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LPV APPROACH FOR $\mathcal{H}_\infty$ FILTER DESIGN FOR A CLASS OF NONLINEAR SYSTEMS

Gérard Benjamin * **Souley Ali Harouna** * **Zasadzinski Michel** * **Darouach Mohamed** *

* *CRAN, Nancy Université, CNRS,*
186 rue de Lorraine F-54400 Cosnes et Romain, FRANCE,
(e-mail: bgerardr@ iut-longwy.uhp-nancy.fr).

Abstract: This paper deals with the $\mathcal{H}_\infty$ filtering problem for a class of nonlinear systems. This class of nonlinear systems is composed of a bilinear part and of a lipschitzian one. Using an unbiasedness condition for the bilinear part permits to parameterize the filter matrices through a single gain. Two LPV extensions of the bounded real lemma are used to solve the $\mathcal{H}_\infty$ filtering problem. This approach reduces the conservatism inherent to the boundedness of the inputs. Then the filtering solution is expressed in terms of LMI (Linear Matrix Inequalitiy) to be verified at the vertices of a polytope.

Keywords: Bilinear systems, lipschitzian nonlinearities, $\mathcal{H}_\infty$ filtering, unbiasedness, LMI, LPV.

1. INTRODUCTION

As many physical processes cannot be modeled satisfactorily by a linear system, a lot of works have been accorded to the state estimation of some class of nonlinear systems as bilinear ones ((Gérard *et al.*, 2007)) or lipschitzian ones ((Zhu and Han, 2002), (Lu and Ho, 2002)). In this work, we present an alternative approach that consists in melting this two kinds of nonlinearities to act with more accuracy on each one. The approach proposed here is to consider the control inputs as varying parameters as in (Gérard *et al.*, 2007), so the system is described in an LPV form ((Apkarian *et al.*, 1995)).

The linear parameters varying approach (LPV) can enable us to find solutions easily, by adding some degree of freedom for the LMI, nevertheless we must have the bounds of the inputs derivatives. Using Lyapunov method, we express the quadratic stability and the $\mathcal{H}_\infty$ performance through an inequality that is transformed into LMI as we have an affine system.

The paper is organized as follows. The conditions for the unbiasedness of $\mathcal{H}_\infty$ filter for continuous-time bilinear part of the system and for the parametrization of the filter matrices through an unique gain are studied in section 2. Two LPV lipschitzian real bounded lemmas to ensure quadratic stability and $\mathcal{L}_2$ attenuation of the filter are stated in section 3. These lemma are applied to the estimation problem in section 4. Finally an illustrative example is given in section 5.

Notations. Throughout this paper, $\|x\| = \sqrt{x^T x}$ is the Euclidean vector norm. $A^\dagger$ is a generalized inverse of matrix A satisfying $A = AA^\dagger A$ ((Rao and Mitra, 1971)).

2. PARAMETRIZATION OF THE $\mathcal{H}_\infty$ FILTER

2.1 Problem Formulation

Consider the following class of nonlinear systems

$$\begin{cases} \dot{x} = A_0 x + \sum_{i=1}^{m} A_i u_i x + Ru + Bw + f(t,x,u(t)) \\ y = Cx + Dw \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) = [\, u_1(t) \ \dots \ u_m(t) \,]^T \in \mathbb{R}^m$ is the known control input vector and $y(t) \in \mathbb{R}^p$ is the measured output. $f(t,x,u(t))$ is time-varying nonlinear function and satisfies the following LPV-Lipschitz condition for all (t,x,u) and $(t,x',u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$

$$\|f(t,x,u) - f(t,x',u)\| \leq \|K(u)(x - x')\| \quad (2)$$

where $K(u(t))$ is a $\mathbb{R}^{n\times n}$ matrix linear in $u(t)$.

Remark 1. The nonlinear systems studied in this article are constituted by a linear part, a bilinear part and a lipschitzian part.

The vector $w(t) \in \mathbb{R}^q$ represents the unknown disturbance vector. The problem is to estimate the vector $x(t)$ from the measurements $y(t)$ and the control inputs $u(t)$. As in the most cases for physical processes, the bilinear system (1) has known bounded control inputs, moreover their derivatives are assumed to be bounded too. So let us define the following set

$$\Omega = \{u : t \to \mathbb{R}^m \mid \forall\, t \in \mathbb{R}^+, \begin{matrix} u_{i\min} \leq u_i(t) \leq u_{i\max} \\ \mu_{i\min} \leq \dot{u}_i(t) \leq \mu_{i\max} \end{matrix} \quad i = 1,\dots,m\} \quad (3)$$

As $u(t)$ is bounded, LPV-Lipschitz or k-Lipschitz are equivalent properties. Indeed if $u(t)$ is bounded, then $\max_{u\in\Omega}(\sigma(K(u))$ exists and is a lipschitz scalar gain for f. Moreover if f is k-lipschitz, $K(u) = kI_n$ can be the LPV matrix lipschitz gain.

The proposed filter is given by

$$\begin{cases} \dot{\eta} = H_0\eta + \sum_{i=1}^{m} H_i u_i \eta + J_0 y + \sum_{i=1}^{m} J_i u_i y \\ \qquad + Gu + (I_n - EC)f(t,\widehat{x},u) \\ \widehat{x} = \eta + Ey \end{cases} \tag{4}$$

where $\widehat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. The filtering error is defined as follow

$$e = x - \widehat{x} = \overline{e} - EDw \tag{5}$$

where $\overline{e} = \Psi x - \eta$ and $\Psi = I_n - EC$.

