_a-\alpha\eta^{-1}*I_a*\eta+(M_VE_V+M_BE_B)\\ \dot{\beta} &= \dot{q}*(\hat{\omega}_b-\omega_b)*q^{-1}-q*(\hat{\omega}_b-\omega_b)*q^{-1}*\dot{q}*q^{-1}\\ &\quad+q*(\dot{\hat{\omega}}_b-\dot{\omega}_b)*q^{-1}\\ &= (\eta^{-1}*I_\omega*\eta)\times\beta+\eta^{-1}*(N_VE_V+N_BE_B)*\eta\\ \dot{\alpha} &= \frac{\dot{\hat{a}}_s}{a_s}=\alpha(O_VE_V+O_BE_B).\end{aligned}$$

Since we can write

$$E_V=\nu \qquad\text{and}\qquad E_B=B-\eta*B*\eta^{-1},$$

we find as expected that the error system

$$\dot{\eta} = -\frac{1}{2}\eta*\beta+(L_AE_A+L_CE_C)*\eta \tag{12}$$

$$\nu = I_a-\alpha\eta^{-1}*I_a*\eta+(M_VE_V+M_BE_B) \tag{13}$$

$$\dot{\beta} = (\eta^{-1}*I_\omega*\eta)\times\beta+\eta^{-1}*(N_VE_V+N_BE_B)*\eta \tag{14}$$

$$\dot{\alpha} = \alpha(O_VE_V+O_BE_B) \tag{15}$$

depends only on the invariant state error (η,ν,β,α) and the "free" known invariants I_ω and I_a, but not on the trajectory of the observed system (1)–(4).

The linearized error system around $(\overline{\eta},\overline{\nu},\overline{\beta},\overline{\alpha})=(1,0,0,1)$, i.e. the estimated state equals the actual state, is given by

$$\delta\dot{\eta} = -\frac{1}{2}\delta\beta + (L_V \delta E_V + L_B \delta E_B)$$
$$\delta\dot{\nu} = -2I_a \times \delta\eta - \delta\alpha I_a + (M_V \delta E_V + M_B \delta E_B)$$
$$\delta\dot{\beta} = I_\omega \times \delta\beta + (N_V \delta E_V + N_B \delta E_B)$$
$$\delta\dot{\alpha} = (O_V \delta E_V + O_B \delta E_B),$$

where

$$\delta E_V = \delta\nu$$
$$\delta E_B = -\delta\eta * B * \overline{\eta}^{-1} - \overline{\eta} * B * (-\overline{\eta}^{-1} * \delta\eta * \overline{\eta}^{-1}) = 2(\overline{\eta} * B * \overline{\eta}^{-1}) \times (\delta\eta * \overline{\eta}^{-1}) = 2B \times \delta\eta.$$

5. CHOICE OF THE OBSERVER PARAMETERS

5.1 Choice of the gain matrices

The linearized error system without correction terms turns out to be decoupled into 4 independent subsystems (see §5.3) when I_a is constant and $I_\omega = 0$ (in particular when the aircraft is in level flight). To ensure a simple tuning, the gain matrices should respect this decoupling. On the other hand, the Earth magnetic field is quite perturbed in urban areas, which are usual sites for a small UAV. We do not want these magnetic disturbances – which unavoidably corrupt the heading estimation – to affect too much the attitude and velocity estimations. The idea is thus to rely on the magnetic measurement y_B as little as possible.

Therefore we choose

$$\begin{array}{ll} L_V E_V = -l_V I_a \times E_V & L_B E_B = l_B \langle B \times E_B, I_a \rangle I_a \\ M_V E_V = -m_V E_V & M_B E_B = 0 \\ N_V E_V = n_V I_a \times E_V & N_B E_B = -n_B \langle B \times E_B, I_a \rangle I_a \\ O_V E_V = o_V \langle I_a, E_V \rangle & O_B E_B = 0 \end{array}$$

with $(l_V, l_B, m_V, n_V, n_B, o_V) > 0$.

