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Output feedback stabilization for systems presenting sector-bounded nonlinearities and saturating inputs

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Abstract: In the present work a systematic methodology for computing output stabilizing feedback control laws for nonlinear systems subject to saturating inputs is presented. In particular, the class of L'ure type nonlinear systems is considered. Based on absolute stability tools and a modified sector condition to take into account input saturation effects, an LMI framework is proposed to design the controller. Both regional (local) and global stabilization results are presented. The controller structure is composed by a linear part, an anti-windup loop and a term associated to the output of the dynamic nonlinearity. Convex optimization problems are proposed in order to compute the controller matrices aiming at the maximization of the basin of attraction, or the performance enhancement with a guaranteed region of stability. A numerical example illustrates the potentialities of the methodology.

Keywords: Nonlinear control systems, Sector nonlinearities, Saturation, Dynamic output feedback, LMI.

1. INTRODUCTION

The design of most practical control systems requires to consider the presence of the nonlinearities that are inherent to the plant dynamics and/or to the physical actuator or sensor limitations, both in the analysis and in the synthesis phases. To cope with the presence of such nonlinearities, among the various existing nonlinear control systems approaches, absolute stability theory has been considered in the literature for analysis and synthesis of the so-called Lur'e systems (Khalil [2002], Liberzon [2006]). More recently, the research on absolute stability has been intensified, mainly due to the possibility of using the Linear Matrix Inequality (LMI) framework and the existing related efficient numerical tools for computations (Boyd et al. [1994]).

Considering linear systems with saturating inputs, a large amount of works can be found in the literature. We can cite, for instance the following ones (see also references therein): Pittet et al. [1997], Hindi and Boyd [1998] and Hu et al. [2002], considering state feedback control laws; Kapila and Haddad [2000], Kiyama and Iwasaki [2000] and Gomes da Silva Jr. et al. [2005] regarding dynamic output feedback controller synthesis; Mulder et al. [2001], Grimm et al. [2003] and Gomes da Silva Jr. and Tarbouriech [2005] addressing the anti-windup synthesis. On the other hand, just few works deal with the problem of controlling nonlinear systems with saturating inputs in a systematic way. In the same line of the works cited above, LMI conditions have also been proposed to synthesize stabilizing control laws for nonlinear systems subject to actuator amplitude limitations and for which the dynamics can be decomposed into the feedback interconnection of a linear system with a sector bounded nonlinearity: Castelan et al. [2005, 2006, 2008 (to appear] for precisely-known systems and Castelan et al. [2007] for some uncertain nonlinear systems. It should however be pointed out that, as in Arcak et al. [2003], Arcak and Kokotovic [2001], the considered control law consists of the feedback of the systems states and of the non-linearity associated to the plant dynamics.

In the above context of nonlinear systems subject to control saturations, the present work extends the results in Castelan et al. [2005] by considering the synthesis of a dynamic output feedback controller. The controller structure is composed by a linear compensator presenting the following inputs: the plant output, an anti-windup term (related to the input saturation) and the value of the plant sector bounded nonlinearity. Based on this structure, on the use of a quadratic Lyapunov function, and on sector conditions, LMI sufficient conditions are stated to ensure the local stability of the closed-loop system in a specific region or, provided some additional hypothesis are satisfied by the open-loop system, to ensure global asymptotic stability. From these conditions convex optimization problems are proposed in order to compute the controller matrices aiming at the maximization of the estimates of the basin of attraction of the closed-

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loop system, or the performance enhancement with a guaranteed region of stability.

It should be pointed out that similar control problems have been studied in Kiyama et al. [2005], where the authors separately consider a particular sector bounded nonlinearity (with dead-zone behavior) associated the dynamics of the plant or a static saturation nonlinearity, using in both cases a classical sector condition, which leads to BMI stabilization conditions. Differently from that work, our approach consider simultaneously a dynamic nonlinearity and a static input saturation. Furthermore, the stated conditions will be stated directly in LMI form. Thus, the proposed results can be thought as an additional contribution for the treatment of more realistic nonlinear control systems, also in a more efficient computational way.

The paper is organized as follows. Section 2 presents the problem statement. Some preliminary results are presented in section 3. Section 4 is concerned with the proposition of the local and global stabilization conditions, and also presents some convex optimization problems for synthesis of the dynamic controller. Numerical examples are presented and commented in section 5. The paper finishes with some concluding remarks.

