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Partial Semi-Stability for a Class of Nonlinear Systems ⋆

Eduardo F. Costa * **Alessandro Astolfi** **

* *Depto. de Matemática Aplicada e Estatística, Universidade de São Paulo, C.P. 668, 13560-970, São Carlos, SP, Brazil, currently visiting the Electrical and Electronic Engineering Dept., Imperial College London, SW7 2AZ, London, UK, (e-mail: efcosta@icmc.usp.br).*
** *Electrical and Electronic Engineering Dept., Imperial College London, SW7 2AZ, London, UK, and Dipartimento di Informatica, Sistemi e Produzione, University of Rome "Tor Vergata", 00133 Roma, Italy, (e-mail: a.astolfi@imperial.ac.uk)*

Abstract: This paper studies partial semi-stability for a class of non-linear systems. The system is sufficiently specialised to yield an algebraic test relying on the data A and Σ, describing the dynamics of part of the state of the system and its initial condition. Comments on the applicability of the result to the study of stability properties of Kalman filters are included.

Keywords: Partial stability; nonlinear systems; divergence analysis.

1. INTRODUCTION

Consider a non-linear system described by the equations

$$\Theta : \begin{cases} Z_{k+1} = H_k A Z_k A' H_k', \\ X_{k+1} = A X_k A', \qquad k \geq 0, \\ (Z_0, X_0) = (H_0 V H_0', \Sigma) \end{cases} \tag{1}$$

where (Z_k, X_k) is the state, Z_k and X_k are symmetric positive semidefinite matrices and H_k, $k \geq 0$, stands for the orthogonal projection onto the null space of X_k. The square matrix A is assumed to be known, $V = V' \geq 0$ and $\Sigma = \Sigma' \geq 0$. Note that the dynamics of the Z-component is coupled to the X-component via the projections H_k.

In this paper we are concerned with partial semi-stability (PSS) of system Θ with respect to (w.r.t.) V, that is, semi-stability [1] of the Z-component w.r.t. V, for a fixed Σ. Namely, for each $V = V' \geq 0$, it is required the existence of $\bar{Z}$ such that $\xi^k Z_k \leq \bar{Z}$, $k \geq 0$ and $0 \leq \xi < 1$, meaning that the Z-component can not diverge exponentially (polynomial divergence is allowed). We shall refer to this problem simply as PSS.

PSS is strongly linked, via [Costa and Astolfi, a, Theorem 1], to the exponential divergence of Kalman filters for systems with incorrect noise information. Obtaining a testable condition for PSS allows to obtain a sharp condition for divergence of Kalman filters, as discussed in Costa and Astolfi [b]. This is an important result, in view of the conservativeness of existing conditions, see Price [1968], Fitzgerald [1971], Sangsuk-Iam and Bullock [1990] and Willems and Callier [1992], which are either necessary or sufficient, or rely on additional assumptions, such as the existence of limiting stationary filters.

Partial stability was studied for linear and non-linear systems, see Chellaboina and Haddad [2002], Djaferis [2006], Molchanov et al. [2003], Nersesov and Haddad [2006] and Vorotnikov [1998]. In principle, results concerning general non-linear systems, such as Lyapunov V-functions that are positive definite w.r.t. part of the variables, could be exploited. However, they are too general to yield the easy to test algebraic condition that we are seeking. In addition, there is no available result that takes into account the special features of Θ, mainly the connections with linear systems: the X-component obeys a linear difference equation and the coupling with the Z-component is via an orthogonal projection. Finally, the available results for partial stability of linear systems do not apply directly to Θ, and it is worth mentioning that it is inappropriate to deal with the problem assuming that H_k are general projections, not connected with X, *in order to retrieve linearity*; in fact, in such modified setting, Z_k can diverge exponentially whereas A is stable [2] (A stable implies PSS, see Remark 1).

This paper takes into account the special features of Θ to show that it is PSS if and only if

$$\ker\{J\Sigma J^{-1}\} \cap \mathscr{J} = \{0\}, \tag{2}$$

where J is the similarity transformation such that JAJ^{-1} is in Jordan form and $\mathscr{J}$ stands for the unstable subspace [3] of JAJ^{-1}. Recalling from linear systems theory that (A, Σ) semi-stabilizable can be interpreted as requiring that Σ excites the strictly unstable modes of A, the interpretation of (2) is that Σ has to "completely excite" all the unstable modes of A. Condition (2) is stronger than semi-stabilizability of (A, Σ) and not comparable to stabilizability of (A, Σ), see Remark 1.

The paper is organised as follows. Section 2 presents definitions and preliminary results involving a sequence of bases that allow to derive a simple structure for A and to simplify the evaluation

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[1] Following the terminology of Abou-Kandil et al. [2003].

[2] For example, consider the case $H_k = V = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $k \geq 0$, and $A = \frac{3}{4}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

[3] Please see Section II for definitions.

of the projections H_k. These results allow to obtain the testable condition for PSS is Section 3. Finally, Section 4 provides some conclusions.

2. DEFINITIONS AND PRELIMINARY RESULTS

Let $\mathbb{R}^n$ denote the $n-$th dimensional Euclidean space. Let $\mathbb{D}$ (respectively $\bar{\mathbb{D}}$) be the open (closed) unit disk. Let e_i, $i = 1,\dots,n$ be the canonical basis of $\mathbb{R}^n$. $[v_1,\dots,v_m]$ stands for the vector space spanned by $v_1,\dots,v_m \in \mathbb{R}^n$. For vector subspaces $\mathscr{E}$ and $\mathscr{F}$, $\mathscr{E} \perp \mathscr{F}$ means that $\mathscr{E}$ and $\mathscr{F}$ are orthogonal, $\mathscr{E}^\perp$ is such that $\mathscr{E}^\perp \perp \mathscr{E}$, $\mathscr{E} \oplus \mathscr{F}$ is the direct sum of $\mathscr{E}$ and $\mathscr{F}$, and $\mathscr{E} \ominus \mathscr{F} = \mathscr{E} \cap \mathscr{F}^\perp$. Let $\mathscr{R}^{r,s}$ (respectively, $\mathscr{R}^r$) represent the normed linear space formed by all $r \times s$ real matrices (respectively, $r \times r$) and $\mathscr{R}^{r*}$ ($\mathscr{R}^{r0}$) the cone $\{U \in \mathscr{R}^r : U = U'\}$ (the closed convex cone $\{U \in \mathscr{R}^r : U = U' \geq 0\}$) where U' denotes the transpose of U. For $U \in \mathscr{R}^n$, $\lambda_i(U)$, $i = 1,\dots,n$, stands for an eigenvalue of U. $\lambda_i(U)$ is referred to as a semi-stable (respectively stable) eigenvalue when it lies inside $\bar{\mathbb{D}}$ ($\mathbb{D}$); the associated eigenvector $v \in \mathbb{R}^n$ is semi-stable (stable), otherwise it is unstable. The space spanned by all stable eigenvectors is referred to as the stable subspace of U, and similarly for semi-stable and unstable semi-spaces.