To characterize the disturbance attenuation, the following definition is introduced.

Definition 1. Let $\gamma > 0$, the mapping from $w(t)$ to $e(t)$ is said to have $\mathcal{L}_2$ gain less than or equal to γ if

$$J_{ew} = \int_0^\infty \left(\|e(t)\|^2 - \gamma^2\|w(t)\|^2\right) dt \leq 0 \tag{6}$$

$\forall\, w(t) \in \mathcal{L}_2[0,\infty)$, and with zero initial conditions. ♢

The definition 1 can be seen as a generalization of the $\mathcal{H}_\infty$ norm for linear systems to nonlinear ones (see (van der Schaft, 1992)). The design of the filter (4) for nonlinear system (1) is stated as follows.

Problem 1. In this paper, the problem of the filter design is to determine G, H_0, H_i, J_0, J_i and E such that

- **(i)** the filter (4) is unbiased for the bilinear part if $w(t) = 0$ (see (Seron *et al.*, 1997), p. 176), *i.e.* the filtering error $e(t)$ is independent of $x(t)$,
- **(ii)** the filtering error $e(t)$ is quadratically stable for $u(t) \in \Omega$ and $w(t) = 0$,
- **(iii)** the mapping from the disturbance input $w(t)$ to the filtering error $e(t)$ has $\mathcal{L}_2$ gain less than a given scalar γ for $u(t) \in \Omega$. ♢

The error system (5) can be parameterized as follows

$$\begin{cases} \dot{\overline{e}} = (A_u(u) + \overline{Z}A_z(u))\overline{e} \\ \qquad + (B_u(u) + \overline{Z}B_z(u))w + \Psi(f(t,x,u) - f(t,\widehat{x},u)) \\ e = \overline{C}\overline{e} + \overline{D}w \end{cases} \tag{7}$$

where all matrices A_u, A_z, B_u, B_z, $\overline{C}$ and $\overline{D}$ dependent of matrices of system (1) (see (Gérard *et al.*, 2007)). The parameterization is not given due to a lack of space then the $\mathcal{H}_\infty$ filter synthesis is equivalent to the determination of $\overline{Z}$ in order to stabilize the system (7) and to ensure the $\mathcal{L}_2$ gain attenuation between $w(t)$ and $e(t)$.

3. TWO BOUNDED REAL LEMMA REFORMULATION FOR AN AFFINE LPV SYSTEM WITH LIPSCHITZ TERM

Consider the following LPV system (similar to (7), with $\{u_i(t), i = 1, \ldots, m\}$ as varying parameters

$$\begin{cases} \dot{\overline{e}} = \mathbb{A}(u)\overline{e} + \mathbb{B}(u)w + \Gamma(g(t,x,u) - g(t,\widehat{x},u)) \\ e = \overline{e} + \mathbb{D}w. \end{cases} \tag{8}$$

with $e = x - \widehat{x}$, $g(t,x,u(t)$ a time-varying nonlinear function which satisfies the LPV-Lipschitz condition (see (2)).

To ensure quadratic stability and $\mathcal{H}_\infty$ performance, we can use a kind of bounded-real lemma for LPV systems where Lyapunov matrix is separated from the system dynamics in the Lyapunov inequality. This is done in order to take a Lyapunov matrix with a particular structure.

3.1 First approach using majoration

First, we recall the well-known following lemma

Lemma 2. If M, N, $\mathcal{G}$ are real matrices with appropriated dimensions such that $\mathcal{G}^T\mathcal{G} \leq I$, then

$$2x^T M\mathcal{G}Ny \leq \alpha x^T MM^T x + \frac{1}{\alpha} y^T N^T Ny \tag{9}$$

$\forall\, \alpha \in \mathbb{R}^{+*}$, $\forall\, x, y$ vectors with appropriated dimensions. ▽

Now, we can state the following lemma.

Lemma 3. For given ε and ε', the LPV system (8) is quadratically stable and has a $\mathcal{L}_2$ gain from $w(t)$ to $e(t)$ less than or equal to γ if there exist matrices $P(u) = P(u)^T > 0$ and F such that

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} < 0 \tag{10}$$

with

$$\Lambda_{11} = \begin{bmatrix} (1,1) & (1,2) & F\mathbb{B}(u) & \mathbb{C}^T \\ (1,2)^T & -F - F^T & F\mathbb{B}(u) & 0 \\ (F\mathbb{B}(u))^T & (F\mathbb{B}(u))^T & -\gamma^2 I & \mathbb{D}^T \\ \mathbb{C} & 0 & \mathbb{D} & -I_q \end{bmatrix}$$

$$\Lambda_{12} = \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}}K(u)^T & \frac{1}{\sqrt{\varepsilon'}}K(u)^T & \sqrt{\varepsilon}F\Gamma & 0 \\ 0 & 0 & 0 & \sqrt{\varepsilon'}F\Gamma \\ \frac{1}{\sqrt{\varepsilon}}(K(u)\mathbb{D})^T & \frac{1}{\sqrt{\varepsilon'}}(K(u)\mathbb{D})^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Lambda_{22} = -I_{4n}$$

where $u(t) \in \Omega$ and
$(1,1) = \dot{P}(u) + F\mathbb{A}(u) + \mathbb{A}^T(u)F^T$ and
$(1,2) = P(u) - F + \mathbb{A}^T(u)F^T$. ♢