Notations. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A and $He\{A\} = A + A'$. $A_{(i)}$ denotes the ith row of matrix A. $\star$ stands for symmetric blocks; $\bullet$ stands for an element that has no influence on the development. I denotes an identity matrix of appropriate order. $diag\{A, B\}$ is the block-diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

2. PROBLEM STATEMENT

Consider a nonlinear continuous-time system represented by the Lur'e type system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + G\phi(z(t)) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and the control input, respectively, $y(t) \in \Re^p$ corresponds to the measured output and $z(t) \in \Re^q$ is the input to the nonlinear vector valued function $\phi(\cdot) : \Re^q \to \Re^q$. A, B, C, G and L are real constant matrices of appropriate dimensions.

Regarding system (1) the following assumptions are considered:

A1. The nonlinearity $\phi(z)$ is continuous and verifies a cone bounded sector condition, i.e., $\phi(0) = 0$ and there exists a symmetric positive definite matrix $\Omega \in \Re^{q \times q}$ such that

$$\phi(z)'\Delta(\phi(z) - \Omega z) \leq 0, \quad \forall z \in \mathcal{S}_1 \subseteq \Re^q \tag{2}$$

where $\Delta \in \Re^{q \times q}$ is any diagonal matrix defined as follows:

$$\Delta = \begin{cases} diag(\delta_l), & \delta_l > 0, \ \text{ if } \phi(\cdot) \text{ is decentralized;} \\ \delta I_q, & \delta > 0, \ \text{ otherwise.} \end{cases}$$

The matrix Ω is supposed to be known. On the other hand, as it will be seen in the sequel, the matrix Δ will represent a degree of freedom in the controller design method (Castelan et al. [2007]). If $\mathcal{S}_1 = \Re^q$, then the sector condition (2) is globally verified, otherwise, it is only locally verified, as it will be characterized in the sequel.

A2. The system output $y(t)$ and the output of nonlinearity $\phi(z(t))$ are available for measurement.

A3. The control inputs are supposed to be bounded as follows:

$$-u_{0(i)} \leq u_{(i)} \leq u_{0(i)}, \quad i = 1, \ldots, m \tag{3}$$

In consequence of the control bounds, the actual control signal to be injected in the system is a saturated one, i.e, considering the signal sent to the actuator given by $v(t)$, we have

$$u(t) = sat(v(t)) \tag{4}$$

where each component of $sat(v)$ is defined, $\forall i = 1, ..., m$, by: $sat(v)_{(i)} = sat(v_{(i)}) = sign(v_{(i)}) \min(u_{0(i)}, |v_{(i)}|)$.

Consider now a nonlinear dynamic output feedback controller with the following structure:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u_c(t) + \\ &\qquad E_c(sat(v(t)) - v(t)) + G_c\phi(z(t)) \\ y_c(t) &= C_c x_c(t) + D_c u_c(t) + F_c\phi(z(t)) \end{aligned} \tag{5}$$

where $x_c(t) \in \Re^n$ is the controller state, $u_c(t)$ is the controller input and $y_c(t)$ is the controller output, matrices A_c, B_c, C_c, D_c, E_c, F_c and G_c are of appropriate dimensions. The term $E_c(sat(v(t)) - v(t))$ corresponds to a static anti-windup loop to mitigate the undesirable effects of windup caused by input saturation. The interconnection between the plant and the controller is given by: $v(t) = y_c(t)$, $u_c(t) = y(t)$.

In this paper, we are interested in the synthesis of dynamic output feedback controllers (5) (i.e. the computation of the controller matrices A_c, B_c, C_c, D_c, E_c, G_c and F_c), taking into account the control saturation effects and the nonlinear closed-loop behavior, in order to ensure regional (local) or, when possible, global asymptotic stability of the origin of the closed-loop system. In the regional case, we are interested in ensuring the asymptotic stability for a certain set of admissible initial conditions. An implicit problem in this case consists in computing the controller in order to maximize the domain of attraction of the closed-loop system. Another interesting problem is the performance improvement with a guaranteed region of stability for the closed-loop system. These problems will be addressed in the sequel.

3. PRELIMINARIES

Define the following matrices:

$$\mathbb{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}, \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \mathbb{R} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \mathbb{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}$$

$$\mathbb{K} = [\, D_cC \;\; C_c \,] \; and \; \mathbb{L} = [\, L \;\; 0 \,].$$

Hence, considering an augmented state vector $\xi(t) = \left[\, x(t)' \;\; x_c(t)' \,\right]'$, the closed-loop system composed by the connection of the system (1) and the controller (5) reads:

$$\begin{aligned} \dot{\xi}(t) &= \mathbb{A}\xi(t) + (\mathbb{B}F_c + \mathbb{G} + \mathbb{R}G_c)\phi(z(t)) - \\ &\qquad (\mathbb{B} + \mathbb{R}E_c)\psi(y_c(t)) \end{aligned} \tag{6}$$

where

$$\begin{aligned} y_c(t) &= \mathbb{K}\xi(t) + F_c\phi(z(t)) \\ \psi(y_c(t)) &= y_c(t) - sat(y_c(t)) \end{aligned} \tag{7}$$

with $(\psi(y_c))_{(i)} \triangleq y_{c(i)} - sat(y_c)_{(i)}, \quad i = 1, \ldots, m$. Note that, $\psi(y_c)$ corresponds to a decentralized deadzone nonlinearity.