Definition 1. Consider system Θ. We say that (A,Σ) is partially semi-stable (PSS) if, for each $0 \leq \xi < 1$ and $V \in \mathscr{R}^{n0}$, there exists $\bar{Z} \in \mathscr{R}^{n0}$ for which $\xi^k Z_k \leq \bar{Z}$, $k \geq 0$.

We employ the notation $Z_k(V)$ to emphasise the dependence on V, and similarly for $X_k(\Sigma)$. The next result is obtained using the fact that, for any $U \in \mathscr{R}^{n0}$, $AUA' \in \mathscr{R}^{n0}$.

Proposition 1. Consider $U_0, U_1 \in \mathscr{R}^{n0}$ and $\alpha \in \mathbb{R}$. The following statements hold.

(i) If $U_1 \geq U_0$ and $0 \leq \alpha \leq 1$: $X_k(\alpha U_1) \geq \alpha X_k(U_0)$ and $Z_k(\alpha U_1) \geq \alpha Z_k(U_0)$, $\forall k \geq 0$

(ii) $X_k(U_0) + X_k(U_1) \geq X_k(U_0 + U_1)$, $\forall k \geq 0$.

Lemma 2. (A,C) is PSS if and only if, for each $0 \leq \xi < 1$, there exists $\bar{Z} \in \mathscr{R}^{n0}$ for which $\xi^k Z_k(I) \leq \bar{Z}$, $k \geq 0$.

Proof. Proof. (Necessity) It follows setting $V = I$ in Definition 1.

(Sufficiency) For each $V \in \mathscr{R}^{n0}$ we can pick $\kappa > 0$ such that $\kappa V \leq I$ and Proposition 1 yields $\xi^k Z_k(\kappa V) \leq \xi^k Z_k(I) \leq \bar{Z}$, which leads to $\xi^k Z_k(V) \leq \kappa^{-1}\bar{Z}$.

Consider now the linear time-varying system related to the dynamics of the Z-component of Θ, defined by

$$\Theta_Z : \begin{cases} z_{k+1} = H_k A z_k, & k \geq 0 \\ z_0 = H_0 z. \end{cases} \tag{3}$$

where $z_k \in \mathbb{R}^n$ is the state. Not surprisingly, PSS is strongly connected to semi-stability of Θ_Z, as stated in the following lemma, the proof of which is omitted.

Lemma 3. Consider systems Θ and Θ_Z. (A,Σ) is PSS if and only if for each $z \in \mathbb{R}^n$ and $0 \leq \zeta < 1$ there exist $\alpha \geq 0$ and $0 \leq \beta < 1$ such that $\|\zeta^k z_k\| \leq \alpha\beta^k$.

Similarly to the sequence z_k connected with the Z-component of Θ, we introduce a vector sequence related to X, as follows. Consider the solution $X_k = A^k \Sigma A^{k\prime}$ for the X-component. Introduce the rank-one decomposition

$$\Sigma = \sigma_1\sigma_1' + \dots + \sigma_{r_\Sigma}\sigma_{r_\Sigma}, \tag{4}$$

where r_Σ stands for the rank of Σ, and the linear system defined by

$$\Theta_X : x_k(\sigma) = A^k\sigma.$$

It is simple to check that

$$X_k = x_k(\sigma_1)x_k(\sigma_1)' + \dots + x_k(\sigma_{r_\Sigma})x_k(\sigma_{r_\Sigma})'$$

and H_k is the orthogonal projection onto $[x_k(\sigma_1),\dots,x_k(\sigma_{r_\Sigma})]^\perp$.

Proposition 4. For $u_i, v_j \in \mathbb{R}^n$, $i = 1,\dots,q$, $j = 1,\dots,m$ let $\bar{v}_j$ be the orthogonal projection of v_j onto $[u_1,\dots,u_q]$. $\ker\{v_1v_1' + \dots + v_mv_m'\} \cap [u_1,\dots,u_q] = \{0\}$ if and only if for each $i = 1,\dots,q$, there is at least one j for which $\bar{v}_j' u_i \neq 0$.

2.1 Convenient bases

The spaces spanned by the trajectory $x_k = A^k\sigma$ play an important role in this paper, because they drive the projection H_k. We now present certain characterisations for convergence of these spaces. Note that, taking into account the original basis, there may be no convergence for $[x_k]$, as in the case of a spinning x_k presented in Example 1. To circumvent this difficulty one can use an alternative basis, e.g. associated to the Jordan canonical form of A. In this paper we employ the bases introduced as follows, in view of the fact that they lead to a simpler characterisation for $[x_k]$ (see e.g. Example 1), in spite of the drawback of an inherent time dependence.

Proposition 5. For each $A \in \mathscr{R}^n$ there is a sequence of transformations W_k, $k \geq 0$, such that $A = W_k^{-1}\bar{A}W_{k-1}$, $k \geq 1$, $A^k = W_k^{-1}\bar{A}^kW_0$, $k \geq 0$, and $\bar{A} = \mathrm{diag}(\mathscr{A}_1(\eta_1),\dots,\mathscr{A}_j(\eta_j))$, where $\mathscr{A}_i(\eta_i)$ is an upper triangular Jordan block and η_i, $0 \leq i \leq j$, is a *real positive* number, corresponding to an eigenvalue v_i of A with $|v_i| = \eta_i$, ordered in such a manner that and $\eta_i \geq \eta_j$ whenever $i \geq j$. Moreover, there exists κ, $0 \leq \kappa < 1$, such that $(1-\kappa) \leq \|W_k\| \leq (1+\kappa)$, $k \geq 0$.