5.2 Equilibrium points of the observer equations

When the observer have converged, the two last equations of the observer write

$$N_V E_V + N_B E_B = 0 \quad \text{and} \quad O_V E_V = 0.$$

With the choice of gain matrices above it leads to

$$\overline{I}_a \times E_V = 0 \quad \text{and} \quad \langle \overline{I}_a, E_V \rangle = 0.$$

So even if the model is wrong, for example if the Earth magnetic field is perturbed, the observer equations ensure $\hat{V} = V$ once it has converged. This important property led us to consider the fourth bias a_s in addition to the usual biases ω_b on the gyros. Similar conclusion is not possible with the Euler angles even if we consider another additional biases: $E_B = 0$ does not ensure that the yaw angle, for example, is correctly estimated.

5.3 First-order approximation and coordinate change

Considering the gain matrices described above, the linearized error equations around the equilibrium point $(1, 0, 0, 1)$ write

$$\delta\dot{\eta} = -\frac{1}{2}\delta\beta - l_V I_a \times \delta E_V + l_B \langle B \times \delta E_B, I_a \rangle I_a$$
$$\delta\dot{\nu} = -2I_a \times \delta\eta - \delta\alpha I_a - m_V \delta E_V$$
$$\delta\dot{\beta} = I_\omega \times \delta\beta + n_V I_a \times \delta E_V - n_B \langle B \times \delta E_B, I_a \rangle I_a$$
$$\delta\dot{\alpha} = o_V \langle I_a, \delta E_V \rangle$$

with $\delta E_V = \delta\nu$ and $\delta E_B = 2B \times \delta\eta$.

We change the coordinate in order to have the estimated specific acceleration vector I_a vertical. Let η_0 be this frame rotation defined by

$$\eta_0^{-1} * I_a * \eta_0 = -kA, \quad \text{where} \quad k > 0 \tag{16}$$
$$\eta_0^{-1} * B * \eta_0 = \tilde{B} \quad \text{with} \quad \tilde{B} = (\tilde{B}_1 \; 0 \; \tilde{B}_3). \tag{17}$$

It follows

$$\tilde{\eta} = \eta_0^{-1} * \eta * \eta_0 \qquad \tilde{\nu} = \eta_0^{-1} * \nu * \eta_0$$
$$\tilde{\beta} = \eta_0^{-1} * \beta * \eta_0.$$

The error system then becomes

$$\delta\dot{\tilde{\eta}} = 2\delta\tilde{\eta} \times (\eta_0^{-1} * \dot{\eta}_0) - \frac{1}{2}\delta\tilde{\beta} + kl_V A \times \delta\tilde{E}_V + k^2 l_B \langle \tilde{B} \times \delta\tilde{E}_B, A \rangle A$$
$$\delta\dot{\tilde{\nu}} = 2\delta\tilde{\nu} \times (\eta_0^{-1} * \dot{\eta}_0) + 2kA \times \delta\eta + k\delta\alpha A - m_V \delta\tilde{E}_V$$
$$\delta\dot{\tilde{\beta}} = 2\delta\tilde{\beta} \times (\eta_0^{-1} * \dot{\eta}_0) + \tilde{I}_\omega \times \delta\tilde{\beta} - kn_V A \times \delta\tilde{E}_V - k^2 n_B \langle \tilde{B} \times \delta\tilde{E}_B, A \rangle A$$
$$\delta\dot{\alpha} = -o_V k \langle A, \delta\tilde{E}_V \rangle$$

with $\delta\tilde{E}_V = \delta\tilde{\nu}$ and $\delta\tilde{E}_B = 2\tilde{B} \times \delta\tilde{\eta}$.

