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An Effective Algorithm for Analytical Computation of Flat Outputs over the Weyl Algebra

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Abstract: This paper presents a new and effective algorithm to compute a set of flat outputs for nonlinear implicit systems described in the differential geometric framework of jets of infinite order. The proposed methodology is based on the necessary and sufficient conditions for differential flatness introduced in Lévine [2006]. A procedure is set up to manage the degrees of freedom involved by the polynomial matrix approach. First, a reduced-order basis of the ideal of differential forms generated by the differentials of all possible Lie-Bäcklund isomorphisms is obtained by computing reduced order bases of the left nullspace of Ore polynomial matrices and the weak Popov form over a Weyl algebra. Then, the generalized Cartan moving frame, including additional structural constraints, is used in order to characterize the strong closedness of the latter ideal.

Keywords: Nonlinear Dynamic Inversion; Differential flatness; flat outputs; nonlinear systems; polynomial matrices; weak Popov form; differential forms.

1. INTRODUCTION

When it comes to nonlinear dynamic inversion (NDI) control methods, the choice of a set of outputs for input-to-state linearization appears to be a critical phase of the process. The flatness approach, introduced first by Fliess et al. [1995], proposes a quite appealing solution to the above problem: a set of particular outputs (the so-called flat outputs) such that a NDI based on these outputs does not yield any nonlinear unobservable subsystem. The stability analysis of the nonlinear model is reduced to validate the root locus of the linearized model obtained by the NDI technique Betts [1985]. Necessary and suffcient conditions for differential flatness are now well-established for both linear and nonlinear multidimensional systems. However, direct application of the flatness necessary and sufficient conditions may lead to a large number of candidate flat outputs. Consequently, the choice of a particular set of them, well adapted to sensor measurements and/or endowed with a physical meaning, appears to be a delicate and so far unsolved problem. The computation of some tractable flat outputs represents an important challenge which is partially addressed in this paper. In order to increase the accessibility of the flatness theory and associated tools to the control system designers, a library of Maple$^{\text{TM}}$formal functions is currently under development. The library is devoted to check if a nonlinear system is flat, to propose some flat outputs satisfying a specific criteria corresponding to some minimality or simplicity objectives and also to characterize some invariant subgroups associated to candidate flat outputs of the system (the most complex topic).

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The methodology proposed in this paper focuses on a better management of the degrees of freedom involved by the polynomial matrix approach. First, a reduced order basis of the left nullspace of the variational system described on the adjoint differential Ore ring is computed Beckermann et al. [2006]. Then a weak Popov form can be computed on the original differential ring Cheng and Labahn [2007], Davies and Cheng [2006] in order to obtain a reduced order basis of the ideal of differential forms. In addition, the problematic related to the strong closedness of the ideal is addressed in this paper. A solution associated to the system of partial differential equations involved by the generalized moving frame structure equations is sought in triangular form.

The paper is organized as follows: the first part recalls the necessary and sufficient conditions for differential flatness proposed in Lévine [2006]. A critical point of view related to a practical implementation of these results is given as an introduction of the second part. Moreover, an algorithm based on reduced order bases and the weak Popov form is proposed to formally compute a basis of the ideal of differential forms. Next, the integrability conditions are addressed and a formal solution is proposed based on a triangular matrix structure. Finally, the proposed methodology is illustrated on a numerical example.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR DIFFERENTIAL FLATNESS

Let X denote a smooth manifold of dimension n, $\mathrm{T}_x X$ its tangent space at an arbitrary point $x \in X$ and $\mathrm{T}X = \bigcup_{x\in X} \mathrm{T}_x X$ its tangent bundle. Let also introduce the

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manifold of jets of infinite order $\mathfrak{X}$, conceptually defined by the product of X with an infinite number of copies of $\mathbb{R}^n$ such that $\mathfrak{X} \stackrel{\text{def}}{=} X \times \mathbb{R}^n_\infty \stackrel{\text{def}}{=} X \times \mathbb{R}^n \times \mathbb{R}^n \times \ldots$. From a practical point-of-view, the number of copies of $\mathbb{R}^n$ is finite and application-related. Thus, this field will be denoted as "infinite but countable" in the sequel of the paper. The manifold $\mathfrak{X}$ is endowed with the product topology (the so-called Frechet topology) in which the open subsets can be written as $\mathcal{O} \times \mathbb{R}^n_\infty$, where $\mathcal{O}$ is an arbitrary open subset defined by the product of X and a finite (but arbitrary) number of copies of $\mathbb{R}^n$. In this geometric setting, a nonlinear implicit system is considered under the following form:

$$F(x, \dot{x}) = 0 \tag{1}$$

where x is defined on the manifold X and F is a C^∞ application from $\mathrm{T}X$ to $\mathbb{R}^{n-m}$ where the rank of its Jacobian matrix $\frac{\partial F}{\partial \dot{x}}$ is equal to $n-m$. The main advantage of such a representation is to be naturally invariant by endogeneous dynamic feedback Lévine [2006]. In order to characterize the Lie-Bäcklund isomorphisms Fliess et al. [1999], the notion of local flatness must be introduced to benefit from a linear geometric setting. In this way, a variational model of the implicit nonlinear system (1) can be obtained by computing its Frechet derivative, namely by performing a linearization of a system of jet expressions along a general trajectory.

Definition 1. The variational system associated to (1) around a general trajectory $t \mapsto x(t)$ of class C^∞ on an interval $\mathcal{I}$ of $\mathbb{R}$ corresponds to a linear time-varying implicit system given by:

$$\left(\frac{\partial F}{\partial x}(x(t), \dot{x}(t))\right)\xi(t) + \left(\frac{\partial F}{\partial \dot{x}}(x(t), \dot{x}(t))\right)\dot{\xi}(t) = 0 \tag{2}$$

with $\overline{\xi} = (t, \xi, \dot{\xi}, \ldots) \in \mathrm{T}\mathfrak{X}$.

The following Theorem gives a local charaterization of flatness for variational systems under the form (2).