The bases of Proposition 5 are employed throughout the paper, hence we introduce the following notation. Unless otherwise stated, for any $V \in \mathscr{R}^{n,r}$ and $v \in \mathbb{R}^n$, we define $\bar{V} \in \mathscr{R}^{n,r}$ and $\bar{v} \in \mathbb{R}^n$ as $\bar{V} = W_0V$ and $\bar{v} = W_0v$. For instance, we denote $W_0\sigma$ simply by $\bar{\sigma}$. The matrix A associated with the transformation W_0 is usually clear from the context, otherwise we employ the explicit notation $W_0(A)$. For $\sigma, z \in \mathbb{R}^n$, define $\bar{z}_k, \bar{x}_k \in \mathbb{R}^n$, $k \geq 0$, by

$$\bar{z}_{k+1} = (\bar{H}_k\bar{A})\bar{z}_k,\ k \geq 1, \quad \bar{z}_0 = \bar{H}_0\bar{z}, \quad \bar{x}_k(\sigma) = \bar{A}^k\bar{\sigma}\ k \geq 0, \tag{5}$$

where

$$\bar{H}_k = W_kH_kW_k^{-1}.$$

Convergence of trajectories is preserved, as stated in the next result, the proof of which is omitted.

Lemma 6. The following statements hold.

(i) $z_k = W_{k-1}^{-1}\bar{z}_k$, $x_k(\sigma) = W_k\bar{x}_k(\sigma)$, $k \geq 0$.

(ii) There exists κ, $0 \leq \kappa < 1$, such that $(1+\kappa)^{-1}\|\bar{z}_k\| \leq \|z_k\| \leq (1-\kappa)^{-1}\|\bar{z}_k\|$ and $(1-\kappa)\|\bar{x}_k(\sigma)\| \leq \|x_k(\sigma)\| \leq (1+\kappa)\|\bar{x}_k(\sigma)\|$, $k \geq 0$.

Regarding the eigenvalues η_j, $j = 1,\dots,n$, of $\bar{A}$, let $j = 1,\dots,m_u$, be the indexes corresponding to eigenvalues strictly greater than one and let $e_1,\dots,e_{q_u}$ be the associated eigenvectors; similarly, $j = 1,\dots,m_e$ correspond to eigenvalues greater or equal to one and $e_1,\dots,e_{q_e}$ are the associated eigenvectors. Introduce the subspaces

$$\mathscr{U} = [e_1, \ldots, e_{q_u}]; \ \mathscr{U}_c = [e_{q_u+1}, \ldots, e_n]; \\ \mathscr{E} = [e_1, \ldots, e_{q_e}]; \ \mathscr{E}_c = [e_{q_e+1}, \ldots, e_n]. \tag{6}$$

The block structure of $\bar{A}$ in Proposition 5 allows for the next invariance results; the proof is omitted.

Lemma 7. The following statements hold.

(i) $\mathscr{U}, \mathscr{U}_c, \mathscr{E}$ and $\mathscr{E}_c$ are $\bar{A}$-invariant.

(ii) For $V \in \mathscr{R}^{n0}$, if $\ker\{V\} \cap \mathscr{U}_c = \{0\}$, then $\ker\{\bar{A}^k V \bar{A}'^k\} \cap \mathscr{U}_c = \{0\}, k \geq 0$.

Note that $\bar{A}$ is in Jordan form, leading to several links with available results for Jordan forms. For example there are invariance results similar to the ones of Lemma 7, see e.g. Abou-Kandil et al. [2003]. Another useful connection is as follows. Let J be the similarity matrix for which JAJ^{-1} is the Jordan form of A and let $\mathscr{J}$ stands for the vector subspace spanned by the unstable eigenvectors of JAJ^{-1}. Then, for each $\sigma \in \mathbb{R}^n$, the projection of $J\sigma$ onto $\mathscr{J}$ is zero if and only if the projection of $\bar{\sigma} = W_0 \sigma$ onto $\mathscr{U}$ is zero, yielding the following result, given without proof, which is useful for representing the main results in terms of Jordan forms.

Lemma 8. $\ker\{W_0 \Sigma W_0'\} \cap \mathscr{U} = \{0\}$ if and only if $\ker\{J \Sigma J'\} \cap \mathscr{J} = \{0\}$.

2.2 Approximation results related to the projections $\bar{H}_k$

The spaces spanned by $\bar{x}_k(\sigma)$ may in a sense align with unstable modes of $\bar{A}$, in such a manner that the projections $\bar{H}$ may provide a cancelling effect for such modes. In order to make these notions precise define, for the vector subspaces $\mathscr{U}$ and $\mathscr{V}$, the quantity

$$\theta_{\mathscr{V}}(\mathscr{U}) = \max_{v \in \mathscr{V}, v \neq 0} \ \min_{u \in \mathscr{U}, u \neq 0} 1 - (\|u\| \|v\|)^{-1} u' v. \tag{7}$$

Note that, if $\sigma \in \mathbb{R}^n$ is such that η is the largest eigenvalue for which $\sigma' v \neq 0$, where v is an eigenvector associated with η, and assuming η unique (i.e., no other eigenvalue of A equals η), then there are $\alpha \geq 0$ and $0 \leq \beta < 1$ such that $\theta_{[e]}([x_k(\sigma)]) \leq \alpha \beta^k$, where e is a non-generalised eigenvector associated with η. This implies that $x_k(\sigma)$ and e align with exponential rate. Moreover, for each eigenvector w of $\bar{A}$ (except $w = e$) there is a $\varphi > 0$ such that $\theta_{[w]}([x_k(\sigma)]) \geq \varphi$ for a sufficiently large k. One can explore the convenient block structure of $\bar{A}$ to obtain the more general characterisation given in Lemmas 9 and 10, the proofs of which are omitted; recall that, according to the notation of Section 2.1, $\bar{\sigma} = W_0 \sigma$.

Lemma 9. Consider $\sigma_j \in \mathbb{R}^n$, $j = 1, \ldots, m$. If $\ker\{\bar{\sigma}_1 \bar{\sigma}_1' + \cdots + \bar{\sigma}_m \bar{\sigma}_m'\} \cap \mathscr{E} = \{0\}$ then there exist $\alpha \geq 0$ and $0 \leq \beta < 1$ such that

$$\theta_{\mathscr{E}}([\bar{x}_k(\sigma_1), \ldots, \bar{x}_k(\sigma_m)]) \leq \alpha \beta^k.$$

Conversely to Lemma 9, if σ_j does not "completely excite" the subspace $\mathscr{E}$, then the space spanned by $\bar{x}_k(\sigma_1)$ does not "align" with $\mathscr{E}$. It is convenient for later reference to formalise this in terms of $\mathscr{U}$ rather than $\mathscr{E}$.