Considering a matrix $H = [\, H_\xi \; H_\phi \,] \in \Re^{m\times(2n+q)}$ and defining the polyhedral set

$$\mathcal{S}_2 \triangleq \{ \begin{bmatrix} \xi \\ \phi \end{bmatrix} \in \Re^{2n+q};|\ [(\mathbb{K} - H_\xi) \;\; (F_c - H_\phi)]_{(i)} \begin{bmatrix} \xi \\ \phi \end{bmatrix} | \le u_{0(i)}, \quad i = 1, ..., m\}$$

the following Lemma, concerning the nonlinearity $\psi(y_c)$ can be stated (Gomes da Silva Jr. and Tarbouriech [2005]).

Lemma 1. If $\begin{bmatrix} \xi \\ \phi \end{bmatrix} \in \mathcal{S}_2$ then the relation

$$\psi(y_c)'T\,(\psi(y_c) - H_\xi \xi - H_\phi \phi(z)) \le 0 \tag{8}$$

is verified for any $T \in \Re^{m\times m}$ diagonal and positive definite.

4. STABILIZATION

4.1 Regional (Local) Stabilization

In this case we consider the set $\mathcal{S}_1$ defined as follows:

$$\begin{aligned} \mathcal{S}_1 &\triangleq \{z \in \Re^q \; ; \; |L_{(i)}z| \le \rho_{(i)}, \rho_{(i)} > 0, \; i = 1, \ldots, q\} \\ &= \{\xi \in \Re^{2n} \; ; \; |\mathbb{L}_{(i)}\xi| \le \rho_{(i)}, \rho_{(i)} > 0, \; i = 1, \ldots, q\}. \end{aligned}$$

Theorem 1. If there exist symmetric positive definite matrices $X,\ Y \in \Re^{n\times n}$, positive definite diagonal matrices $S \in \Re^{m\times m}$, $S_\Delta \in \Re^{q\times q}$, matrices $\hat{A} \in \Re^{n\times n}$, $\hat{B} \in \Re^{n\times p}$, $\hat{C}, \hat{H}_{\xi 1}, \hat{H}_{\xi 2} \in \Re^{m\times n}$, $\hat{H}_\phi \in \Re^{m\times q}$, $\hat{D} \in \Re^{m\times p}$, $\hat{F} \in \Re^{m\times q}$, $\hat{G} \in \Re^{n\times q}$, and a scalar $\nu > 0$ such that the following linear matrix inequalities are verified

$$\begin{bmatrix} \mathbf{J_1} & A + \hat{A}' + B\hat{D}C & \mathbf{J_2} & -BS + \hat{H}'_{\xi 1} \\ \star & He\{YA + \hat{B}C\} & \mathbf{J_3} & \hat{E} + \hat{H}'_{\xi 2} \\ \star & \star & -2S_\Delta & \hat{H}'_\phi \\ \star & \star & \star & -2S \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} X & \star & \star & \star \\ I & Y & \star & \star \\ \Omega L X & \Omega L & 2S_\Delta & \star \\ \hat{C}_{(i)} - \hat{H}_{\xi 1(i)} & \hat{D}_{(i)}C - \hat{H}_{\xi 2(i)} & \hat{F}_{(i)} - \hat{H}_{\phi(i)} & \nu\, u^2_{0(i)} \end{bmatrix} \ge 0 \quad i = 1, ..., m \tag{10}$$

$$\begin{bmatrix} X & \star & \star \\ I & Y & \star \\ L_{(i)}X & L_{(i)} & \nu\, \rho^2_{(i)} \end{bmatrix} \ge 0 \quad i = 1, ..., q \tag{11}$$

where $\mathbf{J_1} = He\{AX + B\hat{C}\}$, $\mathbf{J_2} = GS_\Delta + B\hat{F} + XL'\Omega$ and $\mathbf{J_3} = \hat{G} + L'\Omega$, then the dynamic controller (5) with