Theorem 2. The implicit system is locally flat at $(\overline{x}_0, \overline{y}_0)$ with $\overline{x}_0 \in \mathfrak{X}_0 = \{\overline{x} \in \mathfrak{X} | L^k_{\tau_{\mathfrak{X}}} F(\overline{x}) = 0,\ \forall k \geq 0\}$ and $\overline{y}_0 \in \mathbb{R}^m_\infty$ if and only if there exists a locally smooth invertible mapping $\Phi = (\varphi_0, \varphi_1, \ldots) \in C^\infty(\mathcal{Y}_0; \mathcal{X}_0)$ from $\mathbb{R}^m_\infty$ to $\mathfrak{X}_0$, with smooth inverse, satisfying $\Phi(\overline{y}_0) = \overline{x}_0$, and such that

$$\Phi^* dF_i = 0, \quad i = 1, \ldots, n-m. \tag{3}$$

where each differential $dF_i \in \Lambda^1(\mathcal{X})$ is given by:

$$dF_i = \sum_{j=1}^{n} \left(\frac{\partial F_i}{\partial x_j} dx_j + \frac{\partial F_i}{\partial \dot{x}_j} d\dot{x}_j\right). \tag{4}$$

Following the polynomial matrix approach adopted in Lévine [2006] to characterize the flatness property of nonlinear implicit systems, the variational system associated to (1) can be represented by means of a matrix with entries in a particular Ore algebra of functional operators. While Ore algebras provides a general setting to deal with different classes of linear systems (e.g. differential time-delay systems, multidimensional discrete systems, ...), we restrict our study to the particular case of the Weyl algebra $\mathcal{H}[Z] = \mathfrak{K}[t]\,[Z; \sigma, \delta]$, where $\mathfrak{K}[t]$ represents the field of meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$. In this case, the operators σ and δ takes respectively the values $\sigma = id_{\mathfrak{K}[t]}$ and $\delta = \frac{d}{dt} = L_{\tau_X}$. Thus, the trivialization Φ is also restricted to the class of meromorphic functions. Let introduce the following polynomial matrices, whose entries are skew polynomials belonging to $\mathcal{H}[Z]$:

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}} Z, \quad P(\varphi_0) = \sum_{j \geq 0} \frac{\partial \varphi_0}{\partial y^{(j)}} Z^j \tag{5}$$

In this case, the condition $\Phi^* dF = 0$ can be rewritten as:

$$\Phi^* dF_{\big|\overline{y}} = P(F)_{\big|\Phi(\overline{y})} P(\varphi_0)_{\big|\overline{y}}\, dy = 0. \tag{6}$$

or, by using the Lie-Bäcklund equivalence:

$$\Phi^* dF_{\big|\Psi(\overline{x})} = P(F)_{\big|\overline{x}} P(\varphi_0)_{\big|\Psi(\overline{x})} dy = 0. \tag{7}$$

So as to characterize $P(F)$, denote by $\mathcal{M}_{m,n}[Z]$ the module of $m \times n$ matrices over $\mathcal{H}[Z]$, and by $\mathcal{U}_m[Z]$ the set of unimodular matrices $\mathcal{M}_{m,m}[Z]$ whose inverse still belongs to $\mathcal{M}_{m,m}[Z]$ (i.e. whose inverse is also a matrix containing Z-polynomials). Despite the weak algebraic properties of $\mathcal{M}_{m,n}[Z]$, this module admits some invariant directions. Thus, by using similarity transformations, it turns out that the matrices belonging to $\mathcal{M}_{m,n}[Z]$ admit a pseudo-diagonal structure which is more commonly known as the Smith normal form (see e.g. Cohn [1985]).

Definition 3. Given a $(n \times m)$ polynomial matrix M defined over the non-commutative ring $\mathcal{H}[Z]$, there exists some matrices $V \in \mathcal{U}_m[Z]$ and $T \in \mathcal{U}_n[Z]$ such that:

$$VMT = \begin{cases} (\Delta, 0_{n,m-n}) & if\ n < m \\ \begin{pmatrix} \Delta \\ 0_{n-m,m} \end{pmatrix} & if\ n > m \end{cases} \tag{8}$$

where Δ is a diagonal $p \times p$ matrix with $p = \min(n, m)$ whose diagonal elements, $(\delta_1, \ldots, \delta_\sigma, 0, \ldots, 0)$, are such that δ_i is a nonzero Z-polynomial for $i = 1, \ldots, \sigma$, and is a divisor of δ_j for all $i \leq j \leq \sigma$.

Inherently, this decomposition is not unique, only the matrix Δ is uniquely defined. From now on, considering a Smith decomposition given by (8), the unimodular pre- and post-multipliers will be denoted by $V \in \mathsf{L-Smith}\,(M)$ and $T \in \mathsf{R-Smith}\,(M)$ respectively. When the Smith decomposition leads to $(I_n, 0_{m,m-n})$ for $n < m$ and to $(I_m, 0_{n-m,m})^T$ for $m < n$, then the matrix M is called hyper-regular (in the trivial case $n = m$, the Smith decomposition leads to I_m). In the sequel, it is assumed that $P(F) \in \mathcal{M}_{n-m,n}[Z]$ is hyper-regular. This property corresponds roughly to the controlability of the variational system Lévine [2006]. The following theorem gives some necessary and sufficient conditions for the system (1) to be flat at $(\overline{x}_0, \overline{y}_0)$.

Theorem 4. (i) The set of hyper-regular matrices $\Theta \in \mathcal{M}_{n,m}[Z]$ satisfying $P(F)\Theta dy = 0$ is nonempty and given by

$$\Theta = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W \tag{9}$$

with $U \in \mathsf{R-Smith}\,(P(F))$ and $W \in \mathcal{U}_m[Z]$ an arbitrary unimodular matrix.

(ii) There exists a $n \times n$ matrix $Q \in \mathsf{L-Smith}\left(\hat{U}\right)$, with $\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}$ and an arbitrary matrix $R \in \mathcal{U}_m[Z]$ such that

$$Q\hat{U}R = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} \tag{10}$$

(iii) A necessary and sufficient condition for the system (1) to be flat at $(\overline{x}_0, \overline{y}_0)$ is that there exist $U \in \mathsf{R}-\mathsf{Smith}\,(P(F))$ and $Q \in \mathsf{L}-\mathsf{Smith}\left(\hat{U}\right)$ such that the $\mathcal{H}[Z]$-ideal Ω generated by the vector of 1-forms $\omega = (I_m, 0_{m,n-m})\, Qdx$ is strongly closed in a neighborhood $\mathcal{X}_0$ of $\overline{x}_0 \in \mathfrak{X}_0$.