Lemma 10. Consider $\sigma_j \in \mathbb{R}^n$, $j = 1, \ldots, m$. If $\ker\{\bar{\sigma}_1 \bar{\sigma}_1' + \cdots + \bar{\sigma}_m \bar{\sigma}_m'\} \cap \mathscr{U} \neq \{0\}$ then there exist $\alpha \geq 0$, $0 \leq \beta < 1$, $\varphi > 0$ and $\Upsilon = [\upsilon_1, \ldots, \upsilon_q]$, $\Upsilon \subset \mathscr{U}$, where $\upsilon_1, \ldots, \upsilon_q$ are eigenvectors of $\bar{A}$ associated with strictly unstable eigenvalues, such that

$$\theta_{\Upsilon}([\bar{x}_k(\sigma_1), \ldots, \bar{x}_k(\sigma_m)]) \geq \varphi, \quad k \geq 1,$$
$$\theta_{\mathscr{U} \ominus \Upsilon}([\bar{x}_k(\sigma_1), \ldots, \bar{x}_k(\sigma_m)]) \leq \alpha \beta^k, k \geq 0.$$

Example 1. Consider the system Θ_X with

$$A = \begin{bmatrix} 1 & -0.1 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & \rho \end{bmatrix}, \ \sigma = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

Set $W_k = \tilde{A}^{-k}$, with

$$\tilde{A} \approx \begin{bmatrix} 0.9901 & -0.099 & 0 \\ 0.1980 & 0.9901 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \bar{A} = \begin{bmatrix} 1.01 & 0 & 0 \\ 0 & 1.01 & 0 \\ 0 & 0 & \rho \end{bmatrix}.$$

Let $\rho = 1.01$. Figure 1 illustrates how simple is the behaviour of $\bar{x}_k$ when compared to x_k. One can check the conditions of Lemma 10 and, indeed, $\upsilon_1 = [1 \ 0 \ -1]'$, $\upsilon_2 = [0 \ 1 \ 0]'$ and $\Upsilon = [\upsilon_1, \upsilon_2]$ are such that $\theta_{\Upsilon}(\bar{x}(k)) = 1$ and $\theta_{\mathscr{U} \ominus \Upsilon}(\bar{x}(k)) = 0$. As one can infer from the figure, there is no similar characterisation for x_k, as x_k presents an oscillatory behaviour that prevents convergence of $\theta_V(x_k)$ for any V.

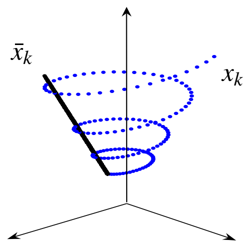

Fig. 1. Trajectories of Example 1.

The projections $\bar{H}$ are not orthogonal (because of the "distortion" introduced by the new bases), but they are similar to H in the sense that $\bar{H}\bar{v} = 0$, whenever $Hv = 0$. We shall need the following related result.

Lemma 11. Consider the rank-one decomposition (4) for Σ. The projections $\bar{H}_k$, $k \geq 0$, are such that $\bar{H}_k v = 0$, for $v \in [\bar{x}_k(\sigma_1), \ldots, \bar{x}_k(\sigma_{r_\Sigma})]$.

Proof. Note that $\bar{H}_k v = \bar{H}_k(\alpha_1 \bar{x}_k(\sigma_1) + \cdots + \alpha_{r_\Sigma} \bar{x}_k(\sigma_{r_\Sigma}))$ for certain scalars α_j, $j = 0, \ldots, r_\Sigma$, and, for each term of this sum, one can employ Lemma 6 (i) to evaluate $\bar{H}_k \bar{x}_k(\sigma_j) = W_k H_k W_k^{-1} W_k x_k(\sigma_j) = W_k H_k x_k(\sigma_j) = 0$, $j = 0, \ldots, r_\Sigma$, since $H_k x_k(\sigma_j) = 0$ by definition of H_k.

As $\bar{x}_k(\sigma_j)$ aligns with $\mathscr{E}$ ($\mathscr{U} \ominus \Upsilon$, respectively) as stated in Lemma 9 (Lemma 10, respectively), we have that the projections $\bar{H}_k$ "tend to align" with the orthogonal projection onto $\mathscr{E}_c$ ($\mathscr{U}_c \oplus \Upsilon$, respectively), which allows to obtain the approximation results that will be useful for Section 3. We present these results in the next lemma, in which S, T and U denote the orthogonal projections onto $\mathscr{E}_c$, $\mathscr{U}_c \oplus \Upsilon$ and Υ, respectively.

Lemma 12. If $\ker\{W_0 \Sigma W_0'\} \cap \mathscr{E} = \{0\}$ then there exist $\alpha \geq 0$, $0 \leq \beta < 1$, such that, for $k \geq 0$:

(i) $\|S(I - \bar{H}_k)v\| \leq \alpha \beta^k \|v\|$ and $\|\bar{H}_k(I - S)v\| \leq \alpha \beta^k \|v\|$.

On the other hand, if $\ker\{W_0 \Sigma W_0'\} \cap \mathscr{U} \neq \{0\}$ then there exist $\alpha \geq 0$, $0 \leq \beta < 1$, $\delta, \varphi, \lambda > 0$ such that, for $k \geq 0$:

(ii) $\|T(I - \bar{H}_k)v\| \leq \alpha \beta^k \|v\|$ and $\|\bar{H}_k(I - T)v\| \leq \alpha \beta^k \|v\|$;

(iii) $\|(U\bar{A})^{k+1} U v\| \geq (1 + \delta)\|Uv\|$;

(iv) $T\bar{A}Uv = U\bar{A}Uv$;

(v) $\|\bar{H}_k U v\| \leq \lambda \|Uv\|$.

Proof. Lemmas 9–11 lead to (i) and (ii). (iii) follows from the fact that U is the projection onto Υ, a subspace spanned by strictly unstable eigenvectors of $\bar{A}$, as stated in Lemma 9; moreover, $(1+\delta)$ equals the minimal of such eigenvalues. As regards to (iv), Υ is not necessarily $\bar{A}$-invariant in general, but one can easily check from the structure of $\bar{A}$ that, for $w \in \Upsilon$, $\bar{A}w \in \mathscr{U}$, in such a manner that the component of $\bar{A}w$ in $\mathscr{U}_c$ is zero and $T\bar{A}w = U\bar{A}w$. (v) follows from the facts that $\bar{H}_k = W_k H_k W_k^{-1}$ and $(1-\kappa) \le \|W_k\| \le (1+\kappa)$ for some $0 \le \kappa < 1$, as in Lemma 5.

