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Design Technique For Multi-Rate Linear Systems

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Abstract: In this paper a design technique for multi-rate, linear digital control systems is described. This technique takes into account all the sampling rates involved in the system, and generates a multi-rate system that mimics the dynamics of a desired single-rate closed-loop system. The desired closed-loop system is referred to as an ideal single-rate system (ISRS) since it operates at the fastest sampling rate present in the system. The multi-rate system is designed to achieve state-matching, at the fastest rate, with the ISRS, and to exhibit a ripple-free response with zero steady-state error in response to a step reference signal. Unlike prior work in the literature, which is applicable only to static feedback ISRS, the proposed state-space design procedure is applicable for any LTI dynamic feedback ISRS. The proposed design is successfully implemented on a hard disk drive (HDD) platform for seek control of the read-write (R/W) arm. A representative sample of the experimental results are shown and discussed to highlight the proposed multi-rate technique.*Copyright©2008 IFAC*

Keywords: Multi-rate discrete-time systems; regulation problem; ripple-free response; hard disk drives.

1. INTRODUCTION

Multi-rate digital systems are widely used in many industries. They have been studied for many decades, and many methods are available in literature for their analysis and control design. Systematic procedures to model and control single-rate digital systems are described in many texts Ogata [1987], Franklin et al. [1990]. Due to the simplicity in the analysis and design of single-rate digital control systems, the frequency decomposition and switch decomposition methods were developed by Coffey and Williams [1966], Kranc [1957a,b], Boykin and Frazier [1975] to model multi-rate systems as single-rate digital systems. Both the approaches lead to the generation of a unique discrete-time transfer function which models the overall multi-rate system. The main drawback of these two methods is the increase in the complexity of the procedure when the ratio of two of the involved sample rates increases. In the early years there was considerable interest in using the state-space approach to model and control multi-rate systems. A state-space technique that simplified the modeling of multi-rate systems involving non-multiple sample rates was introduced in Araki and Yamamoto [1986]. However, this procedure led to potentially higher dimensional state-space systems. The problem of reducing the high dimensionality of the resulting state-space system was solved by Godbout et al. [1990], Jordan and Apostolakis [1991]. The state-space approach to design controllers for multi-rate systems has been mainly restricted to static feedback controllers. A procedure to design optimal static time-varying feedback controllers which minimize a certain functional was developed in Jordan and Apostolakis [1991]. A solution to the pole placement problem by using static feedback controllers is given in Jordan and Apostolakis [1991]. In Lee [2006], a procedure to design a static feedback multi-rate control system achieving a similar step response to a pre-defined single-rate static feedback control system is given.

There are numerous applications in large-scale manufacturing where multi-rate digital control systems are employed. One such application where multi-rate systems are found is in control of the R/W arm of the hard disk drive (HDD). The data on hard-disk drive platters are stored along thousands of concentric circular tracks. Each track is divided into sectors, which are usually the smallest addressable units containing the data stored on the hard-disk drive. Each sector stores a certain amount of user data, while additional bytes are reserved for control and management purposes of the drive. Those additional bytes are usually stored at the beginning of the sectors and include the sector ID information. The sector ID is useful for locating the sector on the disk when certain data have to be retrieved or stored. The angular position of the R/W magnetic head is obtained every time the R/W head passes through the part of the sector containing the sector ID information. Therefore, the angular position is obtained at different rates depending on the rotational speed of the platters. The control input current to the Voice-Coil Motor Actuator (VCMA), which drives the arm containing the R/W head, is provided by the digital board installed on the back of the HDD. Modeling the HDD as a multi-rate system is justified by the fact that the discrete-time control action provided by the digital board updates at a rate much faster than the available angular position feedback update rate. Therefore, a multi-rate control structure can be used to improve and optimize the performance of the overall system.

It is well known that decreasing the sampling time in a single-rate system leads to a better inter-sample behavior. Therefore, the reference model is taken as a single-rate system operating at the fast control update rate. The designed discrete-time controller is provided with time-varying gains periodically changing within every measurement sampling period. The controller gains are designed with the twofold target of guaranteeing no ripples in the steady-state response to a step reference signal, and achieving state matching with the reference model, at each control update instant. The proposed design technique is applied to seek control of the R/W arm of a hard-disk drive. Experimental results on the HDD platform show that increasing

the ratio between the measurement and control update rate leads to an improvement in the inter-sample behavior.

The contribution of this work consists of a new state-space approach to design dynamic controllers for multi-rate systems such that the closed-loop system response matches the response of any given Ideal fast Single-Rate LTI System (ISRS). The proposed design technique leads to a multi-rate system achieving state-matching with the ISRS, at the fastest rate involved. Moreover, it is shown that the response to a step reference change is ripple-free with zero steady-state error if the ISRS is at least a type-one system. A systematic procedure for obtaining the time-varying gains of the multi-rate controller is given. In addition, the proposed design technique has been successfully implemented on a HDD platform.