The reader can refer to Lévine [2006] to find a detailed proof of this theorem.

The strong closedness of the $\mathcal{H}[Z]$-ideal Ω generated by the 1-forms $\omega = (\omega_1, \ldots, \omega_m)$ is ensured iff there exists a matrix $M \in \mathcal{U}_m[Z]$ that verifies $d\,(M\omega) = 0$. In Lévine [2006], the author proposes a characterization of strongly closed ideals based on a generalization, in the framework of manifolds of jets of infinite order, of the well-known moving frame structure equations (see e.g. Chern et al. [2000]). In this case, the reformulation of the initial problem leads to a triple of equations which seems to be more computationally tractable. The following Theorem gives some conditions for the $\mathcal{H}[Z]$-ideal Ω to be strongly closed.

Theorem 5. The $\mathcal{H}[Z]$-ideal Ω generated by the 1-forms $\omega = (\omega_1, \ldots, \omega_m)$ is strongly closed in $\mathcal{X}_0$ if and only if there exists a matrix μ whose entries are polynomials of Z with coefficients in Δ_n^1 , and a matrix $M \in \mathcal{U}_m[Z]$ such that [1] :

$$d\omega = \mu\ \omega \tag{11}$$

$$d\mu = \mu^2 \tag{12}$$

$$dM = -M\mu \tag{13}$$

In addition, if (11)-(13) holds, some flat outputs y are obtained by integrating the system of partial differential equations given by $dy = M\omega$.

3. AN EFFECTIVE ALGORITHM FOR ANALYTICAL COMPUTATION OF FLAT OUTPUTS

3.1 Statement of the problem

Obviously, the basis ω of the $\mathcal{H}[Z]$-ideal Ω is highly not unique. More precisely, considering a group G acting on a set X, the orbit of a point $x \in X$ is defined as the set of elements of X to which x can be moved by the elements g of G such that:

$$G_x = \{g \cdot x | g \in G\} \tag{14}$$

Given an element $y \in X$, we say that x and y belong to the same orbit if and only if there exists a left action $\tilde{g}$ (resp. a right action $\hat{g}$) in G such that $y = \tilde{g}x$ (resp. $y = x\hat{g}$). The set of all orbits of X is written as X/G (resp. $X \setminus G$) for left group actions (resp. right group actions), and is called the orbit space. In our case, the number of solutions of $P(F)\Theta dy = 0$ is made of a finite number of distinct orbits $\mathcal{O}_\Theta = \mathrm{Card}\,(\mathcal{M}_{n,m}[Z]/\mathcal{U}_m[Z])$ such that:

$$\mathcal{O}_\Theta = \frac{1}{6}\left(n(n+1)(n+5) - m(m+1)(m+5)\right) \tag{15}$$

and the number of distints orbits $\mathcal{O}_{\hat{U}} = \mathrm{Card}\,(\mathcal{U}_n[Z] \setminus \mathcal{U}_n[Z])$ of $\mathsf{L}-\mathsf{Smith}\left(\hat{U}\right)$ is also finite and given by:

$$\begin{aligned}\mathcal{O}_{\hat{U}} \ = \mathcal{O}_\Theta \times \tfrac{1}{6}\,(n(n+1)(n+5) - \cdots \\ (n-m)(n-m+1)(n-m+5))\end{aligned} \tag{16}$$

Consequently, the number of admissible solutions Q is quite huge and the obtention of a minimal basis ω of the ideal Ω represents an outstanding challenge that will be addressed in the next section by using reduced order bases and the weak Popov form.

On the other hand, the strong closedness property of the ideal Ω is difficult to verify since the candidate structure of the matrix M satisfying $d\,(M\omega) = 0$ is not initially constrained. The equation (11) is simply a system of linear non-differential equations and the operator $\mu \in \mathcal{L}_1\left((\Lambda(\mathfrak{X}))^m\right)$ such that $d\omega = \mu\ \omega$ can be obtained by componentwise identification. The second equation (12) confines the solution space of μ to a subspace in which the relation $\mathfrak{d}\,(\mu) = \mu^2$ is satisfied. It leads to a nonlinear system of PDE whose order is fixed by the general structure of μ resulting from the first condition. The first two equations have been thoroughly studied in Avanessoff [2005] where an attempt is made to characterize the set of solutions μ by introducing "very" formal series and a filtration on the system of equations. Finally, the equation (13) seems to be the more difficult to satisfy. However, by chosing the matrix $M \in \mathcal{U}_m[Z]$ with a triangular structure, the condition on the basis ω for which a solution of the associated system of PDE exists will be given in the next section.

3.2 Computation of a reduced-order basis of an ideal of differential forms

In this section, an effective algorithm is proposed in order to compute a basis of the $\mathcal{H}[Z]$-ideal of differential forms Ω. The methodology relies on the computation of reduced order bases and the weak Popov form. These tools have been introduced in Beckermann et al. [2006], in which the authors propose a recursive algorithm to compute the rank and a reduced order basis of the left nullspace of a matrix of Ore polynomials. Roughly speaking, it corresponds to a particular kind of Gaussian elimination for non-commutative polynomials including degree constraints to control coefficient growth during intermediate reductions. In Cheng and Labahn [2007], Davies and Cheng [2006], the authors show that the problem of computing the weak Popov form of Ore polynomial matrices and the associated unimodular transformation matrix can be reduced to the problem of computing a reduced order basis of the left nullspace of such matrices.

Recalling that the computations are performed over the field of meromorphic functions $\mathfrak{K}[Z]$, some complex terms may appear in the coefficients of the Z-polynomials when performing row (or even column) reductions. Hence, a reliable management of such terms is of primary interest when characterizing the left or right nullspaces of Ore polynomial matrices. More precisely, these terms must be avoided as such as possible in the rows of the transformation matrix corresponding to the zero rows of the residual matrix. In this way, the fraction-free recursion formulas defined in Beckermann et al. [2006] are modified so as to shift such complex meromorphic functions in the rows of the transformation matrix corresponding to the nonzero rows of the residual $R(Z)$.