An important feature of the case with $\ker\{\bar{\Sigma}\} \cap \mathscr{U} \neq \{0\}$ is that $\mathrm{Im}(\bar{H}) \cap \mathscr{U} \neq \{0\}$, which follows from the fact that $\bar{H}$ can not "cover" $\mathscr{U}$ as stated in Lemma 10. This fact together with the structure of invariant spaces presented in Lemma 7 allows to pick an initial condition $\bar{z}$ for which the associated $\bar{z}_k$ has a non-trivial projection in Υ, as in the next result, the proof of which is omitted.

Proposition 13. There exists $\bar{z} \in \mathscr{E}$ such that $U\bar{z}_k \neq 0$, $k \ge 0$, provided $\ker\{\bar{\Sigma}\} \cap \mathscr{U} \neq \{0\}$.

3. TESTABLE CONDITION FOR PSS OF Θ

This section presents, separately, a sufficient condition for PSS and a necessary one. The results are gathered together in Theorem 17. We start showing that $\bar{z}_k$ defined in (5) converges exponentially if Σ completely excites $\mathscr{E}$.

Lemma 14. Consider W_0 as in Proposition 5, $\mathscr{E}$ as in (6) and $\bar{z}$ as in (5). If $\ker\{W_0 \Sigma W_0'\} \cap \mathscr{E} = \{0\}$, then for each $\bar{z}$ there exist $\chi \ge 0$ and $0 \le \beta < 1$ such that $\|\bar{z}_k\| \le \chi \beta^k$.

Proof. For ease of notation, in this proof we write $\bar{A}$ and $\bar{H}$ as A and H, respectively; for $\ell \ge 0$, $w_{1,\ell}, w_{2,\ell}, w_{3,\ell}$ stand for vectors with $\|w_{j,\ell}\| \le 1$. Recall the orthogonal projections S, T and U used in Lemma 12. From Lemma 7 we have that both $\mathscr{E}_c$ and $\mathscr{E}$ are A-invariant, in such a manner that $ASz_{k+\ell} \in \mathscr{E}_c$ and $A(I-S)z_{k+\ell} \in \mathscr{E}$, $k,\ell \ge 0$. Moreover, $\ker\{W_0 \Sigma W_0'\} \cap \mathscr{E} = \{0\}$, hence the conditions of Lemma 12 (i) hold, allowing to evaluate, for $k, \ell \ge 0$,

$$\begin{aligned} SH_{k+\ell+1}(ASz_{k+\ell}) &= S(ASz_{k+\ell}) + \alpha\beta^{k+\ell+1}\|ASz_{k+\ell}\|w_{1,\ell} \\ &= ASz_{k+\ell} + \alpha\beta^{k+\ell+1}\|ASz_{k+\ell}\|w_{1,\ell} \\ H_{k+\ell+1}A(I-S)z_{k+\ell} &= H_{k+\ell+1}(I-S)(A(I-S)z_{k+\ell}) \\ &= \alpha\beta^{k+\ell+1}\|A(I-S)z_{k+\ell}\|w_{2,\ell} \end{aligned} \tag{9}$$

where α, β are as in Lemma 12. Now we shall show inductively that

$$\begin{aligned} z_{k+\ell+1} &= A^{\ell+1}Sz_k + (I-S)H_{k+\ell+1}A^{\ell+1}Sz_k + \\ &\quad + 2\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1} + \cdots + \beta^{k+\ell+1})w_{3,\ell}, \quad \ell \ge 0. \end{aligned} \tag{10}$$

For $\ell = 0$, from (3) and (9) we have that

$$\begin{aligned} z_{k+1} &= H_{k+1}Az_k = H_{k+1}ASz_k + H_{k+1}A(I-S)z_k \\ &= SH_{k+1}ASz_k + (I-S)H_{k+1}ASz_k + H_{k+1}A(I-S)z_k \\ &= ASz_k + (I-S)H_{k+1}ASz_k \\ &\quad + \alpha\beta^{k+\ell+1}(\|ASz_k\|w_{1,0} + \|A(I-S)z_k\|w_{2,0}) \\ &= ASz_k + (I-S)H_{k+1}ASz_k + \alpha\beta^{k+\ell+1}(2\|A\|\|z_k\|w_{3,0}) \end{aligned}$$

and assuming (10) holds for $\ell - 1$, similarly as above we evaluate from (3)

$$\begin{aligned} z_{k+\ell+1} &= H_{k+\ell+1}Az_{k+\ell} \\ &= H_{k+\ell+1}A^{\ell+1}Sz_k \\ &\quad + H_{k+\ell+1}A(I-S)H_{k+\ell}A^{\ell}Sz_k \\ &\quad + H_{k+\ell+1}A(2\alpha\|A\|^{\ell}\|z_k\|(\beta^{k+1} + \cdots + \beta^{k+\ell})w_{3,\ell-1}) \\ &= SH_{k+\ell+1}A^{\ell+1}Sz_k + (I-S)H_{k+\ell+1}A^{\ell+1}Sz_k \\ &\quad + H_{k+\ell+1}A(I-S)H_{k+\ell}A^{\ell}Sz_k \\ &\quad + H_{k+\ell+1}A(2\alpha\|A\|^{\ell}\|z_k\|(\beta^{k+1} + \cdots + \beta^{k+\ell})w_{3,\ell-1}) \end{aligned}$$

and, from (9),

$$\begin{aligned} z_{k+\ell+1} &= A^{\ell+1}Sz_k + \alpha\beta^{k+\ell+1}\|A^{\ell+1}Sz_k\|w_{1,k+\ell+1} \\ &\quad + (I-S)H_{k+\ell+1}A^{\ell+1}Sz_k \\ &\quad + \alpha\beta^{k+\ell+1}\|A(I-S)H_{k+\ell}A^{\ell}Sz_k\|w_{2,k+\ell+1} \\ &\quad + H_{k+\ell+1}A(2\alpha\|A\|^{\ell}\|z_k\|(\beta^{k+1} + \cdots + \beta^{k+\ell})w_{3,\ell-1}) \\ &= A^{\ell+1}Sz_k + (I-S)H_{k+\ell+1}A^{\ell+1}Sz_k \\ &\quad + 2\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1} + \cdots + \beta^{k+\ell} + \alpha\beta^{k+\ell+1})w_{3,\ell}, \end{aligned}$$

completing the inductive proof of (10). Then we can write, for $k, \ell \ge 0$,

$$\begin{aligned} \|z_{k+\ell+1}\| &\le \|A^{\ell+1}Sz_k\| + \|(I-S)H_{k+\ell+1}A^{\ell+1}Sz_k\| \\ &\quad + 2\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1} + \cdots + \beta^{k+\ell+1}) \\ &\le 2\|A^{\ell+1}Sz_k\| + 2\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1} + \cdots + \beta^{k+\ell+1}). \end{aligned} \tag{11}$$