$$\begin{aligned} &A_c = V^{-1}[\hat{A} - (YAX + YB\hat{C} + VB_cCX)](U')^{-1} \\ &B_c = V^{-1}(\hat{B} - YB\hat{D}), \quad C_c = (\hat{C} - \hat{D}CX)(U')^{-1} \\ &D_c = \hat{D}, \quad E_c = -V^{-1}(\hat{E}S^{-1} + YB) \\ &G_c = -V^{-1}(-\hat{G}S_\Delta^{-1} + YG + YBF_c), \quad F_c = \hat{F}S_\Delta^{-1} \end{aligned} \tag{12}$$

where matrices U and V verify $VU' = I - YX$, guarantees that the region $\mathcal{E}(P, \nu^{-1}) = \{\xi \in \Re^{2n}; \xi'P\xi \le \nu^{-1}\}$ with $P = \begin{bmatrix} Y & V \\ V' & \bullet \end{bmatrix}$ is a domain of asymptotic stability for the closed-loop system (6).

Proof: Define a candidate Lyapunov function

$$V(t) = \xi'(t)P\xi(t) \tag{13}$$

with $P = \begin{bmatrix} Y & V \\ V' & \bullet \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} X & U \\ U' & \bullet \end{bmatrix}$. It follows that

$$\begin{aligned} \dot{V}(t) = \; & 2\xi'(t)\mathbb{A}'P\xi(t) - 2\psi'(y_c(t))(\mathbb{B} + \mathbb{R}E_c)'P\xi(t) + \\ & 2\phi'(z(t))(\mathbb{B}F_c + \mathbb{G} + \mathbb{R}E_c)'P\xi(t) \end{aligned} \tag{14}$$

From Lemma 1 and A1, provided that $\xi(t) \in \mathcal{S}_1 \cap \mathcal{S}_2$, it follows that:

$$\begin{aligned} \dot{V}(t) \le \; & \dot{V}(t) - 2\psi(y_c(t))'T(\psi(y_c(t)) - H_\xi\xi(t) - \\ & H_\phi\phi(z(t))) - 2\phi(z(t))'\Delta(\phi(z(t)) - \Omega\mathbb{L}\xi(t)) \end{aligned} \tag{15}$$

Re-writing this expression in matrix form, it follows that $\dot{V}(\xi(t)) \le \eta(t)'\Gamma\eta(t)$ with $\eta(t) = [\, \xi(t)' \;\; \phi(z(t))' \;\; \psi(y_c(t))' \,]'$ and

$$\Gamma = \begin{bmatrix} He\{\mathbb{A}'P\} & P\mathbf{J_4} + \mathbb{L}'\Omega\Delta & -P\mathbf{J_5} + H'_\xi T \\ \star & -2\Delta & H'_\phi T \\ \star & \star & -2T \end{bmatrix} \tag{16}$$

where $\mathbf{J_4} = \mathbb{G} + \mathbb{R}G_c + \mathbb{B}F_c$ and $\mathbf{J_5} = \mathbb{B} + \mathbb{R}E_c$.

Define now a matrix $\Pi = \begin{bmatrix} X & I \\ U' & 0 \end{bmatrix}$ (Scherer et al. [1997]). Note that, from condition (10), it follows that $I - YX$ is nonsingular, which implies that is always possible to compute square and nonsingular matrices V and U verifying the equation $VU' = I - YX$. This fact ensures that Π is nonsingular.

Pre and post-multiplying (16) respectively by $Diag(\Pi', S'_\Delta, S')$ and $Diag(\Pi, S_\Delta, S)$, with $S_\Delta = \Delta^{-1}$ and $S = T^{-1}$, one gets:

$$\begin{bmatrix} He\{\Pi'\mathbb{A}'P\Pi\} & \Pi'P\mathbf{J_4}S_\Delta + \Pi'\mathbb{L}'\Omega & -\Pi'P\mathbf{J_5}S + \Pi'H'_\xi \\ \star & -2S_\Delta & S_\Delta H'_\phi \\ \star & \star & -2S \end{bmatrix}. \tag{17}$$

Considering the following change of variables:

$$\begin{aligned} &\hat{A} = YAX + YBD_cCX + VB_cCX + YBC_cU' + VA_cU', \\ &\hat{B} = YBD_c + VB_c, \quad \hat{C} = C_cU' + D_cCX, \quad \hat{D} = D_c, \\ &\hat{E} = -(YBS + VE_cS), \quad \hat{F} = F_cS_\Delta, \\ &\hat{G} = YGS_\Delta + VG_cS_\Delta + YBF_cS_\Delta, \quad \hat{H}_\phi = H_\phi S_\Delta \\ &\hat{H}_{\xi 1} = H_{\xi 1}X + H_{\xi 2}U', \quad \hat{H}_{\xi 2} = H_{\xi 1} \end{aligned}$$