[1] Δ_n^1 denotes the module generated by $(dx_1, \ldots, dx_n)$.

Theorem 6. Consider an order basis $M(Z) \in \mathcal{M}_{m,s}[Z]$ with order $\boldsymbol{\omega}$ and degree $\boldsymbol{\mu}$ over the Weyl algebra, and $\lambda \in \{1, \ldots, s\}$ the set of columns indices of $M(Z)$. Denote by $r_j = c_{\boldsymbol{\omega}^\lambda}\left((M(Z) \cdot F(Z))^{j,\lambda}\right)$ the (j, λ) entry of the first term of the residual $R(Z)$ of $M(Z)$. Finally, set $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} + \boldsymbol{e}_\lambda$.

(i) If $r_1 = \ldots = r_m = 0$ then $\tilde{M}(Z) = M(Z)$ is an order basis of degree $\boldsymbol{\nu} = \boldsymbol{\mu}$ and order $\tilde{\boldsymbol{\omega}}$.

(ii) Otherwise, let π be an index such that $r_\pi \neq 0$. Then an order basis $\tilde{M}(Z)$ of degree $\boldsymbol{\nu} = \boldsymbol{\mu} + \boldsymbol{e}_\lambda$ and order $\tilde{\boldsymbol{\omega}}$ with coefficients in $\mathfrak{K}[Z]$ can be obtained via the formulas:

$$\tilde{M}(Z)^{l,k} = M(Z)^{l,k} - \frac{r_l}{r_\pi} M(Z)^{\pi,k} \tag{17}$$

for $l, k = 1, \ldots, m,\ l \neq \pi$, and

$$\begin{aligned} \tilde{M}(Z)^{\pi,k} &= \left(Z - \tfrac{\delta(r_\pi)}{r_\pi}\right) M(Z)^{\pi,k} \\ &\quad - \textstyle\sum_{l \neq \pi} \tfrac{\sigma(p_l)}{r_\pi} \tilde{M}(Z)^{l,k} \end{aligned} \tag{18}$$

for $k = 1, \ldots, m$, where $p_j = c_{\boldsymbol{\mu}^j + \delta_{\pi,j} - 1}\left(M(Z)^{\pi,j}\right)$.

(iii) In addition, if $M(Z)$ is a Mahler system with respect to $(\boldsymbol{\mu}, \boldsymbol{\omega})$, then $\tilde{M}(Z)$ is also a Mahler system with respect to $(\boldsymbol{\nu}, \tilde{\boldsymbol{\omega}})$.

The algorithm associated to Theorem 6 is particularly well suited for left nullspace computations since the entries of the polynomial matrix containing the lowest degrees are eliminated first. However, it can't be directly adapted to compute right nullspace bases where the highest degrees of the polynomials are not frozen. To overcome this difficulty, it is possible to compute a right nullspace basis on the adjoint Ore ring $\mathcal{H}^*[Z]$. The following theorem provides a tractable algorithm for the characterization of an ideal of differential forms. It constitutes one of the main contributions of this paper.

Theorem 7. Let $P(F) \in \mathcal{M}_{n-m,n}[Z]$ be the variational system associated to the regular implicit system $F(x, \dot{x}) = 0$.

(i) A reduced order basis $\tilde{\Theta}(Z) \in \mathcal{M}_{n,m}[Z]$ satisfying $P(F)\tilde{\Theta}(Z) = 0$ can be obtained by computing a reduced order basis $\tilde{\Theta}^*(Z) \in \mathcal{M}_{m,n}[Z]$ of the left nullspace of $P(F)^*$ defined on the adjoint Ore ring $\mathcal{H}^*[Z]$ such that [2]:

$$\tilde{\Theta}^*(Z) P(F)^* = 0 \tag{19}$$

(ii) A reduced order left multiplier $\tilde{Q}(Z) \in \mathcal{U}_n[Z]$ can be obtained by computing a weak Popov form $T(Z) \in \mathcal{M}_{n,m}$ of $\tilde{\Theta}(Z) \in \mathcal{M}_{n,m}[Z]$ such that:

$$\tilde{Q}(Z) \cdot \tilde{\Theta}(Z) = T(Z) \tag{20}$$

(iii) A reduced order basis ω of the $\mathcal{H}[Z]$-ideal of differential forms Ω is given by $\omega = (I_m, 0_{m,n-m})\,\tilde{Q}dx$

The proof is ommited here to save place. Theorem 7 provides a mean to control the growth of the final transformation matrices by choosing pivots according to row degree constraints. These additional constraints involve a lower number of iterations needed to converge to a solution. In comparison, by using the algorithm based on Smith decompositions, all the combinations of the pivot elements must be explored before to make a decision about the retained transformation matrix. The termination criterion associated to Theorem 6 is based on an estimation of the number of cycles that must be performed to obtain a reduced order basis. Namely, for an input matrix $F(Z) \in \mathcal{M}_{m,s}[Z]$, we consider that $(mN+1)s$ is an upper bound to obtain a system $M(Z)$ of degree $\boldsymbol{\mu}$ and order $\boldsymbol{\omega}$, where N stands for the degree of $F(Z)$, i.e. the largest degree of its polynomial entries Beckermann et al. [2006]. Applying this formula to Theorem 7 and noting that $N = \deg(P(F)) = 1$ due to the structure of (5), a right nullspace basis $\tilde{\Theta}(Z)$ of $P(F)$ can be obtained in at most $\overline{\mathcal{O}}_{\tilde{\Theta}}$ iterations such that:

$$\overline{\mathcal{O}}_{\tilde{\Theta}} = (n+1)(n-m) \tag{21}$$

and a left nullspace basis $\tilde{Q}(Z)$ of $\tilde{\Theta}_{aug}$ can be computed in $\overline{\mathcal{O}}_{\tilde{Q}}$ iterations such that:

$$\overline{\mathcal{O}}_{\tilde{Q}} = ((n+m)N_{\tilde{\Theta}} + 1)\, m \tag{22}$$

where $N_{\tilde{\Theta}}$ corresponds to the degree of the matrix $\tilde{\Theta}$ previously obtained. Note that in the case of early convergence of the process, the termination criterion can be rebounded downwards. These bounds must be compared to those associated to the Smith normal forms which are given by (15) and (16).