We suppose now that the system is moving along "smooth" trajectory, i.e $\dot{\eta}_0$ and I_ω are first order terms. All the terms of the form $\cdot \times (\eta_0^{-1} * \dot{\eta}_0)$ then disappear and k is now constant. The error system splits into three decoupled subsystems and one cascaded subsystem:

- the longitudinal subsystem

$$\begin{pmatrix} \delta\dot{\tilde{\eta}}_2 \\ \delta\dot{\tilde{\nu}}_1 \\ \delta\dot{\tilde{\beta}}_2 \end{pmatrix} = \begin{pmatrix} 0 & kl_V g & -\frac{1}{2} \\ -2kg & -m_V & 0 \\ 0 & -kn_V g & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{\eta}_2 \\ \delta\tilde{\nu}_1 \\ \delta\tilde{\beta}_2 \end{pmatrix}$$

- the lateral subsystem

$$\begin{pmatrix} \delta\dot{\tilde{\eta}}_1 \\ \delta\dot{\tilde{\nu}}_2 \\ \delta\dot{\tilde{\beta}}_1 \end{pmatrix} = \begin{pmatrix} 0 & -kl_V g & -\frac{1}{2} \\ 2kg & -m_V & 0 \\ 0 & kn_V g & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{\eta}_1 \\ \delta\tilde{\nu}_2 \\ \delta\tilde{\beta}_1 \end{pmatrix}$$

- the vertical subsystem

$$\begin{pmatrix} \delta\dot{\tilde{\nu}}_3 \\ \delta\dot{\alpha} \end{pmatrix} = \begin{pmatrix} -m_V & kg \\ -kgo_V & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{\nu}_3 \\ \delta\alpha \end{pmatrix}$$

- the heading subsystem

$$\begin{pmatrix} \delta\dot{\tilde{\eta}}_3 \\ \delta\dot{\tilde{\beta}}_3 \end{pmatrix} = \begin{pmatrix} -2k^2 g^2 l_B \tilde{B}_1^2 & -\frac{1}{2} \\ 2k^2 g^2 n_B \tilde{B}_1^2 & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{\eta}_3 \\ \delta\tilde{\beta}_3 \end{pmatrix} + \begin{pmatrix} 2k^2 g^2 l_B \tilde{B}_3 \tilde{B}_1 \\ -2k^2 g^2 n_B \tilde{B}_3 \tilde{B}_1 \end{pmatrix} \delta\tilde{\eta}_1.$$

Thanks to this decoupled structure, the tuning of the gains $l_V, l_B, m_V, n_V, n_B, o_V$ is straightforward. Obviously the lateral, longitudinal and vertical subsystems do not depend on the magnetic measurements, so will not be affected if the magnetic field is perturbed.

5.4 Influence of magnetic disturbances on static behavior

We now investigate how the equilibrium point $(1, 0, 0, 1)$ is modified when the magnetic field is (statically) perturbed. We show that only the yaw angle ψ is affected while all the other variables, in particular the attitude angles

$\phi, \theta,$ remain unchanged. Here the Euler angles ϕ, θ, ψ correspond to the error quaternion $\tilde{\eta}$ in the new frame (16)-(17).

The equilibrium points $(\overline{\tilde{\eta}}, \overline{\tilde{\nu}}, \overline{\tilde{\omega}}, \overline{\tilde{\alpha}})$ are defined by

$$\begin{aligned}(l_V kA \times \overline{\tilde{\nu}} + l_B \langle \tilde{B} \times \tilde{E}_B, kA\rangle kA) * \overline{\tilde{\eta}} - \frac{1}{2}\overline{\tilde{\eta}} * \overline{\tilde{\beta}} &= 0\\ -kA + \overline{\tilde{\alpha}}\overline{\tilde{\eta}}^{-1} * kA * \overline{\tilde{\eta}} - m_V \overline{\tilde{\nu}} &= 0\\ \overline{\tilde{\eta}}^{-1} * (-n_V kA \times \overline{\tilde{\nu}} - n_B \langle \tilde{B} \times \tilde{E}_B, kA\rangle kA) * \overline{\tilde{\eta}} &\\ +(\overline{\tilde{\eta}}^{-1} * \overline{\tilde{I}}_\omega * \overline{\tilde{\eta}}) \times \overline{\tilde{\beta}} &= 0\\ o_V \langle kA, \overline{\tilde{\nu}}\rangle \overline{\tilde{\alpha}} &= 0.\end{aligned}$$

We ensure $\overline{\tilde{\beta}} = 0$ by choosing $L_V E_V + L_B E_B$ colinear to $N_V E_V + N_B E_B$, that is $\frac{n_V}{l_V} = \frac{n_B}{l_B} \triangleq \sigma$. This implies $\overline{\tilde{\nu}} = 0$, $\overline{\tilde{\alpha}}\overline{\tilde{\eta}} * A * \overline{\tilde{\eta}}^{-1} - A = 0$ and $\langle \tilde{B} \times \tilde{E}_B, kA\rangle = 0$.