Proof. As in (Chughtai and Munro, 2004), we consider system (8) with the following "descriptor formulation" (with $\phi = \dot{\overline{e}}$)

$$\begin{cases} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\overline{e}} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \mathbb{A}(u) & -I \end{bmatrix} \begin{bmatrix} \overline{e} \\ \phi \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbb{B}(u) \end{bmatrix} w \\ \qquad + \begin{bmatrix} 0 \\ \Gamma(g(t,x,u) - g(t,x',u)) \end{bmatrix} \\ e = [\mathbb{C}\ \ 0] \begin{bmatrix} \overline{e} \\ \phi \end{bmatrix} + \mathbb{D}w. \end{cases} \tag{11}$$

Let $V(\overline{e})$ be the candidate Lyapunov function

$$V(\overline{e}) = \overline{e}^T P(u)\overline{e} = \begin{bmatrix} \overline{e} \\ \dot{\overline{e}} \end{bmatrix}^T \begin{bmatrix} P(u) & F \\ 0 & F \end{bmatrix} \overline{E} \begin{bmatrix} \overline{e} \\ \dot{\overline{e}} \end{bmatrix}$$

with

$$\overline{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, P(u) = P^T(u) > 0.$$

The time derivative of the Lyapunov function is given by

$$\begin{aligned}\dot V(\bar e) &= 2\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix}^T\begin{bmatrix}P(u) & F\\ 0 & F\end{bmatrix}\overline{E}\begin{bmatrix}\dot{\bar e}\\ \ddot{\bar e}\end{bmatrix}+\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix}^T\begin{bmatrix}\dot P(u) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix}\\
&= 2\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix}^T\begin{bmatrix}P(u) & F\\ 0 & F\end{bmatrix}\begin{bmatrix}\dot{\bar e}\\ 0\end{bmatrix}+\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix}^T\begin{bmatrix}\dot P(u) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix}\\
&= 2\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix}^T\begin{bmatrix}P(u) & F\\ 0 & F\end{bmatrix}\begin{bmatrix}\dot{\bar e}\\ \begin{pmatrix}-\dot{\bar e}+\mathbb{A}(u)\bar e+\mathbb{B}(u)w\\ +\Gamma\widetilde g(t,x,\widehat x,u)\end{pmatrix}\end{bmatrix}\\
&\quad+\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix}^T\begin{bmatrix}\dot P(u) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\bar e\\ \dot{\bar e}\end{bmatrix} \qquad (12)\end{aligned}$$

with all (t,x,u) and $(t,x',u)\in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$

$$\widetilde g(t,x,\widehat x,u)=g(t,x,u)-g(t,\widehat x,u). \qquad (13)$$

Substituting (11) in (12) and using the facts that $V(0)=0$ and $V(\infty)\geqslant 0$, reminding equation (6), the following inequality is obtained

$$\begin{aligned}J_{ew} &\le \int_0^\infty\left(e^Te-\gamma^2w^Tw\right)dt+V(\infty)-V(0)\\
&\le \int_0^\infty\left(e^Te-\gamma^2w^Tw+\dot V(\bar e)\right)dt \qquad (14)\end{aligned}$$

Using (2) and (13), we can write

$$\widetilde g^T(t,x,\widehat x,u)\widetilde g(t,x,\widehat x,u)-e^TK^T(u)K(u)e\le 0 \qquad (15)$$

which is equivalent to (from (5)

$$\begin{aligned}&\widetilde g^T(t,x,\widehat x,u)\widetilde g(t,x,\widehat x,u)-\bar e^TK^T(u)K(u)\bar e\\
&-w^T(K(u)\mathbb{D})^T(K(u)\mathbb{D})w+2\bar eK^T(u)K(u)\mathbb{D}w\le 0 \qquad (16)\end{aligned}$$

In equation (12), we have two terms to major (14) : $2\bar e^TF\Gamma\widetilde g$ and $2\dot{\bar e}^TF\Gamma\widetilde g$. Using lemma 2, we can write

$$\begin{aligned}2\bar e^TF\Gamma\widetilde g &\le \varepsilon\bar e^TF\Gamma\Gamma^TF^T\bar e+\frac{1}{\varepsilon}\widetilde g^T\widetilde g\\
&\le \varepsilon\bar e^TF\Gamma\Gamma^TF^T\bar e+\frac{1}{\varepsilon}e^TK^T(u)K(u)e\\
&\le \varepsilon\bar e^TF\Gamma\Gamma^TF^T\bar e\\
&\quad+\frac{1}{\varepsilon}(\bar e-\mathbb{D}w)^TK^T(u)K(u)(\bar e-\mathbb{D}w) \qquad (17)\end{aligned}$$

We obtain the same kind of relation with $\dot{\bar e}$.