3.3 Integrability conditions

Recall that the $\mathcal{H}[Z]$-ideal Ω generated by the 1-forms $\omega = (\omega_1, \ldots, \omega_m)$ is strongly closed in $\mathcal{X}_0$ if and only if there exists a matrix $M \in \mathcal{U}_m[Z]$ such that $d(M\omega) = 0$. This condition has been rewritten previously using the generalized moving frame structure equations, leading to the triple of equations given by (11)-(13). As previously stated, the equation (13) is the more difficult to verify. Namely, it is not easy to define a generic structure of the matrices $M \in \mathcal{U}_m[Z]$ (and therefore the less conservative) whose inverses still belong to $\mathcal{U}_m[Z]$. However, if the solution space is restricted to the subset $\mathcal{U}_{\Delta m}[Z]$ of the $m \times m$ matrices M having a triangular structure, these matrices can be chosen to be unimodular by construction. In the sequel, a computationally tractable algorithm is proposed in order to integrate the basis ω whose solutions belong to $\mathcal{U}_{\Delta m}[Z]$.

$$\mu = \begin{pmatrix} \sum_{j=0}^{J} \mu_{1,1}^j Z^j & \ldots & \sum_{j=0}^{J} \mu_{1,m}^j Z^j \\ \vdots & & \vdots \\ \sum_{j=0}^{J} \mu_{m,1}^j Z^j & \ldots & \sum_{j=0}^{J} \mu_{m,m}^j Z^j \end{pmatrix} \tag{23}$$

where J represents the greatest exponent in $Z = \frac{d}{dt}$ of the elements of μ, and $\mu_{p,q}^j$ are arbitrary 1-forms in $\Lambda^1(\mathfrak{X})$. By factorizing each form of the matrix μ in the basis $\omega = (\omega_1, \ldots, \omega_m)$, we obtain for all $(p, q) \in \{1, \ldots, m\}^2$ and $j \in \{0, \ldots, J\}$:

$$\mu_{p,q}^j = \sum_{i=1}^{m} \sum_{k=0}^{K} a_{p,q,i,k}^j \omega_i^k \tag{24}$$

with $a_{p,q,i,k}^j \in \mathfrak{K}[t]$ for all p, q, j, i and k. Since the values of the integers J and K are not initially known, (11)-(12) corresponds to a system of PDE where neither the order nor the number of unknown functions are bounded

[2] In the case of the Weyl algebra, the adjoint Ore ring is given by $\mathcal{H}^*[Z] = \mathfrak{K}[t]\,[Z; \sigma^*, \delta^*]$ where $\sigma^* = id_{\mathfrak{K}[t]}$ and $\delta^* = -\frac{d}{dt} = -L_{\tau_X}$.

by advance. However, if the integers J and K are arbitrary fixed, it is normally possible to determine if the system (11)-(12) admits a set of solutions. More particularly, when the solution space is restricted to $\mathcal{U}_{\Delta m}[Z]$, the ideal Ω is strongly closed if there exists a set of m independent 1-forms $\pi_1, \ldots, \pi_m$ generating Ω with a triangular structure, such that $d\pi_i = \sum_{j=i+1}^{m} \sum_{k=0}^{s_{i,j}} \tau_{i,j}^k \wedge \pi_j^{(k)}$ for $i = 1, \ldots, m-1$ and $d\pi_m = 0$. In this case, the desired matrix M may be sought in triangular form such as:

$$M = \begin{pmatrix} 1 & M_{11} & \ldots & & M_{1,m} \\ 0 & 1 & \ldots & & M_{2,m} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & M_{m-1,m} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \quad (25)$$

Then M^{-1}, dM, $M^{-1}dM$ are also triangular and therefore $M^{-1}dM \wedge \pi$ has the required structure. Indeed, the strong closedness property is satisfied if the identification

$$M^{-1}dM = \begin{pmatrix} 0 & \sum_{k\geq 0} \tau_{1,2}^k Z^k & \ldots & & \sum_{k\geq 0} \tau_{1,m}^k Z^k \\ 0 & 0 & \ldots & & \sum_{k\geq 0} \tau_{2,m}^k Z^k \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 0 & \sum_{k\geq 0} \tau_{m-1,m}^k Z^k \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \quad (26)$$

is possible, where the functions $\tau_{i,j}^k$ represent general 1-forms such that $\tau_{i,j}^k = \sum_{p=1}^{m} a_{i,j,p}^k dx_p$ with $a_{i,j,p}^k \in \mathfrak{K}[t]$ for all i, j, p, k and where $dx = (dx_1, \ldots, dx_n)$ is a basis of the tangent space of $\mathrm{T}_x X$. Obviously, this construction is strongly related to the choice of the triangularly generating vector π and the previous identification may fail in some cases. However, the main interest of this structure is that the matrix M is unimodular by construction (namely $det(M) = 1$) and its entries can be more easily computed. Clearly, the general structure of the matrix μ can be written with respect to the entries of M. Let $\mathcal{M}_{i,j}$ and $\mathcal{N}_{p,q}$ be two sets of integers such that $\mathcal{M}_{i,j} = \{i, \ldots, j | i < j\}$ and $\mathcal{N}_{p,q} = \{p, \ldots, q | p < q\}$. let $M_{\mathcal{M}_{i,j}\mathcal{N}_{p,q}} \triangleq \{M_{xy} | \forall x \in \mathcal{M}_{i,j}, \forall y \in \mathcal{N}_{p,q}, x < y, (x; y) \neq (i; q)\}$ where the M_{xy} are arbitrary Z-polynomials with meromorphic functions coefficients. If M satisfies the equation (25), then the general structure of the matrix $\mu = M^{-1}dM$ is given by the equation (27). Moreover, the additionnal contraint $d\mu = \mu \wedge \mu$ must be verified, as stated in Theorem 5. In the case of a triangular matrix M, the set of contraints given by (28) must be met.

$$\begin{cases} d\mu_{ij} = \sum_{k=i+1}^{j-1} \mu_{ik} \wedge \mu_{kj} & \text{if } j > i+1 \\ d\mu_{ij} = 0 & \text{if } j = i+1 \end{cases} \quad (28)$$

where $i \in \{1, \ldots, m-1\}$ and $j \in \{1, \ldots, m\}$.
Therefore, the terms dM_{ij} can be computed and integrated to find the desired matrix M, providing that the identification $M^{-1}dM = d\omega$ is possible and that the constraints given by (28) are met. In order to derive the conditions on the basis ω for which the identification $M^{-1}dM = d\omega$ is made possible, some notations are introduced first.