The remainder of the paper is organized as follows. Section 2 gives the problem formulation. Description of the multi-rate design technique is given in Section 3; details on the selection of gains with pertinent discussions are also given. In Section 4, experimental platform is described together with a discussion of the experimental results. Conclusions of the paper and potential future work are given in Section 5.

2. PROBLEM FORMULATION

Consider the continuous-time linear time-invariant system
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad (1)$$
where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, and $y(t) \in \mathbb{R}^{n_y}$ is the measured output to be regulated to a step reference signal $r \in \mathbb{R}^{n_y}$. The discrete-time equivalent of the plant (1) with the sampling time T_u is given by
$$\begin{aligned} x(i+1) &= \Phi_c x(i) + \Gamma_c u(i) \\ y(i) &= C_c x(i) \end{aligned} \qquad (2)$$
where
$$\Phi_c = e^{AT_u}, \quad \Gamma_c = \int_0^{T_u} e^{A(T_u - t)} B\, dt, \quad C_c = C. \qquad (3)$$
Assume that a linear time-invariant digital controller, operating at the sampling time T_u, can be designed to achieve certain closed-loop performance criteria. Let this controller be given by
$$\begin{aligned} \eta(i+1) &= A_\eta \eta(i) + B_\eta e(i) \\ u(i) &= C_\eta \eta(i) + D_\eta e(i) \end{aligned} \qquad (4)$$
where $\eta \in \mathbb{R}^{n_\eta}$ is the controller state vector, $e = r - y$ is the regulation error, which is the input to the controller, and the matrices A_η, B_η, C_η and D_η are known. Notice that, in the definition of controller (4), the feedback measurement e is assumed to be available at the same rate as the control update rate, $1/T_u$. This assumption is intrinsic to any single-rate feedback control system. Denote with
$$\begin{aligned} \zeta(i+1) &= F\zeta(i) + Gr \\ y(i) &= \bar{C}_c \zeta(i) \end{aligned} \qquad (5)$$
the discrete-time closed-loop system, formed by (2) and (4), where $\zeta(\cdot) = [x^T(\cdot), \eta^T(\cdot)]^T$ is the closed-loop state vector. In many practical applications, the plant output $y(t)$ is not available at the control update period T_u, but only at a longer period $T_s = qT_u$ (where q is any positive integer). This may be due to the measurement hardware limitations or to some physical limitations of the controlled model. In these cases it is

common practice to directly design the controller at the longer measurement period T_s. However, increasing the control update period, from T_u to T_s, has a negative effect on the inter-sample behavior of the closed-loop response. Moreover, by adopting a controller operating at the longer sampling time T_s we do not take advantage of the available digital hardware. In order to keep track of both the sampling periods, T_s and T_u, the notation (k, i) will be used in the rest of the paper to refer to the time instant $t = kT_s + iT_u$, with $i = 0, 1, \ldots, q - 1$. By adopting this notation, the discrete-equivalent (2) (at the sampling period T_u) of the continuous-time plant (1) can be rewritten as
$$\begin{aligned} x(k, i+1) &= \Phi_c x(k, i) + \Gamma_c u(k, i) \\ y(k, i) &= C_c x(k, i) \end{aligned} \qquad (6)$$
where the matrices Φ_c, Γ_c and C_c are given in (3). Similarly, the single-rate closed-loop system (5) can be rewritten in the form
$$\begin{aligned} \zeta(k, i+1) &= F\zeta(k, i) + Gr \\ y(k, i) &= \bar{C}_c \zeta(k, i) \end{aligned} \qquad (7)$$
The closed-loop system (7), where the measurement period equals the short control update period T_u, can be considered ideal since it is obtained assuming a measurement rate faster than the one practically obtainable. With the aim of reproducing the same response as that of the ISRS (7), and to avoid the presence of inter-sample ripple in the closed-loop response, we propose a multi-rate digital control system where the controller operates at the faster rate $1/T_u$, and the feedback measurement is available at the slower rate $1/T_s$, which are constrained by the integral relationship $q \triangleq T_s/T_u$, $q \in \mathbb{N}^+$.