Using (17), the equation (14) is rewritten as (see (12))

$$J_{ew}\le\int_0^\infty\left(\begin{bmatrix}\bar e\\ \dot{\bar e}\\ w\end{bmatrix}^T\begin{bmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{12}^T & \Theta_{22}\end{bmatrix}\begin{bmatrix}\bar e\\ \dot{\bar e}\\ w\end{bmatrix}\right)dt \qquad (18)$$

where

$$\begin{aligned}\Theta_{11} &= \begin{bmatrix}P(u) & F\\ 0 & F\end{bmatrix}\begin{bmatrix}0 & I\\ \mathbb{A}(u) & -I\end{bmatrix}\cdot+\begin{bmatrix}0 & I\\ \mathbb{A}(u) & -I\end{bmatrix}^T\begin{bmatrix}P(u) & 0\\ F^T & F^T\end{bmatrix}\\
&+\begin{bmatrix}\mathbb{C}^T\mathbb{C}+\dot P(u) & 0\\ 0 & 0\end{bmatrix}\\
&+\begin{bmatrix}(\frac{1}{\varepsilon}+\frac{1}{\varepsilon'})K^T(u)K(u)+\varepsilon F\Gamma\Gamma^TF^T & 0\\ 0 & \varepsilon' F\Gamma\Gamma^TF^T\end{bmatrix}\\
\Theta_{12} &= \begin{bmatrix}P(u) & F\\ 0 & F\end{bmatrix}\begin{bmatrix}0\\ \mathbb{B}(u)\end{bmatrix}+\begin{bmatrix}\mathbb{C}^T\mathbb{D}-(\frac{1}{\varepsilon}+\frac{1}{\varepsilon'})K^T(u)K(u)\mathbb{D} & 0\\ 0 & 0\end{bmatrix}\\
\Theta_{22} &= \mathbb{D}^T\mathbb{D}-\gamma^2I+(\frac{1}{\varepsilon}+\frac{1}{\varepsilon'})(K(u)\mathbb{D})^TK(u)\mathbb{D}\end{aligned}$$

So if the following relation

$$\begin{bmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{12}^T & \Theta_{12}\end{bmatrix}<0 \qquad (19)$$

is satisfied, then we have $J_{ew}<0$. Applying Schur lemma ((Apkarian *et al.*, 1995)) to the relation (19), inequality (10) is obtained. □

Notice that matrices P,F and Y are differents in different lemma or theorems of this article.

3.2 Second approach using a new variable

We state the following lemma in order to give a second formulation of a LPV bounded real lemma with less terms, moreover all variables to be determined can be some variables of the LMI.

Lemma 4. For a given ε, the LPV system (8) is quadratically stable and has a $\mathcal{L}_2$ gain from $w(t)$ to $e(t)$ less than or equal to γ if there exist matrices $P(u)=P(u)^T>0$, F such that

$$\begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12}^T & \Pi_{22}\end{bmatrix}<0 \qquad (20)$$

with

$$\Pi_{11}=\begin{bmatrix}(1,1) & (1,2) & F\mathbb{B}(u)\\ (1,2)^T & -F-F^T & F\mathbb{B}(u)\\ (F\mathbb{B}(u))^T & (F\mathbb{B}(u))^T & -\gamma^2I\end{bmatrix} \qquad (21)$$

$$\Pi_{12}=\begin{bmatrix}F\Gamma & \mathbb{C}^T & \sqrt{\epsilon}K(u)^T\\ F\Gamma & 0 & 0\\ 0 & \mathbb{D}^T & \sqrt{\epsilon}(K(u)\mathbb{D})^T\end{bmatrix} \qquad (22)$$

$$\Pi_{22}=\begin{bmatrix}-\sqrt{\epsilon}I_n & 0 & 0\\ 0 & -I_q & 0\\ 0 & 0 & -I_n\end{bmatrix} \qquad (23)$$

where $u(t)\in\Omega$ and
$(1,1)=\dot P(u)+F\mathbb{A}(u)+\mathbb{A}^T(u)F^T$ and
$(1,2)=P(u)-F+\mathbb{A}^T(u)F^T$. ◇

Proof.

We consider the following state vector

$\left[\, e(t)^T \; \dot{e}(t)^T \; \widetilde{g}(t,x,\widehat{x},u) \,\right]^T$ with the same approach as in section 3.1 equation (12) (*i.e.* $\widetilde{g}(t,x,\widehat{x},u)$ is consider as a variable).

4. FILTER SYNTHESIS

We define

$$\begin{aligned}
\mathbb{A}(u) &= A_u(u) + \overline{Z}A_z(u) && (24a)\\
\mathbb{B}(u) &= B_u(u) + \overline{Z}B_z(u) && (24b)\\
\mathbb{C} &= \overline{C} = I_r && (24c)\\
\mathbb{D} &= \overline{D} = -E_1 D && (24d)\\
\Gamma &= \Psi = I - E_1 C - \overline{Z}\overline{E}_2 C, && (24e)
\end{aligned}$$

the two systems (7) and (8) are equivalent. So the system (7) can be considered as an affine LPV system with Lipschitz terms where the control inputs $u_i(t)$ are considered as affine time-varying parameters thus we can apply the LPV bounded real lemma 3 and 4 to the system (7).

First application of lemma 3 to system (7).