it follows that

$$\begin{aligned} &\Pi'P\Pi = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}; \; \Pi'P\mathbf{J_5}S = \begin{bmatrix} BS \\ -\hat{E} \end{bmatrix}; \\ &\Pi'P\mathbf{J_4}S_\Delta = \begin{bmatrix} GS_\Delta + B\hat{F} \\ \hat{G} \end{bmatrix}; \Pi'H'_\xi = \begin{bmatrix} \hat{H}'_{\xi 1} \\ \hat{H}'_{\xi 2} \end{bmatrix}; \\ &\Pi'P\mathbb{A}\Pi = \begin{bmatrix} AX + B\hat{C} & A + B\hat{D}C \\ \hat{A} & YA + \hat{B}C \end{bmatrix}. \end{aligned} \tag{18}$$

Hence, since Π, S_Δ and S are nonsingular, it follows that (9) is equivalent to $\Gamma < 0$, which, from (15), implies that $\dot{V}(t) < 0$ holds with the matrices A_c, B_c, C_c, D_c, E_c, F_c and G_c defined as in (12).

Consider now $\mathcal{E}(P,\nu^{-1})$. Pre and post-multiplying inequalities (10) respectively by $\begin{bmatrix} (\Pi^{-1})' & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and its transpose, and since $\mathbb{K}\Pi = [D_cCX + C_cU' \quad D_cC] = [\hat{C} \quad \hat{D}C]$, it follows that condition (10) ensures that $\mathcal{E}(P,\nu^{-1}) \subset \mathcal{S}_2$. By similar reasoning, (11) implies that $\mathcal{E}(P,\nu^{-1}) \subset \mathcal{S}_1$ (Castelan et al. [2005]). Thus, if relations (9),(10),(11) are verified, one effectively obtains $\dot{V}(t) < 0$, $\forall \xi \in \mathcal{E}(P,\nu^{-1})$, which concludes the proof □

Remark 1. The result of the Theorem can be straightforwardly extended to treat the local stabilization when the nonlinearity $\phi(z(t))$ globally satisfies the sector condition (2), i.e., when $\mathcal{S}_1 = \Re^q$. For this, it suffices to consider $H_\phi = F_c$ in condition (9), to eliminate the third row and third column matrices in (10) and to eliminate (11).

4.2 Global Stabilization

In this case we consider the set $\mathcal{S}_1 = \Re^q$ and that the open-loop matrix A is Hurwitz.

Corollary 1. If there exist symmetric positive definite matrices X, $Y \in \Re^{n\times n}$, positive definite diagonal matrices $S \in \Re^{m\times m}$, $S_\Delta \in \Re^{q\times q}$ and matrices $\hat{A} \in \Re^{n\times n}$, $\hat{B} \in \Re^{n\times p}$, $\hat{C}$, $\hat{H}_{\xi 1}$, $\hat{H}_{\xi 2} \in \Re^{m\times n}$, $\hat{H}_\phi \in \Re^{m\times q}$, $\hat{D} \in \Re^{m\times p}$, $\hat{F} \in \Re^{m\times q}$ and $\hat{G} \in \Re^{n\times q}$ such that the following linear matrix inequalities are verified

$$\begin{bmatrix} \mathbf{J_1} & A+\hat{A}'+B\hat{D}C & \mathbf{J_2} & -BS+\hat{C}' \\ \star & He\{YA+\hat{B}C\} & \mathbf{J_3} & \hat{E}+C'\hat{D} \\ \star & \star & -2S_\Delta & \hat{F}' \\ \star & \star & \star & -2S \end{bmatrix} < 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{19}$$

then the dynamic controller (5) with the matrices defined as in (12), where matrices U and V verify $VU' = I - YX$, guarantees the global asymptotic stability of the origin of the closed-loop system (6).

Proof: Consider $H_\xi = \mathbb{K}$ and $H_\phi = F_c$. It follows that the sector condition $\psi(y_c(t))'T\,(\psi(y_c(t)) - \mathbb{K}\xi(t) - F_c\phi(z(t))) \leq 0$ is verified for all $\xi(t) \in \Re^{2n}$ and $\phi(z(t)) \in \Re^q$. In this case, it is easy to see that (19) corresponds to (9) and the global asymptotic stability of the origin follows.