The proposed controller has the following form:
$$\begin{aligned} \varphi(k, i+1) &= K_\varphi(i)\varphi(k, 0) + K_x(i)x(k, 0) + L(i)r \\ u(k, i) &= C_\varphi \varphi(k, i) \end{aligned} \qquad (8)$$
where the unknown matrices K_φ, K_x and L change at every control update period T_u, and the controller state vector φ has the same dimension as η (i.e., $\dim(\varphi) = n_\eta$). Notice that the controller (8) utilizes the state feedback $x(k, 0)$ only at every sampling period T_s, while the controller output updates at every sampling period T_u. The additional feed-forward term, $L(i)r$, is utilized to eliminate the steady-state regulation error and to guarantee a ripple-free closed-loop response. The problem addressed in this paper consists of developing conditions under which it is possible to find a set of periodically varying matrices $K_\varphi(i)$, $K_x(i)$, $L(i)$, $i = 0, 2, \ldots, q - 1$, to achieve closed-loop state matching, at the faster rate $1/T_u$, with the ISRS (7). Moreover, the matrices $K_\varphi(i)$, $K_x(i)$ and $L(i)$ must be designed in order to exhibit a ripple-free closed-loop response to a step reference. By "ripple-free" it is meant that the regulation error, $e(t) = y(t) - r$, has to be zero at steady-state within two consecutive control update instants, or in other words
$$\lim_{k \to \infty} \int_{kT_u}^{(k+1)T_u} e^T(t)e(t)\, dt = 0.$$

A summary of the problem statement is the following. Given the continuous-time system (1), and the discrete-time controller (4), let T_u and T_s be the control and measurement update period respectively, such that $T_s = qT_u$ for a positive integer q. It is desired to design a controller of the form (8) to achieve closed-loop state matching, at the faster rate $1/T_u$, with the ISRS (7) which represents the given ideal model. Moreover, it is desired to achieve a ripple-free closed-loop response with no steady-state error to a step reference change.

3. MULTI-RATE CONTROL DESIGN

The discrete-time system (6) and the proposed time-varying controller (8) can be rewritten in the following compact form:

$$\xi(k, i+1) = \bar{\Phi}_c \xi(k, i) + \bar{\Gamma}_c w(k, i)$$
$$y(k, 0) = \bar{C}_c \xi(k, 0) \qquad (9)$$

where $\xi(k, i) \triangleq [x^T(k, i) \quad \varphi^T(k, i)]^T$ is the extended state vector, the matrices $\bar{\Phi}_c, \bar{\Gamma}_c, \bar{C}_c$ are given by

$$\bar{\Phi}_c \triangleq \begin{bmatrix} \Phi_c & \Gamma_c C_\varphi \\ 0_{n_\eta \times n_x} & 0_{n_\eta \times n_\eta} \end{bmatrix}, \quad \bar{\Gamma}_c \triangleq \begin{bmatrix} 0_{n_x \times n_\eta} \\ I_{n_\eta \times n_\eta} \end{bmatrix}$$

$$\bar{C}_c \triangleq \begin{bmatrix} C_c & 0_{n_y \times n_\eta} \end{bmatrix},$$

and the pseudo-input $w(k, i)$ is given by

$$w(k, i) \triangleq K(i)\xi(k, 0) + L(i)r \qquad (10)$$

where

$$K(i) \triangleq \begin{bmatrix} K_x(i) & K_\varphi(i) \end{bmatrix}.$$

Notice that the closed-loop system (9)-(10) is a multi-rate system since it involves signals sampled at different rates. In fact, even though the pseudo-input $w(k, i)$ is updated at every period T_u, it depends on the feedback $\xi(k, 0)$ which is updated at the measurement period T_s.

3.1 Discrete-time lifted systems

The lifting technique described by Godbout et al. [1990] can be used to evaluate the behavior of a discrete-time system at a slower sampling time. As explained in the following, this technique can also be utilized to convert a multi-rate time-varying system, such as the one formed by (8) and (6), into a single-rate time invariant-system. This technique is applied to the closed-loop system formed by (9) and (10). Notice that the value of $\xi(k, i)$ within two consecutive measurement instants (i.e., in the interval $[kT_s, kT_s + (q - 1)T_u]$) can be obtained by recursively applying (9):

$$\xi(k, 1) = \bar{\Phi}_c \xi(k, 0) + \bar{\Gamma}_c w(k, 0)$$
$$\xi(k, 2) = \bar{\Phi}_c^2 \xi(k, 0) + \bar{\Phi}_c \bar{\Gamma}_c w(k, 0) + \bar{\Gamma}_c w(k, 1)$$
$$\cdots$$
$$\xi(k+1, 0) = \bar{\Phi}_c^q \xi(k, 0) + \sum_{j=0}^{q-1} \bar{\Phi}_c^j \bar{\Gamma}_c w(k, q-1-j). \qquad (11)$$