Lemma 5. The LPV system (7) is quadratically stable and satisfies the $\mathcal{H}_\infty$ performance (6) if there exist matrices $P(u) = P(u)^T > 0$, F and Y and two strictly positive scalars ε and ε' such that

$$\begin{bmatrix} \widetilde{\Lambda}_{11} & \widetilde{\Lambda}_{12} \\ \widetilde{\Lambda}_{12}^T & \widetilde{\Lambda}_{22} \end{bmatrix} < 0 \tag{25}$$

where $u(t) \in \Omega$ and

$$\widetilde{\Lambda}_{11} = \begin{bmatrix} (1,1) & (1,2) & (1,3) & I_n \\ (1,2)^T & -F - F^T & (1,3) & 0 \\ (1,3)^T & (1,3)^T & -\gamma^2 I_q & \overline{D}^T \\ I_n & 0 & \overline{D} & -I_n \end{bmatrix} \tag{26}$$

$$\widetilde{\Lambda}_{12} = \begin{bmatrix} \dfrac{K(u)^T}{\sqrt{\varepsilon}} & \dfrac{K(u)^T}{\sqrt{\varepsilon'}} & \sqrt{\varepsilon}(1,7) & 0 \\ 0 & 0 & 0 & \sqrt{\varepsilon'}(1,8) \\ \dfrac{(K(u)\mathbb{D})^T}{\sqrt{\varepsilon}} & \dfrac{(K(u)\mathbb{D})^T}{\sqrt{\varepsilon'}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

$$\widetilde{\Lambda}_{22} = I_{4n} \tag{28}$$

with

$$\begin{aligned}
(1,1) &= \dot{P}(u) + FA_u(u) + YA_z(u)) \\
&\quad + A_u^T(u)F^T + A_z^T(u)Y^T \\
(1,2) &= P(u) - F + A_u^T(u)F^T + A_z^T(u)Y^T \\
(1,3) &= FB_u(u) + YB_z(u) \\
(1,7) &= (1,8) = F(I - E_1C) + Y\overline{E}_2C
\end{aligned}$$

and $\overline{Z} = F^{-1}Y$. ♢

Proof. If (25) has a solution, then $F + F^T > 0$ and F is invertible. That is if LMI (25) has a solution, then $\overline{Z}$ is always computable. Using expression of $\overline{Z}$ and inserting (24) in LMI (25), we obtain inequality (20). Since system (8) with equations (24) represents system (7), using lemma 4, lemma 6 is proved. □

Second application of 4 to system (7).

Lemma 6. The LPV system (7) is quadratically stable and satisfies the $\mathcal{H}_\infty$ performance (6) if there exist matrices $P(u) = P(u)^T > 0$, F and Y such that

$$\begin{bmatrix} \widetilde{\Pi}_{11} & \widetilde{\Pi}_{12} \\ \widetilde{\Pi}_{12}^T & \widetilde{\Pi}_{22} \end{bmatrix} < 0 \tag{29}$$

where $u(t) \in \Omega$ and

$$\widetilde{\Lambda}_{11} = \begin{bmatrix} (1,1) & (1,2) & (1,3) \\ (1,2)^T & -F - F^T & (1,3) \\ (1,3)^T & (1,3)^T & -\gamma^2 I_q \end{bmatrix} \tag{30}$$

$$\widetilde{\Lambda}_{12} = \begin{bmatrix} F(I - E_1C) + Y\overline{E}_2C & I_n & K(u)^T \\ F(I - E_1C) + Y\overline{E}_2C & 0 & 0 \\ 0 & \overline{D}^T & (K(u)\overline{D})^T \end{bmatrix} \tag{31}$$

$$\widetilde{\Lambda}_{22} = -I_{3n} \tag{32}$$

with $(1,1) = \dot{P}(u) + FA_u(u) + YA_z(u)) + A_u^T(u)F^T + A_z^T(u)Y^T$, $(1,2) = P(u) - F + A_u^T(u)F^T + A_z^T(u)Y^T$, $(1,3) = FB_u(u) + YB_z(u)$ and $\overline{Z} = F^{-1}Y$. ♢

Proof. Similar to the proof of lemma 5. □

To be able to solve problem 1, we use the fact that $u(t) \in \Omega$ and thus we can solve the LMIs of the two last theorem on the vertices of a convex polytope. In order to state the final theorems, we set the following notations.

In this paper, we consider that matrix $P(u)$ has the following bilinear structure $P(u) = P_0 + \sum_{i=1}^{m} u_i P_i$ where P_i are constant matrices. So we have $\dot{P}(u) = \sum_{i=1}^{m} \dot{u}_i P_i$. Consider the following ρ,

$$\rho(t) = \begin{bmatrix} \rho^1(t) \\ \vdots \\ \rho^m(t) \\ \rho^{m+1}(t) \\ \vdots \\ \rho^{2m}(t) \end{bmatrix} = \begin{bmatrix} u^1(t) \\ \vdots \\ u^m(t) \\ \dot{u}^1(t) \\ \vdots \\ \dot{u}^m(t) \end{bmatrix} \tag{33}$$

Thus we can define $\widetilde{P}$ and $\overline{P}$ as follows

$$\begin{aligned}
\widetilde{P}(\rho) &= P_0 + \sum_{i=1}^{m} \rho_i P_i = P(u) && (34a)\\
\overline{P}(\rho) &= \sum_{i=1}^{m} \rho_{m+i} P_i = \dot{P}(u). && (34b)
\end{aligned}$$

and we have from (24)

$$\begin{aligned}
\widehat{A}_\rho(\rho) &= H_{01} + \sum_{i=1}^{m} \rho_i H_{i1} = A_u(u) && (35a)\\
\widehat{A}_z(\rho) &= \overline{H}_{02} + \sum_{i=1}^{m} \rho_i \overline{H}_{i2} = A_z(u) && (35b)\\
\widehat{B}_\rho(\rho) &= LB - E_1CB - \Bigg(\Upsilon_{01} + \sum_{i=1}^{m} \rho_i \Upsilon_{i1} \\
&\quad + (H_{01} + \sum_{i=1}^{m} \rho_i H_{i1})E_1\Bigg) D = B_u(u) && (35c)\\
\widehat{B}_z(\rho) &= -\overline{E}_2CB - \Bigg(\overline{\Upsilon}_{02} + \sum_{i=1}^{m} \rho_i \overline{\Upsilon}_{i2} \\
&\quad + (\overline{H}_{02} + \sum_{i=1}^{m} \rho_i \overline{H}_{i2})E_1\Bigg) D = B_z(u) && (35d)\\
\overline{K}(\rho) &= K(u). && (35e)
\end{aligned}$$