Let $x = (x_1, \ldots, x_n)$ be the states of the nonlinear implicit system (1) and $\overline{x} = (x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \ldots)$ the set of global coordinates of $\mathfrak{X}$. In addition, denote by ρ the set of integers defined by $\rho = \{\max_{\alpha} dx_i^{(\alpha)} \neq 0, \forall i \in \{1, \ldots, n\}\}$. Let $\mathfrak{J}$ be the finite set of integers such that:

$$\mathfrak{J} = \{i | f_{i,k,m}\left(x_i^{(k)}\right) = 0, \forall k \in \{0, \ldots, \nu\}, \forall i \in \{1, \ldots, n\}\} \quad (29)$$

and ${}^c\mathfrak{J}$ its complement. Let $\tilde{\mathfrak{J}}$ be the finite set of integers defined by:

$$\tilde{\mathfrak{J}} = \{i | f_{i,k,m}\left(x_i^{(k)}\right) = 0, k = 0, \forall i \in \{1, \ldots, n\}\} \quad (30)$$

and ${}^c\tilde{\mathfrak{J}}$ its complement. Finally, denote by $\hat{\mathfrak{J}}$ the finite set of integers such that $\hat{\mathfrak{J}} = \tilde{\mathfrak{J}} \backslash \mathfrak{J}$ and ${}^c\hat{\mathfrak{J}}$ its complement given by ${}^c\hat{\mathfrak{J}} = {}^c\tilde{\mathfrak{J}} \cup \mathfrak{J}$. The terms $\delta_{p,q}$ with $p \in \{1, \ldots, m-1\}$ and $q \in \{2, \ldots, m\}$ represent arbitrary integers. The following Theorem gives a necessary and sufficient condition for the ideal Ω to be strongly closed in a neighborhood $\mathcal{X}_0$ of $\overline{x}_0 \in \mathfrak{X}_0$.

Theorem 8. A necessary and sufficient condition for the $\mathcal{H}[Z]$-ideal Ω to be strongly closed in a neighborhood $\mathcal{X}_0$ of $\overline{x}_0 \in \mathfrak{X}_0$ is that the two following propositions are satisfied:

(i) There exists a generating vector of 1-forms $\omega = (\omega_1, \ldots, \omega_m)$ satisfying the equation (31).
(ii) The system of partial differential equations involved by (27) and (28) admits a solution.

To save place, the proof of the previous Theorem is ommited here. If the identification $M^{-1}dM = d\omega$ fails, this technique is no longer applicable and the problem of strong closedness of the ideal is far more complex since the matrix M may have an arbitrary structure and the unimodularity property must be additionally ensured.

The overall procedure described in the previous sections is implemented as a library of formal functions under Maple$^{\mathrm{TM}}$software. In the following section, we illustrate step by step the proposed methodology on a 3-tank process.

3.4 Application to a 3-tank process

In order to illustrate the proposed methodology, we consider the nonlinear model of an hydraulic system. It consists of three vertical tanks T_1, T_2 and T_3 with cross-section S_c, a storage tank T_0 and two pumps P_1 and P_2. Each vertical tank is connected to the storage tank by means of a duct of section S_n whose flow may be modulated by means of a manual gate. In addition, two ducts with the same section S_n, whose flow can also be modulated by

$$\mu = \begin{pmatrix} 0 & dM_{12} & f_{13}(dM_{13}, M_{\mathcal{M}_{1,2}\mathcal{N}_{2,3}}, dM_{\mathcal{M}_{1,2}\mathcal{N}_{2,3}}) & \ldots & f_{1m}(dM_{1m}, M_{\mathcal{M}_{1,m-1}\mathcal{N}_{2,m}}, dM_{\mathcal{M}_{1,m-1}\mathcal{N}_{2,m}}) \\ 0 & 0 & dM_{23} & \ldots & f_{2m}(dM_{2m}, M_{\mathcal{M}_{2,m-1}\mathcal{N}_{3,m}}, dM_{\mathcal{M}_{2,m-1}\mathcal{N}_{3,m}}) \\ \vdots & \vdots & & & \vdots \\ 0 & & \ldots & \ldots & dM_{m-1,m} \\ 0 & & \ldots & \ldots & 0 \end{pmatrix} \quad (27)$$