4.3 Optimization Problems

Enlargement of the stability region

An implicit objective in the synthesis of the stabilizing controller (5) can be the maximization of estimates of the basin of attraction associated to the closed-loop system. In other words, we want to compute (5) such that the associated region of asymptotic stability is as large as possible considering some size criterion. This can be addressed if, for instance, we consider a set Ξ with a given shape and a scaling factor β. This shape set can be easily defined as a polyhedral set described by the convex hull of its vertices:

$$\Xi \stackrel{\triangle}{=} Co\{v_1, v_2, \ldots, v_{n_r}\},\ v_l \in \Re^{2n},\ l = 1, \ldots, n_r$$

Hence, recalling Theorem 1, we aim at searching for matrices $X, Y, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{H}_\xi$ and $\hat{H}_\phi$ in order to obtain $\beta\,\Xi \subset \mathcal{E}(P,\nu^{-1})$ with β as large as possible.

The vectors v_l can therefore be viewed as directions in which we want to maximize the region of attraction. Considering that $\beta v_l \in \mathcal{E}(P,\nu^{-1})$ is equivalent to

$$\beta v_l' P v_l \beta \leq \nu^{-1} \tag{20}$$

and considering $\eta = 1/\beta^2$, it follows that the maximization of the ellipsoid $\mathcal{E}(P,\nu^{-1})$ along the directions v_l is equivalent to the minimization of η. Hence, for a given value $\nu > 0$, $\mathcal{E}(P,\nu^{-1})$ can be maximized along the directions given by generic vectors $v_l = \begin{bmatrix} v_{l1}' & v_{l2}' \end{bmatrix}'$ where $v_{l1} \in \Re^n$ and $v_{l2} \in \Re^n$, by solving the following convex optimization problem:

$$\min_{V,\eta} \eta$$
$$\text{subject to}$$
$$(i) \begin{bmatrix} \nu^{-1}\eta & v_{l1}' & v_{l1}'Y + v_{l2}'V' \\ v_{l1} & X & I \\ Yv_{l1} + Vv_{l2} & I & Y \end{bmatrix} \geq 0 \tag{21}$$
$$l = 1, \ldots, r$$
$$(9),\ (10) \text{ and } (11),$$

where X and Y are given matrices verifying the conditions of Theorem 1.

In order to prove this, it suffices to apply Schur's complement in (20) and, to pre and post multiply the obtained matrix inequality respectively by $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & Y & V \end{bmatrix}$ and F'.

It is worth noticing that matrix V appears explicitly in (21). In this case, once V is obtained, it should be verified *a posteriori* if it is indeed invertible. Alternatively, a constraint of type $V + V' > 0$ (or < 0) can be incorporated in the optimization problem to ensure that V will not be singular.

On the other hand, in practice, we are mainly interested in maximizing the region of stability in directions associated to the states of the plant. In this case the vectors v_l assume the form $\begin{bmatrix} v_{l1}' & 0 \end{bmatrix}'$ and, from (20), it follows that (i) in (21) can be replaced by the constraint:

$$v_{l1}' Y v_{l1} \leq \nu^{-1}\eta, \quad l = 1, \ldots, r \tag{22}$$

Performance Improvement

Let Ξ be a given set in the state space, for which we want to ensure that $\forall \xi(0) \in \Xi$, $\xi(t) \to 0$ as $t \to \infty$. Among all the feasible controllers ensuring that, one may be interested in improving the performance of the closed-loop system.

A natural performance measure is given by the following quadratic criterion on plant states:

$$\mathcal{J} = \int_0^\infty x(t)'Qx(t)dt \quad \text{where} \quad Q = Q' \geq 0, \quad Q \in \Re^{n\times n}.$$

If we are now able to satisfy

$$\dot{V} + \frac{1}{\gamma}\xi' \begin{bmatrix} I \\ 0 \end{bmatrix} Q\,[\,I \ \ 0\,]\,\xi < 0, \tag{23}$$

it follows that $\mathcal{J} < \gamma V(0) < \gamma\,\nu^{-1}$, $\quad \forall \xi\ (0) \in \mathcal{E}(P,\nu^{-1})$.

Note that (23) is satisfied if, for $\mu = \frac{1}{\gamma}$:

$$\begin{bmatrix} \mathbf{J_1} & A+\hat{A}'+B\hat{D}C & X(Q^{1/2}) & \mathbf{J_2} & -BS+\hat{H}'_{\xi 1} \\ \star & He\{YA+\hat{B}C\} & (Q^{1/2}) & \mathbf{J_3} & \hat{E}+\hat{H}'_{\xi 2} \\ \star & \star & -\frac{1}{\mu}I & 0 & 0 \\ \star & \star & \star & -2S_\Delta & \hat{H}'_\phi \\ \star & \star & \star & \star & -2S \end{bmatrix} < 0 \quad (24)$$