Using the above notations, we have $\rho(t) \in \mathcal{P}$ where $\mathcal{P}$ is a convex polytope given by

$$\mathcal{P} = [u_{1,\min}, u_{1,\max}] \times \ldots \times [u_{m,\min}, u_{m,\max}] \times [\mu_{1,\min}, \mu_{1,\max}] \times \ldots \times [\mu_{m,\min}, \mu_{m,\max}]. \quad (36)$$

Let $\mathcal{S}$ be the set of vertices of polytope $\mathcal{P}$ given by

$$\mathcal{S} = \Big\{\beta = [\beta_1 \ldots \beta_{2m}]^T \in \mathbb{R}^{2m} | \forall\, i \in [1, m], \beta_i \in \{u_{i,\min}, u_{i,\max}\} \text{ and } \forall\, i \in [m+1, 2m], \beta_i \in \{\mu_{i,\min}, \mu_{i,\max}\}\Big\}. \quad (37)$$

$\nu = 2^{2m}$ is the number of elements of $\mathcal{S}$.

Then the two following theorems give two different approaches to obtain the gain matrix $\overline{Z}$ from (7) through a resolution of some LMI.

Theorem 7. Given two strictly positive scalars ε and ε'. Problem 1 has a solution if there exist matrices $P_i = P_i^T > 0$, $P_i \in \mathbb{R}^{n\times n}$ (for $i = 0, \ldots, m$), $F \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times\alpha}$ and a real $\gamma > 0$ such that, for $j = 1, \ldots, \nu$,

$$\begin{bmatrix} \overline{\Lambda}_{11j} & \widetilde{\Lambda}_{12j} \\ \widetilde{\Lambda}_{12j}^T & \widetilde{\Lambda}_{22j} \end{bmatrix} < 0 \quad (38)$$

where

$$\overline{\Lambda}_{11j} = \begin{bmatrix} (1,1)_j & (1,2)_j & (1,3)_j & I_n \\ (1,2)_j^T & -F-F^T & (1,3)_j & 0 \\ (1,3)_j^T & (1,3)_j^T & -\gamma^2 I_q & \overline{D} \\ I_n & 0 & \overline{D}^T & -I_q \end{bmatrix} < 0, \quad (39)$$

with

$$\begin{aligned} (1,1)_j &= \overline{P}(\overline{\beta_j}) + F\widehat{A}_\rho(\overline{\beta_j}) + Y\widehat{A}_z(\overline{\beta_j}) \\ &\quad + \widehat{A}_\rho^T(\overline{\beta_j})F^T + \widehat{A}_z^T(\overline{\beta_j})Y^T \\ (1,2)_j &= \widetilde{P}(\overline{\beta_j}) - F + \widehat{A}_\rho^T(\overline{\beta_j})F^T + \widehat{A}_{\overline{\beta_j}}^T Y^T \\ (1,3)_j &= F\widehat{B}_\rho(\overline{\beta_j}) + Y\widehat{B}_z(\overline{\beta_j}). \end{aligned}$$

and

$$\widetilde{\Lambda}_{12j} = \begin{bmatrix} \dfrac{\overline{K}(\overline{\beta_j})^T}{\sqrt{\varepsilon}} & \dfrac{\overline{K}(\overline{\beta_j})^T}{\sqrt{\varepsilon'}} & \sqrt{\varepsilon}(1,7) & 0 \\ 0 & 0 & 0 & \sqrt{\varepsilon'}(2,8) \\ \dfrac{(\overline{K}(\overline{\beta_j})\overline{D})^T}{\sqrt{\varepsilon}} & \dfrac{(\overline{K}(\overline{\beta_j})\overline{D})^T}{\sqrt{\varepsilon'}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{\Lambda}_{22j} = -I_{4n}$$

with $(1,7) = (2,8) = F(I - E_1C) + Y\overline{E}_2C$ and $\overline{\beta_j} \in \mathcal{S}$. The gain matrix $\overline{Z}$ is given by $\overline{Z} = F^{-1}Y$. ♢

Proof. As the system (7) represents the filtering error of filter (4), using equation (34) and definition (33), LMI (25) is linear in $\rho(t)$. Using (34) to (37), LMI (25) holds if LMI (38) is satisfied for the ν vertices of the convex polytope $\mathcal{P}$, *i.e.* for each element $\overline{\beta}_j$ of $\mathcal{S}$ (see (Apkarian *et al.*, 1995)). Thus lemma 6 holds, the system (7) is quadratically stable for $u(t) \in \Omega$ and $w(t) = 0$ and the mapping from the disturbance input $w(t)$ to the filtering error $e(t)$ has $\mathcal{L}_2$ gain less than a given scalar γ for $u(t) \in \Omega$.