Finally $(\overline{\tilde{\eta}}, \overline{\tilde{\nu}}, \overline{\tilde{\beta}}, \overline{\tilde{\alpha}}) = (\overline{\tilde{\eta}}, 0, 0, 1)$; moreover $\overline{\phi} = \overline{\theta} = 0$ and $\overline{\psi}$ is determined by $\langle \tilde{B} \times \tilde{E}_B, kA\rangle = 0$, where the Euler angles $\overline{\phi}, \overline{\theta}, \overline{\psi}$ correspond to the error quaternion $\overline{\tilde{\eta}}$.

6. SIMULATION RESULTS

We first illustrate on simulation the behavior of the invariant observer

$$\begin{aligned}\dot{\hat{q}} &= \frac{1}{2}\hat{q} * (\omega_m - \hat{\omega}_b) + (L_V E_V + L_B E_B) * \hat{q} + \lambda(1 - \|\hat{q}\|^2)\hat{q}\\ \dot{\hat{V}} &= \frac{1}{\hat{a}_s}\hat{q} * a_m * \hat{q}^{-1} + A + (M_V E_V + M_B E_B)\\ \dot{\hat{\omega}}_b &= \hat{q}^{-1} * (N_V E_V + N_B E_B) * \hat{q}\\ \dot{\hat{a}}_s &= \hat{a}_s (O_V E_V + O_B E_B)\end{aligned}$$

with the choice of gain matrices described in §5.1. The added term $\lambda(1 - \|\hat{q}\|^2)\hat{q}$ is a well-known numerical trick to keep $\|\hat{q}\| = 1$. Notice this term is invariant.

We choose here time constants around 10s by taking $l_V = n_V = 4e-2$, $l_B = n_B = 2e-3$, $m_V = 5$, $o_V = 1e-2$ and $\lambda = 1$. The system follows the trajectory defined by

$$a_s = 1.1 \qquad \omega_b = \begin{pmatrix} .01 \\ -.012 \\ .08 \end{pmatrix}$$

$$V = \begin{pmatrix} 3 - 2\cos(.3t) \\ 3 - 2.8\cos(.25t + \pi/4) \\ -1 - 1.7\sin(.3t) \end{pmatrix} \quad \omega_m = \begin{pmatrix} \sin(.5t) \\ \sin(.3t) \\ -\sin(.5t) \end{pmatrix},$$

which is quite representative of a small UAV flight. The states are initialized far from their true values.

At $t = 30s$, the magnetic field is changed from $B = (1\ 0\ 1)^T$ to $B = (1\ 0.4\ 1)^T$.

Though we have no proof of convergence but local, the domain of attraction seems to be large enough, see Fig. 1–3. As expected, only the estimated yaw angle ψ is strongly affected by the magnetic disturbance. Because of the coupling terms $\tilde{I}_w$ and I_a, there is some dynamic influence on the other variables.

7. EXPERIMENTAL RESULTS

We now compare the behavior of our observer with the commercial INS-GPS device MIDG II from Microbotics Inc. We feed the observer with the raw measurements from the MIDG II gyros, acceleros and magnetic sensors, and the velocity provided by the navigation solutions of its GPS engine. The estimations of the observer are then compared to the estimations given by the MIDG II (computed according to the user manual by some Kalman filter). In order to have similar behaviors, we have chosen $l_V = 2.8e-3$, $l_B = 7e-3$, $n_V = 4e-5$, $n_B = 1e-4$, $m_V = 0.9$, $o_V = 9.4e-5$ and $\lambda = 1$.