Another interesting performance criterion is the maximization of the exponential convergence of the trajectories. Note that if we ensure that

$$\dot{V} + \mu \xi' P \xi < 0 \quad (25)$$

it follows that $V(t) < e^{-\mu t}V(0), \ \ \forall \xi\ (0) \in \mathcal{E}(P, \nu^{-1})$. This fact guarantees exponential convergence of the trajectories to the origin with a rate given by μ. The relation (25) is satisfied if

$$\begin{bmatrix} \mathbf{J_1}+\mu X & A+\hat{A}'+B\hat{D}C+\mu I & \mathbf{J_2} & -BS+\hat{H}'_{\xi 1} \\ \star & He\{YA+\hat{B}C\}+\mu Y & \mathbf{J_3} & \hat{E}+\hat{H}'_{\xi 2} \\ \star & \star & -2S_\Delta & \hat{H}'_\phi \\ \star & \star & \star & -2S \end{bmatrix} < 0, \quad (26)$$

Note that (26) ensures that all the eigenvalues of matrix $\mathbb{A}$ have real part smaller than $-\mu$.

The following convex optimization problem can therefore be formulated to take into account performance issues with a guaranteed region of stability:

$$\begin{aligned} &\max \mu \\ &\text{subject to} \\ &\begin{bmatrix} \nu^{-1} & v'_{l1} & v'_{l1}Y + v'_{l2}V' \\ v_{l1} & X & I \\ Yv_{l1}+Vv_{l2} & I & Y \end{bmatrix} > 0 \quad (27) \\ &\qquad l = 1, \ldots, r \\ &\text{(11) and (24) (or (26))} \end{aligned}$$

In the case where the objective is the maximization of the exponential convergence of the trajectories by using (26), problem (27) can be efficiently solved as a GEVP (Boyd et al. [1994]).

Remark 2. Condition (26) can also be used to adapt the convex optimization problem (21) in order to maximize the size of $\mathcal{E}(P, \nu^{-1})$ or $\mathcal{E}(Y, \nu^{-1})$ while guaranteeing a pre-specified degree of exponential convergence inside it. For that, it suffices to replace condition (9) by (26), with a fixed value $\mu > 0$.

5. NUMERICAL EXAMPLES

Consider the following data for the nonlinear system (1):

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ C = [\,1\ 0\,],$$
$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ L = [\,1\ 1\,], \Omega = 1.4, \ u_0 = 5.$$

Since A is not Hurwitz, only local (regional) stabilization is possible. Thus, we first solve the convex problem (21), with $\mu = 1$ (see Remark 2), and with $\Xi = Co\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ [1].

[1] An additional constraint on the real parts of the closed-loop poles, $\Re(\lambda(\mathbb{A})) > -10$, has been considered to guarantee well-conditioned solutions of the presented numerical examples.

ν	$\phi \in \mathcal{S}_1$, with $\rho = 1.8$		$\phi \in \Re^p$	
	β	Area	β	Area
10	1.7278	17.6198	4.9919	88.0903

Table 1. Enlargements for $\nu = 10$

For $\nu = 10$, Table 1 shows the obtained scaling factor of Ξ and the area of $\mathcal{E}(Y, \nu^{-1})$ (given by $\pi\sqrt{det((\nu Y)^{-1})}$), by considering that the nonlinearity $\phi(\cdot)$ is either locally or globally verified. $\mathcal{E}(Y, \nu^{-1})$ corresponds to the intersection of $\mathcal{E}(P, \nu^{-1})$ with the hyperplane defined by the plant states, *i.e.:* $\xi' = [\,x'\ 0\,]$. As expected, a larger domain of stability is obtained when $\phi(\cdot)$ is globally verified. Figure 1 shows the ellipsoidal sets $\mathcal{E}(Y, \nu^{-1})$ obtained for $\nu = 10$. The corresponding obtained compensator parameters are