Then problem 1 is solved and the filter matrices H_i, Υ_i,J_i, E and G are given thanks to $\overline{Z}$ (see (Gérard *et al.*, 2007)). □

Theorem 8. The problem 1 has a solution if there exist matrices $P_i = P_i^T > 0$, $P_i \in \mathbb{R}^{n\times n}$ (for $i = 0, \ldots, m$), $F \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times\alpha}$, and a real $\gamma > 0$ such that, for $j = 1, \ldots, \nu$,

$$\begin{bmatrix} \overline{\Pi}_{11j} & \overline{\Pi}_{12j} \\ \overline{\Pi}_{12j}^T & \overline{\Pi}_{22j} \end{bmatrix} < 0 \quad (40)$$

where

$$\overline{\Pi}_{11j} = \begin{bmatrix} (1,1)_j & (1,2)_j & (1,3)_j \\ (1,2)_j^T & -F-F^T & (1,3)_j \\ (1,3)_j^T & (1,3)_j^T & -\gamma^2 I_q \end{bmatrix} < 0, \quad (41)$$

with

$$\begin{aligned} (1,1)_j &= \overline{P}(\overline{\beta_j}) + F\widehat{A}_\rho(\overline{\beta_j}) + Y\widehat{A}_z(\overline{\beta_j}) \\ &\quad + \widehat{A}_\rho^T(\overline{\beta_j})F^T + \widehat{A}_z^T(\overline{\beta_j})Y^T \\ (1,2)_j &= \widetilde{P}(\overline{\beta_j}) - F + \widehat{A}_\rho^T(\overline{\beta_j})F^T + \widehat{A}_{\overline{\beta_j}}^T Y^T \\ (1,3)_j &= F\widehat{B}_\rho(\overline{\beta_j}) + Y\widehat{B}_z(\overline{\beta_j}). \end{aligned}$$

and

$$\Pi_{12j} = \begin{bmatrix} (1,4) & I_n & \overline{K}(\overline{\beta_j})^T \\ (2,4) & 0 & 0 \\ 0 & \overline{D}^T & (\overline{K}(\overline{\beta_j})\overline{D})^T \end{bmatrix} \quad (42)$$

$$\Pi_{22j} = -I_{p+2n} \quad (43)$$

with $(1,4) = (2,4) = F(I - E_1C) + Y\overline{E}_2C$ and $\overline{\beta_j} \in \mathcal{S}$. The gain matrix $\overline{Z}$ is given by $\overline{Z} = F^{-1}Y$. ♢

Proof. Similar to the proof of theorem 7. □

5. ILLUSTRATIVE EXAMPLE

In order to illustrate our approach, we design a filter for the following bilinear system

$$\begin{cases} \dot{x} = \begin{bmatrix} -0.146 & 0 \\ -0.1763 & -1.197 \end{bmatrix} x + \begin{bmatrix} -0.097 & 0.09 \\ 0.08 & 0.05 \end{bmatrix} u_1 x \\ + \begin{bmatrix} 0.2 & -0.3 \\ 0.1 & 0.1 \end{bmatrix} u_2 x + \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \begin{bmatrix} 0.2 * sin(x_1) \\ 0.2 * cos(x_2) \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix} w \\ y = [1\ \ 0]\, x + 0.9w \end{cases}$$

where $x(t)$, $y(t)$, $w(t)$ and u are defined as in section 2.1, with (for $i = 1, 2$) $-1 \le u_i(t) \le 1$, $-10 \le \dot{u}_i(t) \le 10$ and $\gamma = 1$.

5.1 First approach : with majoration

Applying theorem 7 yields $\varepsilon = 0.7$ and the gain $\overline{Z}$ is

$$\overline{Z} = 10^8 \begin{bmatrix} 0.2585 & 1.8962 & 0.8861 & -1.2384 & -0.3338 & 1.0386 & -0.6089 & 1.8962 & -1.2384 & 1.0386 \\ -0.0063 & -0.0186 & -0.0107 & 0.0213 & 0.0012 & -0.0146 & 0.0137 & -0.0186 & 0.0213 & -0.0146 \end{bmatrix}.$$

Finally, we obtain the following filter matrices

$$\begin{aligned} H_0 &= \begin{bmatrix} -3.1571 & 0 \\ 0.0093 & -1.197 \end{bmatrix}, H_1 = 10^{-2} \begin{bmatrix} -0.12 & 7.94 \\ 0.89 & 5.05 \end{bmatrix}, \\ H_2 &= 10^{-2} \begin{bmatrix} -0.39 & -26.47 \\ 1.34 & 9.83 \end{bmatrix}, J_0 = \begin{bmatrix} 2.6565 \\ -0.1785 \end{bmatrix}, \\ J_1 &= 10^{-2} \begin{bmatrix} -8.5 \\ 7.13 \end{bmatrix}, J_2 = 10^{-2} \begin{bmatrix} 18.14 \\ 8.87 \end{bmatrix}, \\ E &= 10^{-2} [11.77\ \ -0.57]. \end{aligned}$$

The obtained attenuation gain is given by $\gamma_{\text{opt}} = 1.182$.