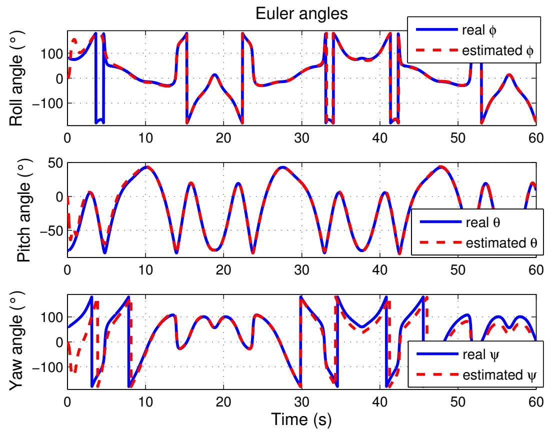

Fig. 1. Estimated Euler angles (simulation)

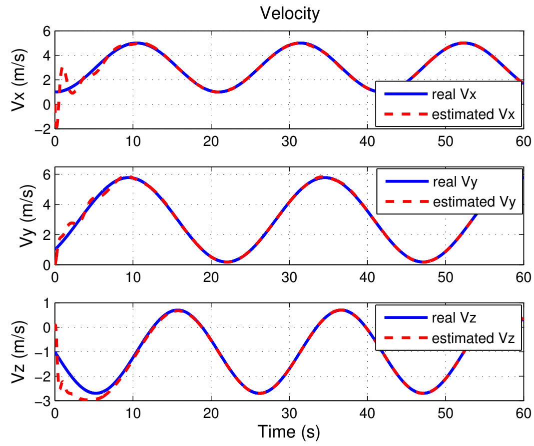

Fig. 2. Estimated velocity (simulation)

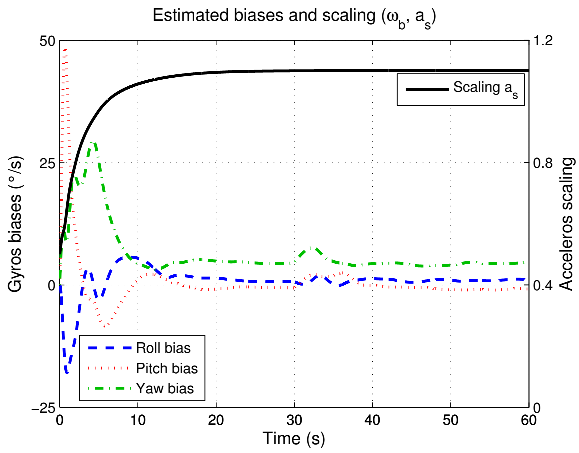

Fig. 3. Estimated biases (simulation)

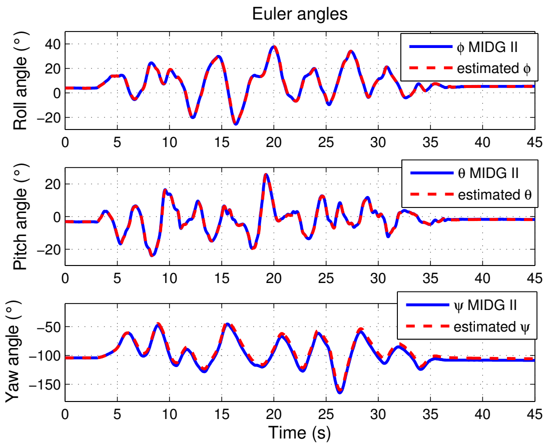

Fig. 4. Estimated Euler angles (experiment)

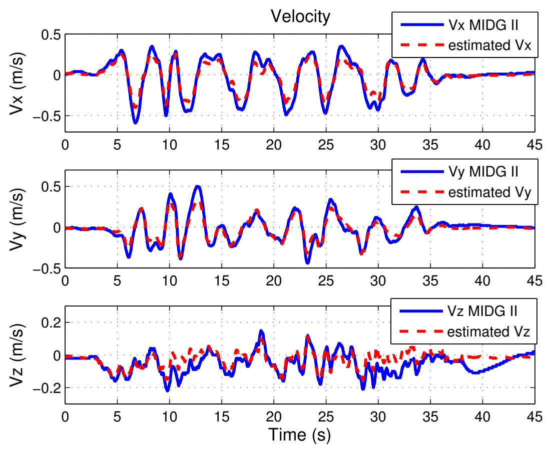

Fig. 5. Estimated velocity (experiment)