Now consider the term $A^{\ell}Sz_k$, $\ell \ge 1$. Since $\mathscr{E}_c$ is A-invariant and corresponds to the subspace spanned by eigenvectors associated to eigenvalues (strictly) inside the unit disk, one has that $\|A^{\ell}Sz_k\| \le \eta\gamma^{\ell}\|Sz_k\| \le \eta\gamma^{\ell}\|z_k\|$ for some scalars $\eta \ge 0$ and $0 \le \gamma < 1$. Then, we set

$$\ell_0 \ : \ \eta\gamma^{\ell_0} \le 1/4$$

and from (11) with $\ell = \ell_0 - 1$ we obtain:

$$\begin{aligned} \|z_{k+\ell_0}\| &\le (2\eta\gamma^{\ell_0} + 2\alpha\|A\|^{\ell_0}(\beta^{k+1} + \cdots + \beta^{k+\ell_0}))\|z_k\| \\ &\le (1/2 + 2\alpha\|A\|^{\ell_0}(\beta^{k+1} + \cdots + \beta^{k+\ell_0}))\|z_k\|, \quad k \ge 0. \end{aligned} \tag{12}$$

Now we set k_0 such that $2\alpha\|A\|^{\ell_0}(\beta^{k_0+1} + \cdots + \beta^{k_0+\ell_0}) < 1/2$. From (12) with $k = k_0$, we obtain $\|z_{k_0+\ell_0}\| \le \pi\|z_{k_0}\|$, where $\pi = (1/2 + 2\alpha\|A\|^{\ell_0}(\beta^{k_0+1} + \cdots + \beta^{k_0+\ell_0})) < 1$; similarly, from (12) with $k = k_0 + m\ell_0$, $m \ge 0$, we obtain

$$\begin{aligned} \|z_{k_0+m\ell_0+\ell_0}\| &\le \pi\|z_{k_0+m\ell_0}\| \\ &\le \pi^2\|z_{k_0+(m-1)\ell_0}\| \le \cdots \le \pi^{m+1}\|z_{k_0}\|. \end{aligned}$$

Finally, we have that each $k \ge k_0$ can be written in the form $k = k_0 + m\ell_0 + r$ for some $0 \le r < \ell_0$ and m with $(k-k_0)/\ell_0 \le m \le (k-k_0)/\ell_0 + 1$, leading to

$$\begin{aligned} \|z_k\| &\le \|A\|^r\|z_{k_0+m\ell_0}\| \\ &\le \|A\|^{\ell_0}\pi^m\|z_{k_0}\| \le \|A\|^{\ell_0}\|z_{k_0}\|\pi^{-1}(\pi^{1/\ell_0})^k, \quad k \ge k_0, \end{aligned}$$

and since $\|z_k\| \le \|A\|^k\|z_0\|$, $k < k_0$, it is a simple matter to check that we can set $\beta = \pi^{1/\ell_0} < 1$ and find $\chi \ge 0$ for which $\|z_k\| \le \chi\beta^k$, $k \ge 0$.

Lemma 14 can be easily extended to show semi-stability of the system Θ_Z, by employing $\zeta < 1$ as a "scaling factor" that converts $\mathscr{E}$ associated with the matrix $\bar{A}$ into $\mathscr{U}$ associated with $\zeta\bar{A}$. Convergence for $\bar{z}$ is also related to the convergence of z by Lemma 6.

Corollary 15. Consider the system Θ_Z, W_0 as in Proposition 5 and $\mathscr{U}$ as in (6). If $\ker\{W_0\Sigma W_0'\} \cap \mathscr{U} = \{0\}$ then for each

z and $0 \leq \zeta < 1$ there exist $\alpha \geq 0$ and $0 \leq \beta < 1$ such that $\|\zeta^k z_k\| \leq \alpha\beta^k$.

Conversely to Corollary 15, if Σ does not completely excite $\mathscr{U}$, then exponential divergence takes place.

Lemma 16. Consider the system Θ_Z, W_0 as in Proposition 5 and $\mathscr{U}$ as in (6). If $\ker\{W_0\Sigma W_0'\} \cap \mathscr{U} \neq \{0\}$ then there exist $z \in \Re^n$ and $0 \leq \zeta < 1$ such that for all $\chi \geq 0$ and $0 \leq \psi < 1$, $\|\zeta^k z_k\| > \chi\psi^k$.

Proof. In this proof we shall need an evaluation that is in analogy with (10) of Lemma 14. In fact, (10) involves projections onto $\mathscr{E}_c$ and $\mathscr{E}$ via S and $I-S$ respectively, and now we consider projections onto $\mathscr{U}_c$, $\mathscr{U} \ominus \Upsilon$ and Υ via $(I-U)T$, $(I-T)$ and U respectively. Using Lemma 12 (ii) and (iii) yields

$$\begin{aligned} z_{k+\ell+1} = &(TA)^{\ell+1}Uz_k + A^{\ell+1}(I-U)Tz_k \\ &+ (I-T)H_{k+\ell+1}(TA)^{\ell+1}Uz_k \\ &+ (I-T)H_{k+\ell+1}A^{\ell+1}(I-U)Tz_k \\ &+ 4\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1})w_{3,\ell}, \;\; \ell \geq 0, \end{aligned}$$

and, since Lemma 12 (v) provides $(TA)^{\ell+1}U = (UA)^{\ell+1}U$, this can be written as

$$\begin{aligned} z_{k+\ell+1} = &(UA)^{\ell+1}Uz_k + A^{\ell+1}(I-U)Tz_k \\ &+ (I-T)H_{k+\ell+1}(UA)^{\ell+1}Uz_k \\ &+ (I-T)H_{k+\ell+1}A^{\ell+1}(I-U)Tz_k \\ &+ 4\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1})w_{3,\ell}, \;\; \ell \geq 0. \end{aligned} \tag{13}$$