5.2 Second approach : with augmented state variable

Applying theorem 8 yields $\epsilon = 1.5$ and the gain $\overline{Z}$ is

$$\overline{Z} = 10^8 \begin{bmatrix} 0.1951 & 1.3745 & 0.5530 & -0.7898 & -0.2714 \\ -0.0072 & -0.0134 & -0.0087 & 0.0180 & 0 \end{bmatrix.$$
$$\begin{matrix} 0.7238 & -0.4851 & 1.3745 & -0.7898 & 0.7238 \\ -0.0112 & 0.0162 & -0.0134 & 0.0180 & -0.0112 \end{bmatrix}.$$

Finally, we obtain the following filter matrices

$$H_0 = \begin{bmatrix} -3.2012 & 0 \\ 0.0138 & -1.197 \end{bmatrix}, \; H_1 = 10^{-2} \begin{bmatrix} -0.04 & 7.94 \\ 0.93 & 5.05 \end{bmatrix},$$
$$H_2 = 10^{-2} \begin{bmatrix} -0.66 & -26.47 \\ 1.43 & 9.83 \end{bmatrix}, \; J_0 = \begin{bmatrix} 2.6955 \\ -0.1824 \end{bmatrix},$$
$$J_1 = 10^{-2} \begin{bmatrix} -8.57 \\ 7.09 \end{bmatrix}, \; J_2 = 10^{-2} \begin{bmatrix} 18.38 \\ 8.79 \end{bmatrix},$$
$$E = 10^{-2} \, [11.77 \; -0.57] \, .$$

The obtained attenuation gain is given by $\gamma_{\text{opt}} = 1.16$.

5.3 Results

The inputs are a mix of steps and sinusoids, the state of the system (44) are presented in figure 1. The filtering errors and the disturbance are given in figures 2 and 3. The difference between the two filters is negligible.

6. CONCLUSION

This paper has presented a computationally tractable solution to the $\mathcal{H}_\infty$ unbiased filtering problem via a LPV approach for bilinear systems with lipschitz terms. The proposed design is shown to be efficient via a numerical example.

REFERENCES

Apkarian, P., P. Gahinet and G. Becker (1995). Self-scheduled $\mathcal{H}_\infty$ control of linear parameter-varying systems: a design example. *Automatica* **31**, 1251–1261.

Chughtai, S.S. and N. Munro (2004). LMI based gain-scheduled control. In: *Proc. Control*. University of Bath, UK.

Gérard, B., H. Souley Ali, M. Zasadzinski and M. Darouach (2007). LPV approach for $\mathcal{H}_\infty$ reduced-order filters for bilinear systems. In: *Proc. European Contr. Conf.*. Kos, Greece.

Lu, G. and D.W.C Ho (2002). Full-order and reduced-order observers for lipschitz descriptor systems: The unified lmi approach. *IEEE Trans. Circ. Syst.* **53**, 563–567.

Rao, C.R. and S.K. Mitra (1971). *Generalized Inverse of Matrices and its Applications*. Wiley. New York.

Seron, M.M., J.H. Braslavsky and G.C. Goodwin (1997). *Fundamental Limitations in Filtering and Control*. Springer-Verlag. New York.

van der Schaft, A.J. (1992). $\mathcal{L}$2-gain analysis of nonlinear systems and nonlinear state-feedback $\mathcal{H}_\infty$ control. *IEEE Trans. Aut. Contr.* **37**, 770–784.

Zhu, F. and Z. Han (2002). A note on observers for Lipschitz nonlinear systems. *IEEE Trans. Aut. Contr.* **47**, 1751–1754.

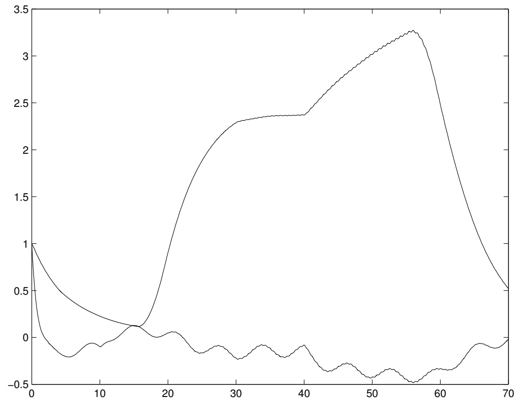

Fig. 1. States $x_1(t)$ and $x_2(t)$

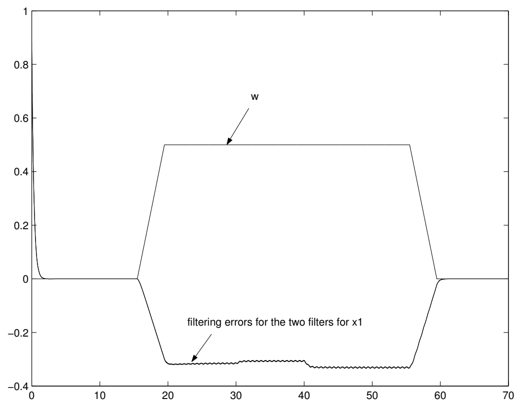

Fig. 2. Filtering error $e_1(t)$ for the two approaches and disturbance $w(t)$

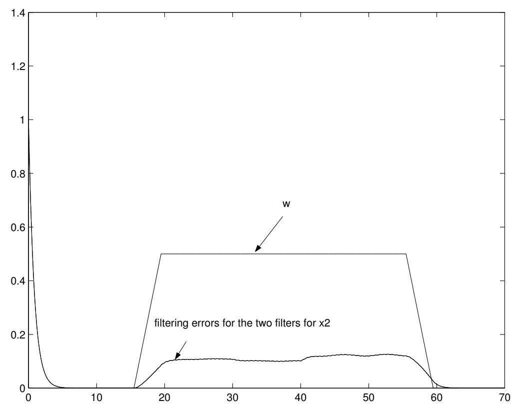

Fig. 3. Filtering error $e_2(t)$ for the two approaches and disturbance $w(t)$