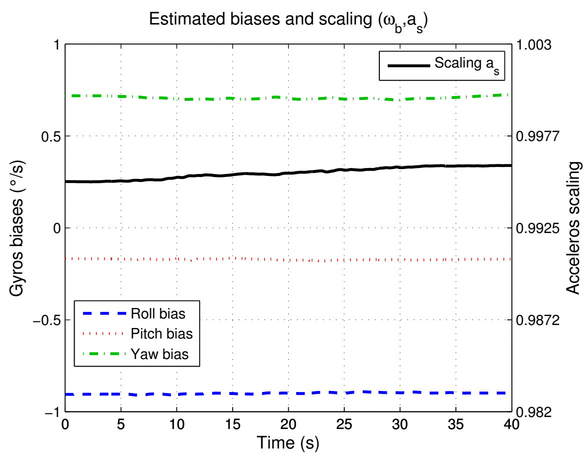

Fig. 6. Estimated biases (experiment)

7.1 *Dynamic behavior (Fig. 4–6)*

We wait a few minutes until the biases reach constant values, then move the system in all directions. The observer and the MIDG II give very similar results (Fig. 4 and 5).

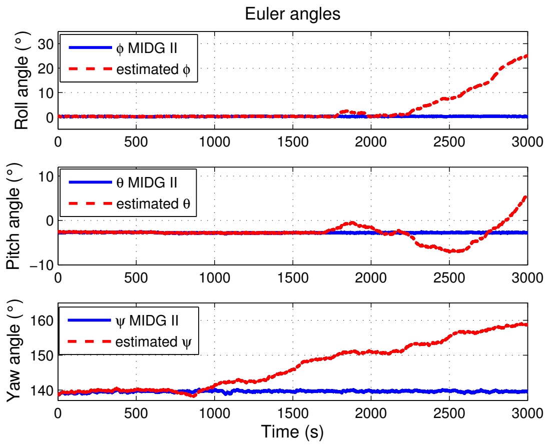

Fig. 7. Usefulness of correction terms (experiment)

7.2 *Usefulness of the observer terms (Fig. 7–10)*

As explained in §5.1 we have chosen the correction terms so as the magnetic measurements correct essentially the yaw angle and its corresponding bias, whereas the velocity measurements act on the other variables.

We highlight this property as well as the importance of correction terms on the following experiment. Once the biases have reached constant values, the system is left at rest during 50 minutes:

- for $t < 700s$ the results are very similar for the observer and the MIDG II (Fig. 7 and 8)
- at $t = 700s$ the "magnetic correction terms" are switched off, i.e. the gains l_B and n_B are set to 0. The yaw angle estimated by the observer diverges because the corresponding bias is not perfectly estimated. Indeed, these variables are not observable without the magnetic measurements. The other variables are not affected (Fig. 7 and 8)
- at $t = 1700s$ the "velocity correction terms" are also switched off, i.e. l_V, m_V, n_V and o_V are set to 0. All the estimated angles and velocities now diverge (Fig. 7 and 8). Zooming around $t = 1700s$, we see on Fig. 10 that the estimated pitch angle diverges with a slope corresponding to the almost-constant difference between the estimated and actual pitch biases. This explains why the estimated velocity V_x diverges quadratically in time.

7.3 *Influence of magnetic disturbances (Fig. 11-12)*

Once the biases have reached constant values, the system is left motionless for $60s$. At $t = 13s$ a magnet is put close to the sensors for $10s$. As expected only the estimated yaw angle is affected by the magnetic disturbance (Fig. 11- 12); the MIDG II exhibits a similar behavior.