Notice also that the state $\xi(k, 0)$, at the measurement instant $t = kT_s$, carries information about the system evolution in between two consecutive measurement instants. In fact, all the other inter-sample states $\xi(k, i)$, $i = 1, \ldots, q - 1$, can be rewritten in terms of $\xi(k, 0)$ and $w(k, j)$, $j = 0, \ldots, i - 1$. Therefore, there is no need to keep track of the first $(q - 1)$ equations in (11), and the evolution of the system (9), at the measurement period T_s, can be described by the last equation in (11):

$$\xi(k+1, 0) = \tilde{\Phi}_c \xi(k, 0) + \tilde{\Gamma}_c \tilde{w}(k, 0)$$
$$y(k, 0) = \tilde{C}_c \xi(k, 0) \qquad (12)$$

where

$$\tilde{\Phi}_c \triangleq \bar{\Phi}_c^q, \quad \tilde{\Gamma}_c \triangleq \begin{bmatrix} \bar{\Phi}_c^{q-1} \bar{\Gamma}_c & \bar{\Phi}_c^{q-2} \bar{\Gamma}_c & \bar{\Phi}_c^{q-3} \bar{\Gamma}_c & \cdots & \bar{\Gamma}_c \end{bmatrix},$$

$$\tilde{C}_c \triangleq \bar{C}_c, \quad \tilde{w}(k, 0) \triangleq \begin{bmatrix} w(k, 0) \\ w(k, 1) \\ \cdots \\ w(k, q-1) \end{bmatrix}. \qquad (13)$$

The system (12) is usually referred to as a simplified *lifted model* of the original system (9). Similarly, the extended input

vector $\tilde{w}(k, 0)$ is referred to as the *lifted pseudo-input*. By closing the loop, through (10), the lifted pseudo-input $\tilde{w}$ takes the form

$$\tilde{w}(k, 0) = \tilde{K}\xi(k, 0) + \tilde{L}r \qquad (14)$$

where

$$\tilde{K} \triangleq \begin{bmatrix} K(0) \\ K(1) \\ \cdots \\ K(q-1) \end{bmatrix}, \quad \tilde{L} \triangleq \begin{bmatrix} L(0) \\ L(1) \\ \cdots \\ L(q-1) \end{bmatrix}$$

are two constant matrices. The result is that the *multi-rate time-varying* closed-loop system formed by (8) and (6), is now equivalently described by the lifted *single-rate time-invariant* system formed by (12) and (14). Hence, the problem of finding the set of periodically time-varying matrices $K_x(i)$, $K_\varphi(i)$, and $L(i)$, $i = 0, \ldots, q - 1$, reduces to that of designing the static time-invariant feedback gain matrix $\tilde{K}$, and the feed-forward constant gain matrix $\tilde{L}$, for the simplified closed-loop system formed by (12) and (14).

The lifting technique can be similarly applied also to the closed-loop ISRS (7) to evaluate its evolution at the slower measurement sampling time T_s. In this case, the lifted ISRS takes the form

$$\zeta(k+1, 0) = \tilde{F}\zeta(k, 0) + \tilde{G}r$$
$$y(k, 0) = \tilde{C}_c \zeta(k, 0) \qquad (15)$$

where

$$\tilde{F} = F^q, \quad \tilde{G} = \sum_{i=0}^{q-1} F^i G, \quad \tilde{C}_c = \bar{C}_c$$

3.2 Design of the gain matrices $\tilde{K}$ and $\tilde{L}$

The simplified lifted closed-loop system formed by (12) and (14) fully describes the dynamics of the closed-loop system formed by (6) and (8). Therefore, the problem of designing the time-varying matrices $K(i)$ and $L(i)$, $i = 0, 1, \ldots, q - 1$, can be converted into that of designing the constant matrices $\tilde{K}$ and $\tilde{L}$ for the lifted closed-loop system (12) and (14).

To achieve state matching with the ISRS (7), the lifted closed-loop system formed by (12) and (14), given by

$$\xi(k+1, 0) = (\tilde{\Phi}_c + \tilde{\Gamma}_c \tilde{K})\xi(k, 0) + \tilde{\Gamma}_c \tilde{L}r$$
$$y(k, 0) = \tilde{C}_c \xi(k, 0), \qquad (16)$$

must match the lifted ISRS (15). This corresponds to requiring the state variable ξ be equal to the state variable ζ at every control update period T_u (notice that ξ and ζ have the same dimension by construction). This is possible if the matrices $\tilde{K}$ and $\tilde{L}$ are designed such that

$$\tilde{\Phi}_c + \tilde{\Gamma}_c \tilde{K} = \tilde{F} \qquad (17)$$
$$\tilde{\Gamma}_c \tilde{L} = \tilde{G}. \qquad (18)$$