Similarly to (9), we have from Lemma 12 (ii)

$$\begin{aligned} TH_{k+\ell+1}(UA)^{\ell+1}Uz_{k+\ell} = &T(UA)^{\ell+1}z_{k+\ell} \\ &+ \alpha\beta^{k+\ell+1}\|(UA)^{\ell+1}Uz_{k+\ell}\|w_{1,\ell}, \end{aligned}$$

allowing to collect the first and third term on the right hand side of (13) and to write

$$\begin{aligned} z_{k+\ell+1} = &H_{k+\ell+1}(UA)^{\ell+1}Uz_k - \alpha\beta^{k+\ell+1}\|(UA)^{\ell+1}Uz_{k+\ell}\|w_{1,\ell} \\ &+ A^{\ell+1}(I-U)Tz_k + (I-T)H_{k+\ell+1}A^{\ell+1}(I-U)Tz_k \\ &+ 4\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1})w_{3,\ell} \\ = &H_{k+\ell+1}(UA)^{\ell+1}Uz_k \\ &+ A^{\ell+1}(I-U)Tz_k + (I-T)H_{k+\ell+1}A^{\ell+1}(I-U)Tz_k \\ &+ 5\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1})w_{4,\ell}, \;\; \ell \geq 0. \end{aligned} \tag{14}$$

Regarding the second and third terms on the right hand side of (14), note that $(I-U)Tz_z \in \mathscr{U}_c$, yielding

$$\begin{aligned} &\|A^{\ell+1}(I-U)Tz_k + (I-T)H_{k+\ell+1}A^{\ell+1}(I-U)Tz_k\| \\ &\quad \leq 2\|A^{\ell+1}(I-U)Tz_k\| \leq 2\eta\gamma^{\ell+1}\|z_k\|. \end{aligned} \tag{15}$$

Note that (14), (15) and Lemma 12 (vi) lead to

$$\begin{aligned} \|z_{k+\ell+1}\| \leq &\|H_{k+\ell+1}(UA)^{\ell+1}Uz_k\| + 2\eta\gamma^{\ell+1}\|z_k\| + \\ &+ 5\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1}) \\ \leq &\lambda\|(UA)^{\ell+1}Uz_k\| + 2\eta\gamma^{\ell+1}\|z_k\| + \\ &+ 5\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1}). \end{aligned} \tag{16}$$

Moreover, for $v \in \mathbb{R}^n$, $(I-T)v \in (U) \ominus \Upsilon$, i.e., $(I-T)v \perp \Upsilon$ in such a manner that $U(I-T) = 0$, as above, employing (13) and Lemma 12, we obtain:

$$\begin{aligned} \|Uz_{k+\ell+1}\| = &\|(UA)^{\ell+1}Uz_k + UA^{\ell+1}(I-U)Tz_k \\ &+ 4\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1})Uw_{3,\ell}\| \\ \geq &\|(UA)^{\ell+1}Uz_k\| - \eta\gamma^{\ell+1}\|z_k\| \\ &- 4\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1}). \end{aligned} \tag{17}$$

By substituting (17) in (16) we get that

$$\begin{aligned} \|z_{k+\ell+1}\| \leq &\lambda\big(\|Uz_{k+\ell+1}\| + \eta\gamma^{\ell+1}\|z_k\| \\ &+ 4\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1})\big) \\ &+ 2\eta\gamma^{\ell+1}\|z_k\| + 5\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1}) \\ = &\lambda\|Uz_{k+\ell+1}\| + (2+\lambda)\eta\gamma^{\ell+1}\|z_k\| \\ &+ (5+4\lambda)\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1}), \;\; \ell \geq 0, \end{aligned}$$

or equivalently, for $\ell \geq 0$,

$$\begin{aligned} \|Uz_{k+\ell+1}\| \geq &\lambda^{-1}\|z_{k+\ell+1}\| - \lambda^{-1}(2+\lambda)\eta\gamma^{\ell+1}\|z_k\| \\ &- \lambda^{-1}(5+4\lambda)\alpha\|A\|^{\ell+1}\|z_k\|(\beta^{k+1}+\cdots+\beta^{k+\ell+1}). \end{aligned} \tag{18}$$

Now we employ the fact that Υ is associated with unstable eigenvalues of A. We proceed similarly as above, employing (13) with k replaced by $k+\ell+1$ and ℓ replaced by $m-1$, and Lemma 12 (ii)–(v), to evaluate:

$$\begin{aligned} \|z_{k+\ell+m+1}\| \geq &\|(UA)^m Uz_{k+\ell+1} \\ &+ (I-T)H_{k+\ell+m+1}(UA)^m Uz_{k+\ell+1} \\ &+ (I-T)H_{k+\ell+m+1}A^m(I-U)Tz_{k+\ell+1}\| \\ &- \|A^m(I-U)Tz_{k+\ell+1}\| \\ &- 4\alpha\|A\|^m\|z_k\|(\beta^{k+\ell+2}+\cdots+\beta^{k+\ell+m+1}). \end{aligned}$$

Recalling that $(I-T)$ and U are orthogonal projections onto $\mathscr{U} \ominus \Upsilon$ and Υ, respectively, and employing Lemma 12 (iii), the above inequality leads to

$$\begin{aligned} \|z_{k+\ell+m+1}\| \geq &(1+\delta)^m\|Uz_{k+\ell+1}\| - \eta\gamma^m\|z_{k+\ell+1}\| \\ &- 4\alpha\|A\|^m\|z_{k+\ell+1}\|(\beta^{k+\ell+2}+\cdots+\beta^{k+\ell+m+1}) \end{aligned}$$

and, from (18),

$$\begin{aligned} \|z_{k+\ell+m+1}\| \geq &(1+\delta)^m\lambda^{-1}\Big(\|z_{k+\ell+1}\| - (2+\lambda)\eta\gamma^{\ell+1}\|z_k\| \\ &- (5+4\lambda)\alpha\|A\|^{\ell+1}(\beta^{k+1}+\cdots+\beta^{k+\ell+1})\|z_k\|\Big) \\ &- \eta\gamma^m\|z_{k+\ell+1}\| \\ &- 4\alpha\|A\|^m\|z_{k+\ell+1}\|(\beta^{k+\ell+2}+\cdots+\beta^{k+\ell+m+1}) \\ \geq &(1+\delta)^m\lambda^{-1}\|z_{k+\ell+1}\| - \beta^k(1+\delta)^m\bar{\alpha}\bar{\kappa}^{\ell+1}\|z_k\| \\ &- \beta^{k+\ell}\tilde{\alpha}\tilde{\kappa}^m\|z_{k+\ell+1}\| \end{aligned}$$

where we set $\bar{\alpha}, \tilde{\alpha}, \bar{\kappa}, \tilde{\kappa} \geq 0$ conveniently. For an arbitrary $\bar{m} > 0$, let m be such that $(1+\delta)^m\lambda^{-1} > 6\bar{m}$ and ℓ be such that $\beta^\ell\tilde{\alpha}\tilde{\kappa}^m < 3\bar{m}$, yielding