$$\left\{\begin{array}{rcl}\omega_m &=& \sum_{i=1}^{\mathrm{Card}(^c\tilde{\mathfrak{J}})} f_{i,m}\left(x_{^c\tilde{\mathfrak{J}}_i}\right)dx_{^c\tilde{\mathfrak{J}}_i} + \sum_{j=1}^{\mathrm{Card}(\hat{\mathfrak{J}})} g_{j,m}\left(x_{\hat{\mathfrak{J}}_j}^{(\rho_{\hat{\mathfrak{J}}_j})}\right)dx_{\hat{\mathfrak{J}}_j}^{(\rho_{\hat{\mathfrak{J}}_j})}\\ \omega_{m-1} &=& \sum_{k=1}^{\mathrm{Card}(\tilde{\mathfrak{J}})} f_{k,m-1}\left(x_{\tilde{\mathfrak{J}}_k}, x_{^c\mathfrak{J}}^{(\rho_{^c\mathfrak{J}},\ldots,\rho_{^c\mathfrak{J}}+\delta_{m-1,m})}\right)dx_{\tilde{\mathfrak{J}}_k} + \sum_{p=1}^{\mathrm{Card}(^c\tilde{\mathfrak{J}})} g_{p,m-1}\left(x_{^c\tilde{\mathfrak{J}}}, x_{^c\mathfrak{J}}^{(\rho_{^c\mathfrak{J}},\ldots,\rho_{^c\mathfrak{J}}+\delta_{m-1,m})}\right)dx_{^c\tilde{\mathfrak{J}}_p}\\ && + \sum_{k=1}^{\mathrm{Card}(\hat{\mathfrak{J}})}\sum_{s=\rho_{\hat{\mathfrak{J}}_k}}^{\rho_{\hat{\mathfrak{J}}_k}+\delta_{m-1,m}} h_{k,s,m-1}\left(x_{\hat{\mathfrak{J}}_k}^{(s)}, x\right)dx_{\hat{\mathfrak{J}}_k}^{(s)} + \sum_{k=1}^{\mathrm{Card}(^c\tilde{\mathfrak{J}})}\sum_{s=1}^{\delta_{m-1,m}} e_{k,s,m-1}\left(x_{^c\tilde{\mathfrak{J}}_k}^{(s)}, x\right)dx_{^c\tilde{\mathfrak{J}}_k}^{(s)}\\ \vdots & \vdots & \\ \omega_{m-i} &=& \sum_{k=1}^{n} f_{k,m-i}\left(x^{(0,\ldots,\sum_{p=m-i+1}^{m-1}\delta_{m-i,p})}, x_{\hat{\mathfrak{J}}}^{(\rho_{\hat{\mathfrak{J}}},\ldots,\rho_{\hat{\mathfrak{J}}}+\sum_{q=m-i+1}^{m}\delta_{m-i,q})}, x_{^c\tilde{\mathfrak{J}}}^{(0,\ldots,\sum_{q=m-i+1}^{m}\delta_{m-i,q})}\right)dx_k\\ && + \sum_{k=1}^{n}\sum_{s=0}^{\sum_{p=m-i+1}^{m-1}\delta_{m-i,p}} g_{k,s,m-i}\left(x_k^{(s)}, x\right)dx_k^{(s)} + \sum_{k=1}^{\mathrm{Card}(\hat{\mathfrak{J}})}\sum_{s=\rho_{\hat{\mathfrak{J}}_k}}^{\rho_{\hat{\mathfrak{J}}_k}+\sum_{q=m-i+1}^{m}\delta_{m-i,q}} h_{k,s,m-i}\left(x_{\hat{\mathfrak{J}}_k}^{(s)}, x\right)dx_{\hat{\mathfrak{J}}_k}^{(s)}\\ && + \sum_{k=1}^{\mathrm{Card}(^c\tilde{\mathfrak{J}})}\sum_{s=0}^{\sum_{q=m-i+1}^{m}\delta_{m-i,q}} e_{k,s,m-i}\left(x_{^c\tilde{\mathfrak{J}}_k}^{(s)}, x\right)dx_{^c\tilde{\mathfrak{J}}_k}^{(s)}\end{array}\right. \quad (31)$$

where $x_{\mathfrak{J}}$ is defined by $x_{\mathfrak{J}} = (x_{\mathfrak{J}_1}, \ldots, x_{\mathfrak{J}_{end}})$ and also $x_{\mathfrak{J}}^{(\rho_{\mathfrak{J}},\ldots,\rho_{\mathfrak{J}}+\delta_{m-1})} = x_{\mathfrak{J}_1}^{(\rho_{\mathfrak{J}_1},\ldots,\rho_{\mathfrak{J}_1}+\delta_{m-1})}, \ldots, x_{\mathfrak{J}_{end}}^{(\rho_{\mathfrak{J}_{end}},\ldots,\rho_{\mathfrak{J}_{end}}+\delta_{m-1})}$ and so on.

some gates, are used to connect the tanks T_1 and T_3 in the one hand, and the tanks T_2 and T_3 in the other hand. The control inputs of the process are the flows Q_1 and Q_2 of the pumps P_1 and P_2. Each az_{ij} term refer to the flow of the duct from the element i to the element j through the gate V_{ij}. Moreover, we consider that $h_1 > h_3 > h_2$, that the gates V_{13}, V_{32} and V_{20} are open and all the other closed. Under these assumptions, a nonlinear model of the 3-tank process is given by:

$$\left\{\begin{array}{rcl} S_c\frac{dh_1}{dt} &=& -az_{10}S_n\sqrt{2gh_1} - az_{13}S_n\sqrt{2g\left(h_1-h_3\right)} + Q_1\\ S_c\frac{dh_2}{dt} &=& -az_{20}S_n\sqrt{2gh_2} + az_{32}S_n\sqrt{2g\left(h_3-h_2\right)} + Q_2\\ S_c\frac{dh_3}{dt} &=& -az_{30}S_n\sqrt{2gh_3} - az_{32}S_n\sqrt{2g\left(h_3-h_2\right)}\\ && +az_{13}S_n\sqrt{2g\left(h_1-h_3\right)}\end{array}\right. \quad (32)$$

Under these assumptions, the model (32) can be put into an implicit form such that:

$$\frac{dh_3}{dt} - K_1\sqrt{h_1-h_3} + K_2\sqrt{h_3-h_2} = 0 \quad (33)$$

where $K_1 = \frac{az_{13}S_n}{S_c}\sqrt{2g}$ and $K_2 = \frac{az_{32}S_n}{S_c}\sqrt{2g}$. By using equation (5), the variational system associated to (33) is given by:

$$P(F) = \begin{pmatrix} -\frac{1}{2}\frac{K_1}{\sqrt{h_1-h_3}}\\ -\frac{1}{2}\frac{K_2}{\sqrt{h_3-h_2}}\\ \frac{1}{2}\frac{K_1}{\sqrt{h_1-h_3}} + \frac{1}{2}\frac{K_2}{\sqrt{h_3-h_2}} + Z\end{pmatrix}^T \quad (34)$$

The first step of the algorithm consists in computing a basis of the $\mathcal{H}[Z]$-ideal Ω corresponding to the variational system $P(F)$. The equation (15) indicates that there is $\mathcal{O}_\Theta = 9$ possible Smith decompositions of $P(F)$. Using Theorem 7, we can obtain a basis of the ideal Ω in two iterations. We begin by computing the adjoint Ore matrix $P(F)^*$ of $P(F)$ on $\mathcal{H}[Z]$ such as:

$$P(F)^* = \begin{pmatrix} -\frac{1}{2}\frac{K_1}{\sqrt{h_1-h_3}}\\ -\frac{1}{2}\frac{K_2}{\sqrt{h_3-h_2}}\\ \frac{1}{2}\frac{K_2(h_1-h_3)+K_1\sqrt{h_3-h_2}\sqrt{h_1-h_3}}{(h_1-h_3)\sqrt{h_3-h_2}} - Z\end{pmatrix} \quad (35)$$