If a solution $(\tilde{K}, \tilde{L})$ exists for (17) and (18), it is given by

$$\tilde{K} = \tilde{\Gamma}_c^+ (\tilde{F} - \tilde{\Phi}_c) \qquad (19)$$
$$\tilde{L} = \tilde{\Gamma}_c^+ \tilde{G} \qquad (20)$$

where $\tilde{\Gamma}_c^+$ is the left pseudo-inverse of the matrix $\tilde{\Gamma}_c$. Therefore, if the solution (19)-(20) exists, the closed-loop system formed by (6) and (8) exhibits discrete-time state matching at the control update period T_u with the ISRS (7). The existence of the

solution (19)-(20) depends only on the existence of the pseudo-inverse $\tilde{\Gamma}_c^+$. Notice that $\tilde{\Gamma}_c$ can be rewritten as

$$\tilde{\Gamma}_c = \begin{bmatrix} RS & 0_{n_x \times n_\eta} \\ 0_{n_\eta \times n_\eta (q-1)} & I_{n_\eta} \end{bmatrix}$$

where S is defined as a diagonal matrix with $(q-1)$ blocks C_φ on the diagonal and $R \triangleq [\Phi_c^{q-2}\Gamma_c, \cdots, \Phi_c\Gamma_c, \Gamma_c]$. For $\tilde{\Gamma}_c^+$ to exist, $\tilde{\Gamma}_c$ must have full row rank. This corresponds to requiring the product RS to be full row rank, i.e., n_x. Notice that, by choosing C_φ as any full row rank matrix (this is always possible as explained in Remark 2), the matrix S will be full row rank. Therefore, the rank of RS is same as the rank of R. The dimension of the matrix R is $n_x \times n_\eta(q-1)$. Therefore, a necessary and sufficient condition for RS to be full row rank is that R is full row rank. As long as the number of rows of R is less than the number of its columns, i.e., $n_x \leq n_u(q-1)$, the controllability of the couple (Φ_c, Γ_c) ensures that the matrix R has full row rank. Notice that in the opposite case, i.e., $n_x > n_u(q-1)$, it is obviously impossible for R to be full row rank. Therefore, sufficient conditions for RS to be full row rank are: 1) $n_x \leq (q-1)n_u$, 2) the couple (Φ_c, Γ_c) is controllable. If these conditions are satisfied, the gain matrices $\tilde{K}$ and $\tilde{L}$ can be obtained as in (19)-(20). Notice that the first condition poses a lower limit on the number q of inter-samples that can be chosen. In fact, for a given plant (i.e., the values of n_x and n_u are known), the proposed controller can be applied only if $q \geq 1 + n_x/n_u$.

Remark 1. (On the eigenvalues of F)
If the gain matrix $\tilde{K}$ was directly chosen to place the eigenvalues of $(\tilde{\Phi}_c + \tilde{\Gamma}_c\tilde{K})$ at a desired location, it may result in a complex valued closed-loop multi-rate system. Even for a real valued $\tilde{K}$, the matrix $(\tilde{\Phi}_c + \tilde{\Gamma}_c\tilde{K})^{1/q}$ may have complex entries. This would jeopardize the achievement of the state matching condition. As far as this problem is concerned, it has to be pointed out that even by using (19), the resulting closed-loop system can be complex valued if the matrix F has any eigenvalue on the negative real axis of the complex plane. Instead, if the controller (4) is chosen such that the matrix F has all distinct eigenvalues, not on the negative real axis, then $(\tilde{\Phi}_c + \tilde{\Gamma}_c\tilde{K})^{1/q}$ has a real solution, as given by Astin [1967], and that solution equals F through (19). Therefore, under the assumption that F does not have any eigenvalue on the negative real axis, or any coincident eigenvalues, (19) can be used to achieve closed-loop state matching at the control update period T_u.

Remark 2. (On the choice of C_φ)
In the process of designing the gain matrices $\tilde{K}$ and $\tilde{L}$ through (19)-(20), the controller matrix C_φ influences the existence of the left pseudo-inverse $\tilde{\Gamma}_c^+$. However, there is no restriction on the choice of C_φ but that of being full row rank. With regard to this, notice that it is always possible to choose a full-row rank matrix C_φ for the proposed controller (8). Two cases have to be distinguished: $n_u \leq n_\eta$ and $n_u > n_\eta$. In the case that $n_u \leq n_\eta$ it is obviously always possible to choose a full row rank matrix C_φ. In the opposite case, let $n_u = n_\eta + m$, $m \in \mathbb{N}^+$. There are exactly m scalar entries in the vector u which are linearly dependent on the remaining n_η. Therefore, there exists a gain matrix V such that the control signal u can be reordered as $u = [u_{n_\eta}^T, Vu_{n_\eta}^T]^T$, where $\dim(u_{n_\eta}) = n_\eta$. Correspondingly, the matrix Γ_c can be reordered as $\Gamma_c = [\Gamma_{n_\eta}, \Gamma_m]$ where Γ_{n_η} is related to u_η, and Γ_m is related to Vu_η. Therefore, by replacing the matrix Γ_c with the new matrix $\Gamma_{c,new} = \Gamma_{n_\eta} + \Gamma_m V$, the

dimension of the controller output would become n_η, the plant dynamics would not change, and it would be possible to choose a full row rank matrix C_φ for the proposed controller.