$$\begin{aligned} \|z_{k+\ell+m+1}\| \geq &6\bar{m}\|z_{k+\ell+1}\| - 3\bar{m}\|z_{k+\ell+1}\| \\ &- \beta^k(1+\delta)^m\bar{\alpha}\bar{\kappa}^{\ell+1}\|z_k\| \\ \geq &3\bar{m}\|z_{k+\ell+1}\| - \beta^k(1+\delta)^m\bar{\alpha}\bar{\kappa}^{\ell+1}\|z_k\| \end{aligned}$$

hence, setting k such that $\beta^k(1+\delta)^m\bar{\alpha}\bar{\kappa}^{\ell+1} < \bar{m}$,

$$\begin{aligned} \|z_{k+\ell+m+1}\| \geq &3\bar{m}\|z_{k+\ell+1}\| - \bar{m}\|z_k\| \\ \geq &\bar{m}\|z_{k+\ell+1}\| + 2\bar{m}\|z_{k+\ell+1}\| - \bar{m}\|z_k\|. \end{aligned}$$

Using this inequality in a recursive fashion, substituting k with $k+qm$, $q \geq 0$, we obtain

$$\begin{aligned} \|z_{k+\ell+1+(q+1)m}\| \geq &\bar{m}^q\|z_{k+\ell+1+m}\| \\ &+ \sum_{j=0}^{q}(2\bar{m})^j\|z_{k+\ell+1+jm}\| - \sum_{j=0}^{q}\bar{m}^j\|z_{k+qm}\|. \end{aligned} \tag{19}$$

The second term on the right hand side of (19) dominates the third one, leading to exponential divergence. It is important to mention that we can pick an initial condition z for which $z_{k+\ell+1+m} \neq 0$, see Proposition 13.

Theorem 17. Consider the system Θ. Let J represent the similarity transformation for which JAJ^{-1} is in Jordan form and let $\mathscr{J}$ stand for the unstable space of JAJ^{-1}. (A,Σ) is PSS if and only if

$$\ker\{J\Sigma J'\} \cap \mathscr{J} = \{0\}. \tag{20}$$

Proof. Consider W_0 as in Proposition 5 and $\mathscr{U}$ as in (6). It follows from Corollary 15 and Lemma 16 that

$$\ker\{W_0 \Sigma W_0'\} \cap \mathscr{U} = \{0\} \tag{21}$$

is a necessary and sufficient condition for the existence, for each $z \in \mathbb{R}^n$ and $0 \leq \zeta < 1$, of $\alpha \geq 0$ and $0 \leq \beta < 1$ such that $\|\zeta^k z_k\| \leq \alpha\beta^k$. Lemma 3 extends the result to PSS of Θ. Finally, (21) holds if and only if (20) holds, see Lemma 8.

Remark 1. Either $\Sigma > 0$ or semi-stable A imply (A,Σ) is PSS, which implies that (A,Σ) is semi-stabilizable. Indeed, $\Sigma > 0$ provides $\ker\{J\Sigma J'\} = \{0\}$ and stable A yields $\mathscr{J} = \{0\}$, and in both cases (20) holds. Regarding the second implication, (A,Σ) not semi-stabilizable means that Σ does not excite an "entire" unstable mode of A, and (20) does not hold. PSS is not comparable to stabilizability of (A,Σ); indeed, in Example 2 (i) we have that (A,Σ) is stabilizable but Θ is not PSS, whereas with $A = 1$ and $\Sigma = 0$ illustrate the opposite situation.

Example 2. Consider the system Θ with A and σ as in (8) and $\Sigma = \sigma\sigma'$. We consider the following setups. (i) $\rho = 1.01$. It is simple to check that (20) is not satisfyied and, according to Theorem 17, Θ is not PSS. See Fig. 2 (i) for the behaviour of the Z-component of the trajectory of Θ. (ii) Replace A with $1.01^{-1}A$. Now the hypothesis of Theorem 17 holds and the system is PSS, see Fig. 2 (ii).

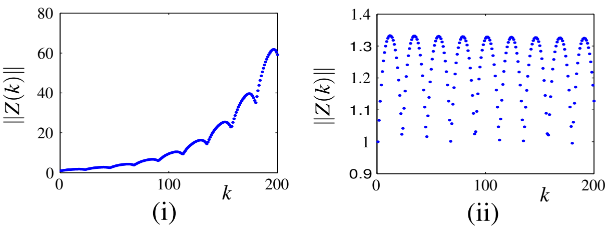

Fig. 2. Norm of the Z-component of the trajectory of Θ for the setups of Example 2.

Example 3. Consider the system Θ with

$$A = \begin{bmatrix} -1.001 & 1 \\ 0 & -1.001 \end{bmatrix}, \ \Sigma \ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Z-component of the state presents a transient with fast decaying norm, in spite of the eigenvalues of A being close to the unit disk, see Fig. 3 (i). However, $\ker\{J\Sigma J'\} \cap \mathscr{J} = [v]$ with $v = [1 \ \ 0]'$ and, according to Theorem 17, Θ is not PSS. In fact, after the initial transient, the trajectory diverges exponentially, see Fig. 3 (ii).

4. CONCLUDING REMARKS

In this paper we have explored the structure of the system Θ in (1), with special attention to the relations among the initial condition Σ of the X-component, its dynamics (governed by A) and the coupling with the Z-component via the orthogonal

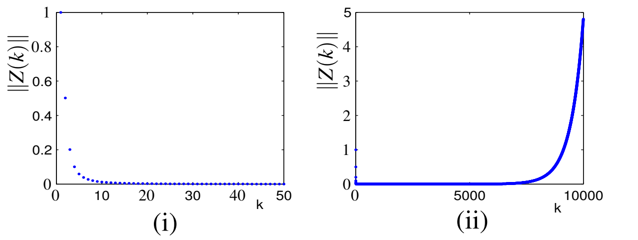

Fig. 3. Norm of the Z-component of the trajectory of system Θ of Example 3.

projection H. We obtain the testable condition in (20) for PSS, with the interpretation that Σ has to completely excite the unstable modes of A. This interpretation is particularly meaningfull in the scenario of Kalman filtering for linear time-invariant systems with initial covariance matrix Σ, meaning that the noise in the initial condition excites the unstable dynamics of the plant; indeed, (20) is essential to obtain, as discussed in Costa and Astolfi [b], a necessary and sufficient condition for avoiding *actual* exponential divergence of estimates under incorrect noise measurements, which is a significant result, taking into account the conservativeness of existing results.

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