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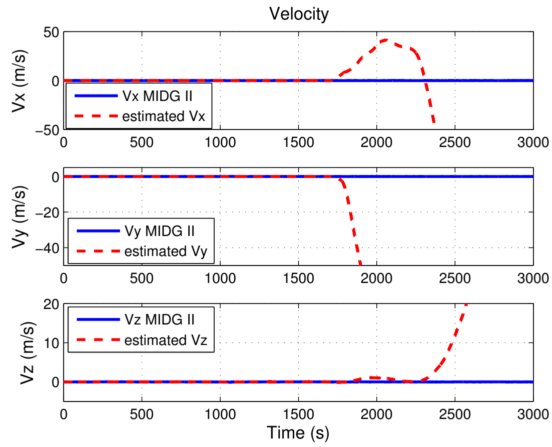

Fig. 8. Usefulness of correction terms (experiment)

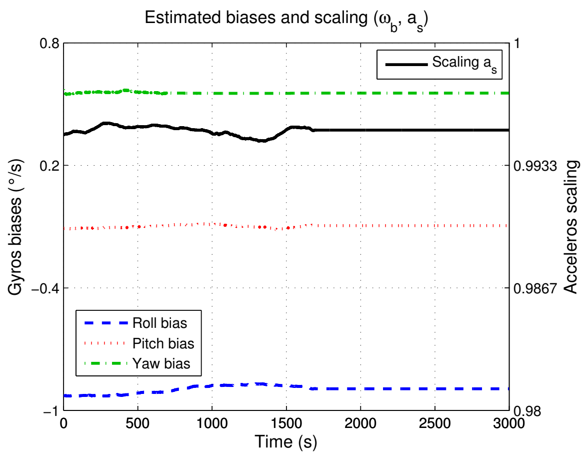

Fig. 9. Usefulness of the correction terms (experiment)

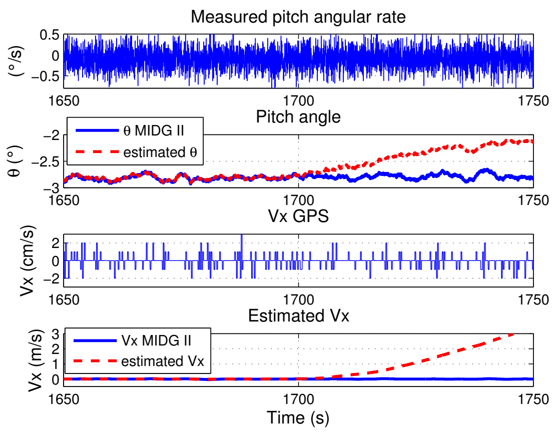

Fig. 10. Zoomed view at t=1700s (experiment)

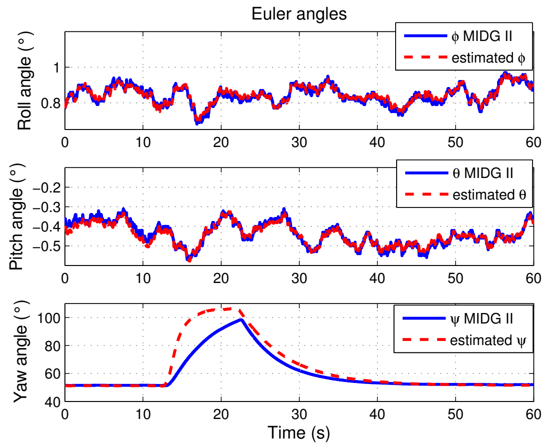

Fig. 11. Influence of magnetic disturbances (experiment)

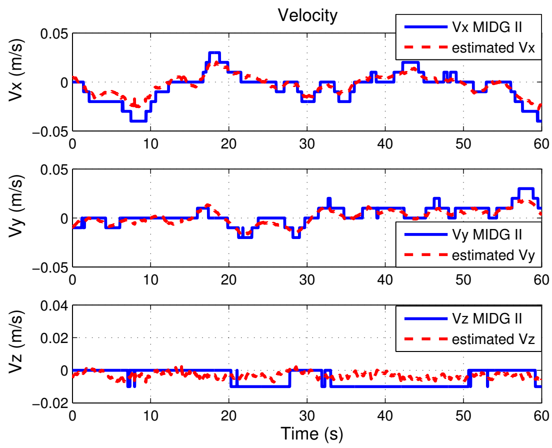

Fig. 12. Influence of magnetic disturbances (experiment)

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