Remark 3. (On the structure of the controller (8))
It is important to point out that the final result would not change if the feedback of the controller state in (8) was taken at the control update rate. This is due to the fact the there is no information of the plant behavior within two consecutive measurement update periods.

3.3 Ripple-free and zero steady-state error

Now it remains to be shown that the system exhibits a ripple-free response to a step reference signal. With regard to the regulation problem, the given ISRS must be at least a type one system in order to exhibit zero steady-state error to a step reference signal. Therefore, there exists a matrix $M \in \mathbb{R}^{(n_x+n_\eta) \times n_y}$ such that $\zeta_r \triangleq Mr$ is the constant reference for the state vector ζ of the ISRS (7). Let us assume that there also exists a matrix $N \in \mathbb{R}^{n_\eta \times n_y}$ such that

$$\bar{\Gamma}_c L(i) = \bar{\Gamma}_c(N - K(i)M). \tag{21}$$

Then, after lifting both the sides of (21), we get

$$\tilde{\Gamma}_c \tilde{L} = \tilde{\Gamma}_c(\tilde{N} - \tilde{K}M)$$

where $\tilde{N} = [N^T, N^T, \ldots, N^T]^T$ is made of q blocks, each one corresponding to the matrix N. Therefore, the dynamics of the closed-loop lifted system (16) can be rewritten as

$$\xi(k+1, 0) = (\tilde{\Phi}_c + \tilde{\Gamma}_c\tilde{K})\xi(k,0) + \tilde{\Gamma}_c(\tilde{N} - \tilde{K}M)r$$
$$= \tilde{\Phi}_c\xi(k,0) + \tilde{\Gamma}_c\left[\tilde{K}(\xi(k,0) - \zeta_r) + \tilde{N}r\right] \tag{22}$$

Hence, if the condition (21) holds, the lifted pseudo-control (14) can be rewritten as

$$\tilde{w} = \tilde{K}(\xi(k,0) - \zeta_r) + \tilde{N}r. \tag{23}$$

Inverting the lifting process, the lifted pseudo-control (23) reduces to the following

$$w(k,i) = K(i)(\xi(k,0) - \zeta_r) + Nr. \tag{24}$$

Notice that, when the state of the ISRS ζ approaches ζ_r (this is possible because the ISRS is at least type one), then by the state matching condition ξ approaches ζ_r, and the pseudo control w becomes constant within two consecutive measurement update instants. Therefore, the output of the multi-rate system is ripple free.

Now the existence of the matrices M and N such that (21) holds is investigated. Let us choose M and N in order to satisfy the following conditions:

$$(\bar{\Phi}_c - I)Mr + \bar{\Gamma}_c Nr = 0$$
$$r - \bar{C}_c Mr = 0. \tag{25}$$

These conditions can be rewritten in the following compact form

$$P \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} 0_{(n_x+n_\eta) \times n_y} \\ I_{n_y \times n_y} \end{bmatrix}, \tag{26}$$

where P is given by:

$$P \triangleq \begin{bmatrix} (\bar{\Phi}_c - I) & \bar{\Gamma}_c \\ \bar{C}_c & 0_{n_y \times n_\eta} \end{bmatrix}. \tag{27}$$

Therefore, the matrices M and N can be uniquely determined if the left pseudo-inverse of the matrix P exists, which translates to the full row rank condition of P. Recall that the system matrix of the system $(\bar{\Phi}_c, \bar{\Gamma}_c, \bar{C}_c, 0)$ is given by

$$\Sigma_c(z) = \begin{bmatrix} (\bar{\Phi}_c - zI) & \bar{\Gamma}_c \\ \bar{C}_c & 0_{n_y \times n_\eta} \end{bmatrix} \tag{28}$$

Since $P = \Sigma_c(1)$, we can conclude that M and N can be determined if and only if $z = 1$ is not an invariant zero of the discrete-time system (9). This corresponds to requiring that $z = 1$ is not an invariant zero of the plant (6) which can be seen considering that P is full row rank if and only if the matrix product

$$\begin{bmatrix} (\Phi_c - I) & \Gamma_c \\ C_c & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_\varphi \end{bmatrix} \qquad (29)$$

is full row rank. The second matrix in the above product is full row rank and the first matrix is full row rank if the system matrix of (2) does not have an invariant zero at $z = 1$.