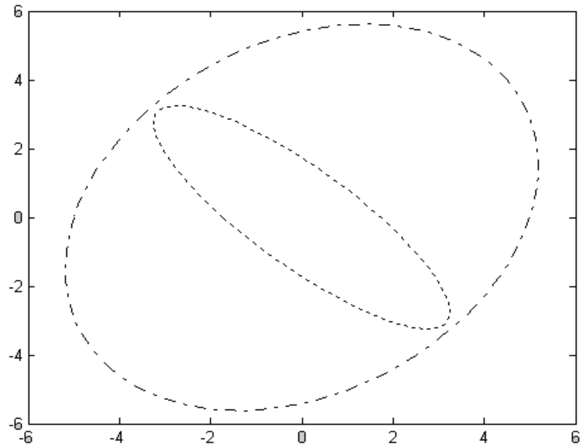

Fig. 1. $\mathcal{E}(Y, 10^{-1})$ for ϕ locally (- -) and globally (-.) verified

$$A_c = \begin{bmatrix} -8297.6 & 8692.6 \\ -8266 & 8285.3 \end{bmatrix}, B_c = \begin{bmatrix} -48.5817 \\ -62.1318 \end{bmatrix},$$
$$C_c = [\,-140.89\ 141.23\,], \ D_c = -3.2068$$
$$E_c = \begin{bmatrix} -6.9411 \\ 0.0286 \end{bmatrix}, \ F_c = -0.3700, \ G_c = \begin{bmatrix} 0.0515 \\ 0.0415 \end{bmatrix}$$

Let us now consider, for $\nu = 1$, the minimization of the upper-bound $\gamma = \gamma\nu^{-1}$ for the quadratic criterion $\mathcal{J}$, with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, by solving the convex optimization problem (27). In this case, we consider $\Xi = Co\left\{ \begin{bmatrix} \kappa \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \kappa \\ 0 \\ 0 \end{bmatrix} \right\}$, where $\kappa > 0$ may assume different values. Tables 2 and 3 show results obtained when $\phi(\cdot)$ is locally and globally verified, respectively. By comparing the two tables, we observe again better results when the sector condition on $\phi(\cdot)$ is globally verified, since smaller guaranteed performances are obtained for greater values of κ in Table 3. Now, in each table it can also be noticed a trade-off between involving the size of the guaranteed region of stability and the upper-bound for $\mathcal{J}$. In fact, the greater is κ and, in consequence, the area of the obtained $\mathcal{E}(Y, \nu^{-1})$, the worse is the upper-bound obtained for $\mathcal{J}$. The tables show also the value of $\sqrt{det(\nu P)^{-1}}$, which is proportional to the volume of the whole stability region $\mathcal{E}(P, \nu^{-1})$.

Table 4 shows, for $\kappa = 1$ and $\nu = 1$, the controllers parameters when $\phi(\cdot)$ is locally or globally verified.

κ	γ	Area	$\sqrt{det(\nu P)^{-1}}$
1	0.1182	3.3155	37.6783
1.5	0.8644	7.6838	875.0125
1.73	182.5002	17.7126	1.4958×10^5
1.74	-	-	-

Table 2. Guaranteed quadratic performance with $\nu = 1$ and for ϕ locally verified

κ	γ	Area	$\sqrt{det(\nu P)^{-1}}$
1	0.0647	3.6020	8.8591
3	3.6963	32.0707	24.6330
4.99	6.3609×10^3	87.9536	645.9406
5	-	-	-

Table 3. Guaranteed quadratic performance with $\nu = 1$ and for ϕ globally verified

	$\phi \in \mathcal{S}_1$, with $\rho = 1.8$	$\phi \in \Re^p$
A_c	$\begin{bmatrix} -6.9325 & 3.4228 \\ -2.3843 & -3.8176 \end{bmatrix}$	$\begin{bmatrix} -3.168 & -2.176 \\ -0.916 & -3.465 \end{bmatrix}$
B_c	$\begin{bmatrix} 0.0074 \\ -2.8429 \end{bmatrix}$	$\begin{bmatrix} 0.9106 \\ -0.3858 \end{bmatrix}$
C_c	$\begin{bmatrix} -3.4074 & 1.3041 \end{bmatrix}$	$\begin{bmatrix} -5.081 & 4.294 \end{bmatrix}$
D_c	-4.5284	-4.5284
E_c	$\begin{bmatrix} 0.0456 \\ 0.3090 \end{bmatrix}$	$\begin{bmatrix} -0.4025 \\ 0.0981 \end{bmatrix}$
F_c	-0.3768	-0.3252
G_c	$\begin{bmatrix} 0.3100 \\ 0.2235 \end{bmatrix}$	$\begin{bmatrix} 0.0230 \\ 0.0602 \end{bmatrix}$

Table 4. Controllers parameters for $\kappa = 1$

6. CONCLUSION

In the present work we have addressed the stabilization problem of Lur'e type nonlinear systems subject to input saturation. Constructive LMI results allowing to compute a nonlinear dynamic controller having as inputs both the plant output and the output of the dynamic nonlinearity have been proposed. From these LMI conditions, convex optimization problems in order to compute the controller aiming at maximize an ellipsoidal estimate of the closed-loop domain of attraction or at improving the performance of the closed-loop system while guaranteeing a pre-specified region of stability have been formulated.

Further investigation points to the synthesis of reduced-order controllers and to the problem of disturbance rejection.

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