Then, by using Theorem 6, a basis of the left nullspace of $P(F)^*$ can be obtained. Next, $p = -\frac{1}{2}\frac{K_1}{\sqrt{h_1-h_3}}$ is chosen twice as the pivoting element during the row reduction process and, by taking the adjoint of the resulting transformation matrix on the adjoint Ore ring $\mathcal{H}^*[Z]$, we obtain a reduced order basis $\tilde{\Theta}(Z)$ of the right nullspace of $P(F)$ such as:

$$\tilde{\Theta}(Z) = \begin{pmatrix} -\frac{K_2}{K_1}\sqrt{\frac{h_1-h_3}{h_3-h_2}} & 1 + \frac{K_2}{K_1}\sqrt{\frac{h_1-h_3}{h_3-h_2}} + \frac{2\sqrt{h_1-h_3}}{K_1}Z\\ 1 & 0\\ 0 & 1\end{pmatrix} \quad (36)$$

Recall that the formula given by (21) is only an upper bound of the number of iterations required to obtain the desired structure and may not be reached in all the cases. Indeed, two iterations are sufficient in this study case while the formula gives $\overline{\mathcal{O}}_{\tilde{Q}} = 4$ iterations. In order to compute a weak Popov form $T(Z)$ of $\tilde{\Theta}(Z)$, a left nullspace basis of the augmented matrix $\tilde{\Theta}_{aug}$ is computed, where

$$\tilde{\Theta}_{aug} = \begin{pmatrix}\tilde{\Theta}(Z)\\ -I_2\end{pmatrix} \quad (37)$$

A straightforward application of Theorem 6 leads to a form $\tilde{M}_{aug}\tilde{\Theta}_{aug} = R_{aug}$ after four iterations of the algorithm, where

$$\tilde{M}_{aug} = \begin{pmatrix} 1 & \frac{K_2}{K_1}\sqrt{\frac{h_1-h_3}{h_3-h_2}} & \tilde{M}_{aug_{13}} & 0 & 0\\ 0 & Z^2 & 0 & 0 & 0\\ 0 & 0 & Z^2 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1\end{pmatrix} \quad (38)$$

and with $\tilde{M}_{aug_{13}} = -1 - \frac{K_2}{K_1}\sqrt{\frac{h_1-h_3}{h_3-h_2}} - \frac{2\sqrt{h_1-h_3}}{K_1}Z$. Moreover, the matrix of residuals is given by:

$$R_{aug} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T \quad (39)$$

To compare with the Smith decomposition algorithm, the formula (16) gives $\mathcal{O}_{\hat{U}} = 126$ candidate solutions while equation (22) estimates as $\overline{\mathcal{O}}_{\tilde{Q}} = 12$ the number of iterations needed to obtain a reduced order basis, with $N_{\tilde{\Theta}} = 1$. Once again, this number is not reached here since the algorithm terminates after four iterations only. Finally, by taking the rows of $\tilde{M}_{aug}$ corresponding to the n zero rows of R_{aug}, the following matrix is obtained:

$$\begin{pmatrix} 1 & \frac{K_2}{K_1}\sqrt{\frac{h_1-h_3}{h_3-h_2}} & -1 - \frac{K_2}{K_1}\sqrt{\frac{h_1-h_3}{h_3-h_2}} - \frac{2\sqrt{h_1-h_3}}{K_1}Z & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad (40)$$

which may be partitionned such as

$$\begin{pmatrix} \tilde{Q}(Z) & T(Z) \end{pmatrix} \cdot \begin{pmatrix} \tilde{\Theta}(Z) \\ -I_m \end{pmatrix} = 0_{n,m} \quad (41)$$

where the requested unimodular matrix satisfying $\tilde{Q}(Z) \cdot \tilde{\Theta}(Z) = T(Z)$ is given by

$$\tilde{Q}(Z) = \begin{pmatrix} 1 & \frac{K_2}{K_1}\sqrt{\frac{h_1-h_3}{h_3-h_2}} & -1 - \frac{K_2}{K_1}\sqrt{\frac{h_1-h_3}{h_3-h_2}} - \frac{2\sqrt{h_1-h_3}}{K_1}Z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (42)$$

and

$$T(Z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \quad (43)$$

Since the matrix $\tilde{\Theta}(Z)$ is hyper-regular, the weak Popov form $T(Z)$ satisfies the equation (20) and comprises $\min(m, n) = 2$ nonzero rows. Then, a reduced order basis ω can be obtained by taking the rows of $\tilde{Q}(Z)$ corresponding to the nonzero rows of $T(Z)$ and by multiplying it with the basis of the cotangent space $T_x^*\mathfrak{X}$:

$$\omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dh_1 \\ dh_2 \\ dh_3 \end{pmatrix} \quad (44)$$

The integrability conditions are trivially satisfied here since $\omega_1 = dh_2$ and $\omega_2 = dh_3$. By choosing M as the identity matrix, it leads to $dy_1 = dh_2$ and $dy_2 = dh_3$, and the flat outputs are then given by:

$$\begin{cases} y_1 = h_2 + C_1 \\ y_2 = h_3 + C_2 \end{cases} \quad (45)$$

where C_1 and C_2 are arbitrary constants. It is straightforward to check that h_2 and h_3 are flat outputs of the 3-tank process. Clearly, all the states and inputs of (32) can be rewritten as functions of these outputs and of a finite number of their time derivatives.

4. CONCLUSIONS AND FUTURE WORKS

In this paper, a computationally tractable algorithm is proposed with a view to the development of a formal library under Maple^{TM}devoted to nonlinear flat systems. The first part of the algorithm deals with the determination of a reduced-order basis of an ideal of differential forms. Especially, it has been shown that the use of reduced order bases leads to a better management of the degrees of freedom involved by the Smith decompositions. In the sesond part of the paper, the condition for which the ideal of differential forms is strongly closed is given in the case of a triangular generating structure. Then, a subspace of reduced-order solutions of the system of PDE involved by the generalized moving frame structure equations is sought in triangular form. So as to improve the robustness property of flatness-based control designs, an on-going research effort focusses on the characterization of some invariant subgroups of flat outputs under parameters variations.

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