Notice that if M and N are chosen in order to satisfy (25), then the closed-loop system formed by (9) and (24) exhibits zero steady-state regulation error. Therefore, by lifting the closed-loop system formed by (9) and (24), it is straightforward to show that the matrices M and N obtained from (25) satisfy the following equation as well

$$\left(\tilde{\Phi}_c - I \right) Mr + \tilde{\Gamma}_c \tilde{N}r = 0 \qquad (30)$$

From (17) and (30), we have

$$\begin{aligned} \tilde{\Gamma}_c(\tilde{N} - \tilde{K}M) &= \tilde{\Gamma}_c\tilde{N} - \tilde{\Gamma}_c\tilde{K}M \\ &= (I - \tilde{\Phi}_c)M - \tilde{\Gamma}_c\tilde{K}M \\ &= (I - \tilde{\Phi}_c)M - (\tilde{F} - \tilde{\Phi}_c)M \\ &= (I - \tilde{F})M. \end{aligned}$$

Assuming that the ISRS (7) is a type one system, at steady-state the following holds for the lifted ISRS

$$(\tilde{F} - I)M + \tilde{G} = 0.$$

Since, from (18), $\tilde{G} = \tilde{\Gamma}_c\tilde{L}$,

$$\tilde{\Gamma}_c(\tilde{N} - \tilde{K}M) = (I - \tilde{F})M = \tilde{\Gamma}_c\tilde{L}.$$

Therefore, there exist matrices M and N such that (21) holds, and hence the designed closed-loop multi-rate system exhibits a ripple-free response to a step reference change as long as the ISRS is at least a type one system.

The results can be summarized as follows. Given any continuous-time system (1), let T_u and T_s be the control and measurement update periods, respectively, such that $T_s = qT_u$ for a positive integer q. Let $(A_\eta, B_\eta, C_\eta, D_\eta)$ be a given controller designed to achieve the desired performance criteria for the desired closed-loop ISRS. If the ISRS system is designed in order to achieve a zero steady-state error to a step reference signal, and if

- $n_x \leq (q-1)n_u$,
- the couple (Φ_c, Γ_c) is controllable,

then it is possible to design a periodically time-varying controller of the form (8) in order to achieve closed-loop state matching with the ISRS at the faster rate $1/T_u$, and a ripple-free response converging to the step reference value.

4. EXPERIMENTS

4.1 Experimental setup

Experiments were conducted on a hard disk drive platform, which is shown in Figure 1. A third order model has been utilized to describe the VCMA dynamics as shown by Ratliff and Pagilla [2005]. Choosing the state vector $x = [\theta, \dot\theta, i]$, and

defining $u = V_s$ as the control input, the VCMA dynamics can be described by the state space equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \dfrac{K_t}{J} \\ 0 & -\dfrac{K_t}{L} & -\dfrac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix} u \qquad (31)$$

where J is the arm inertia, K_t is the torque factor, V_s is the voltage input to the coil with resistance R and inductance L, and i is the current in the coil. The parameters values $J = 1.26e^{-07}$kg-m², $K_t = 5.5e^{-03}$N-m/A, $R = 14\Omega$ and $L = 1.1$mH, were obtained by identification. The experimental setup is shown in Figure 1. Since we do not have the proprietary
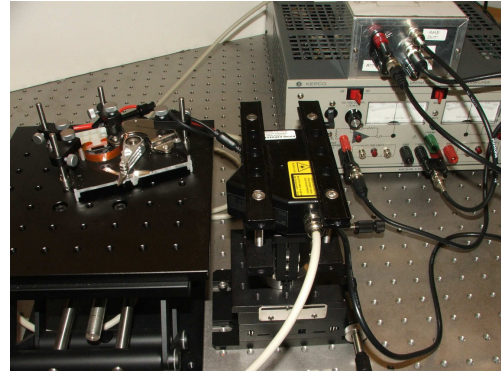

Fig. 1. Sketch of the experimental platform with laser sensor.

software to obtain arm position measurement from the platters, a laser distance sensor has been used to get the angular position feedback of the HDD R/W head. The laser is an LMI LDS90/40 triangulation sensor providing an analog output up to 100 kHz, and a resolution of 0.02 degrees. As shown in the figures, the platters have been removed from the HDD, and the frame of the HDD has been modified in order to avoid any obstruction for the laser beam generated by the laser sensor. A velocity feedback is computed by measurement of current, through the equation:

$$\dot{\theta} \approx \frac{u - Ri}{K_t} \qquad (32)$$

obtained from the electrical dynamics by neglecting the term involving the derivative of the current. As discussed by Ratliff and Pagilla [2006], (32) provides a good approximation of the angular velocity of the arm. The voltage V_s is supplied to the coil through a KEPCO bipolar operational power supply.

4.2 Experimental results and discussion

The controller update period T_u was chosen equal to 50 μs, and the values $q = 4, 14$ were chosen to test the proposed algorithm. This corresponds to the measurement update periods $T_s = 200, 700$ μs respectively. Notice that $n_x = 3$, therefore $q = 4$ corresponds to the smallest integer that satisfies the condition $n_x \leq (q-1)n_u$. The controller (4) was chosen in order to achieve a gain margin greater than 50 dB, a phase margin greater than 40 deg, and a bandwidth of at least 20 rad/s. This resulted in the matrices $A_\eta = 1$, $B_\eta = 19.53e^{-04}$, $C_\eta = 25.6e^{-04}$, $D_\eta = 0.1$, for the controller (4). The first two plots of figure 2 show the behavior of the closed-loop ISRS for step reference changes from 5 deg to 25 deg, and vice versa. When the multi-rate scheme is adopted the HDD R/W head position profile does not change. The last two plots

of figure 2 show the control action for the multi-rate control scheme in the cases $q = 4$ and $q = 14$, for the same step reference changes. The designed multi-rate system exhibits a
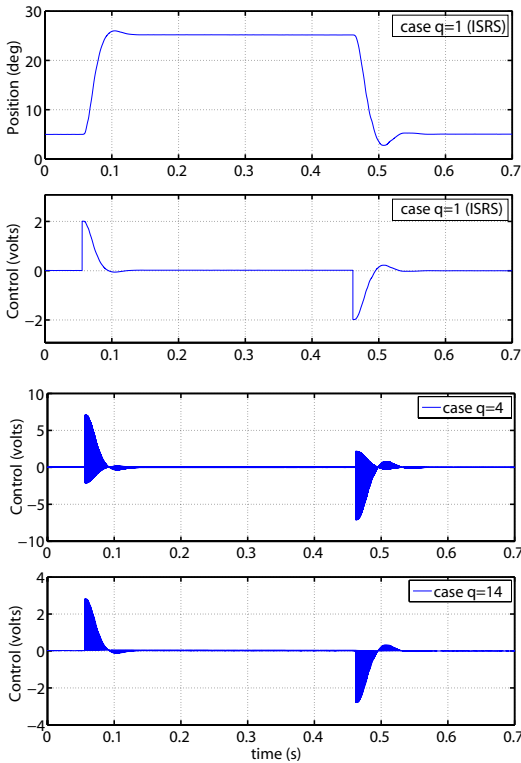

Fig. 2. Experimental results for step reference changes.

ripple-free response that resembles the ISRS response in both the cases $q = 4$, $q = 14$. Even though the identified values of the HDD model parameters may not be precise, and the VCMA dynamics is affected by a nonlinear term due to the flex bias that was not taken into account during the design process, the proposed design leads to satisfactory results. The main difference between the ISRS response and the multi-rate system response is the presence of high frequency content in the control action of the multi-rate scheme. This feature is highlighted in Figure 3 where the voltage input to the coil is shown in a larger time scale for the case $q = 14$. The oscillations in the
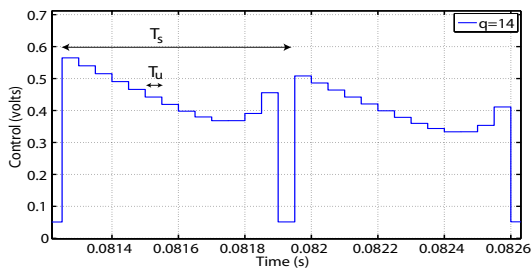

Fig. 3. Zoom of the last plot of Figure 2.

controller output are mainly due to the time varying nature of the controller. However, the control action gets smoother and its magnitude decreases if the value of q increases. This is reasonable considering that an increment of the measurement sampling period T_s acts like a filtering action against possible disturbances or measurement noises. However, even though an increment of q has positive effects on the shape of the

control action, it might have a negative effect on the closed-loop performance if the uncertainty on the model dynamics is larger. Therefore, the lower is the uncertainty of the plant to be controlled, the more similar are the multi-rate system and ISRS responses.

5. CONCLUSIONS

A state-space approach to design discrete-time controllers for multi-rate systems has been developed. The proposed design can be applied not only to single-loop systems where the measurement and the control update rate are different, but also to multi-loop control structures where each loop operates at a different sampling rate. The resulting multi-rate closed-loop system achieves state-matching with the fast single-rate reference model, and exhibits ripple-free response with zero steady-state regulation error if the reference model is at least a type one system. Despite the presence of un-modeled resonance dynamics, and flex bias effect, the experiments show satisfactory results. Future experiments will take into account also the nonlinear windage effect due to the rotation of the platters. Future research should also concentrate on the design of a multi-rate controller when full state feedback is not available.

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