Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Leader-Following Consensus Control for Multi-Agent Systems Under Measurement Noises ⋆

Cui-Qin Ma, Tao Li, Ji-Feng Zhang

Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China (E-mail: cuiqinma@amss.ac.cn)

Abstract: This paper is concerned with leader-following consensus control for multi-agent systems under measurement noises. Time-varying consensus gains are introduced into the network protocol designed. By using the tools of stochastic analysis and algebraic graph theory, a sufficient condition is obtained for the protocol to ensure strong mean square consensus under the fixed topologies. This condition is shown to be necessary and sufficient in the noise-free case. In addition, by using a common Lyapunov function, the result is extended to the switching topology case.

1. INTRODUCTION

In recent years, leader-following consensus problems have attracted many researchers, due to their broad applications in swarms and flocks (Yu *et al.*, 2005; Olfati-Saber, 2006), formation control (Leonard & Fiorelli, 2001; Fax & Murray, 2004), and large scale robotic systems (Belta & Kumar, 2002), etc. In leader-following multi-agent systems, the leaders are usually independent of their followers, but have influence on the followers' behaviors. Therefore, one can realize one's control objective of the agents by only controlling the leaders, which transfers the control of a multi-agent system to that of a single-agent. This not only simplifies the design and implementation of the controls but also helps to save energy and reduce control cost (Ren *et al.*, 2004; Cutts & Speakman, 1994).

Jadbabaie *et al.* (2003) considered the nearest neighborhood principle, and under time-varying topologies, proved that if all the agents were jointly connected with their leader, then their states would converge to the state of the leader as time goes on. Ren & Beard (2005) extended the results of (Jadbabaie *et al.*, 2003) to the directed topology case. Hu & Hong (2005) considered the leader-following consensus with time-varying time-delays under fixed and switching topologies.

A common feature of the above literature on leader-following consensus is that measurement noises are not considered and the state information is assumed to be exactly obtained. However, real communication processes are often disturbed by various random factors, and measurement noises seem always there. Therefore, just as Ren, Beard & Atkins (2005) pointed out, for the consensus problems it is important to investigate how to design consensus protocols applicable to the cases with communication noises.

⋆ This work was supported by the National Natural Science Foundation of China under grants 60221301 and 60674038.

Recently, consensus problems under measurement noises have gradually been studied in (Huang & Manton, 2006; Ren, Beard & Kingston, 2005; Kingston *et al.*, 2005; Li & Zhang, 2007). Ren, Beard & Kingston (2005) designed a consensus protocol based on Kalman filter structure. Kingston *et al.* (2005) showed that the protocol designed in (Ren, Beard & Kingston, 2005) was input-to-state stable (from measurement noises to consensus errors). Li & Zhang (2007) considered average-consensus problem under measurement noises and fixed topologies. All these works are characterized by leader-free. To our knowledge, there is no result of leader-following consensus under measurement noises, which is the main focus of this paper.

In this paper, for simplicity, the state of the leader is assumed to be a constant. However, the follower collects information and updates its state in a decentralized way, based on the measurements of its neighbors corrupted by white noises. To overcome the impact of the measurement noises, time-varying consensus gains are introduced in the followers' consensus protocols, which attenuate the noises effectively, but render the closed-loop system a time-varying stochastic differential equation whose state matrix is neither symmetric nor diagonalizable, since what we consider is a digraph. Thus, the closed-loop system cannot be decoupled, and hence, it is hard to analyze the convergence of the consensus protocol. It is worth pointing out that, different from (Li & Zhang, 2007), here the digraph is not required to be balanced, thus, the method of the symmetrized graph in (Li & Zhang, 2007) is not suitable. It is also different from the noise-free cases in (Jadbabaie *et al.*, 2003; Lin *et al.*, 2004), where the state matrix of the closed-loop equation can be easily transformed to a stochastic matrix, and so, the convergence properties can be analyzed by employing the theory of stochastic matrices. But, here due to the time-varying consensus gains, the state matrix of the closed-loop equation is no longer a stochastic matrix, so the tools of stochastic matrices do not work. We combine stochastic analysis and algebraic graph theory together, and use the common Lyapunov function to do the convergence analysis of the consensus protocol. A sufficient condition is obtained for the state of each follower to converge to that of the leader in mean square, which is also found necessary and sufficient in the noise-free case. In addition, the protocol is shown to be strong mean square consensus under switching topologies if the subgraph formed by the followers is balanced.

The remainder of this paper is organized as follows. In section 2, some concepts in graph theory and the problem to be investigated are formulated. In section 3, the convergence properties of the closed-loop systems are analyzed under fixed and switching topologies, respectively. In section 4, a numerical example is given to illustrate our results. In section 5, some concluding remarks and future research topics are discussed. In addition, due to space limitation only the proof of Theorem 2 is provided.

The following notations will be used throughout this paper: $\mathbf{1}$ denotes a column vector with all ones; $\mathbb{R}^{n\times n}$ denotes the family of all $n\times n$ dimensional matrices; $\lambda_{max}(X)$ and $\lambda_{min}(X)$ denote the maximum and minimum eigenvalues of the real symmetric matrix X, respectively; I_m denotes the m dimensional identity matrix. $P>0$ denotes the matrix P is positive definite. For a given set $\mathcal{S}$, χ_S denotes the indicator function of $\mathcal{S}$. For a given vector or (square) matrix A, A^T denotes its transpose; $tr(A)$ denotes its trace. For a family of random variables $\{\xi_\lambda, \lambda\in\Lambda\}$, $\sigma(\xi_\lambda,\lambda\in\Lambda)$ denotes the σ-algebra [1] generated by $\{\xi_\lambda\in B, B\in\mathfrak{B}, \lambda\in\Lambda\}$ where $\mathfrak{B}$ denotes the 1 dimensional Borel sets.

2. PROBLEM FORMULATION

Before formulating our problem, we first introduce some basic concepts and notions in algebraic graph theory.

2.1 Preliminaries

(Godsil & Royle, 2001; Olfati-Saber & Murray, 2004)

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ be a weighted digraph with the set of vertices $\mathcal{V}=\{1,2,...,n\}$ and the set of edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$. In $\mathcal{G}$, the ith vertex represents the ith agent, and a directed edge from i to j is denoted as an ordered pair $(i,j)\in\mathcal{E}$, which means that agent j can directly receive information from agent i. If there is a directed edge from i to j, then the vertex i is called the parent vertex and the vertex j is called the child vertex. The set of neighbors of the ith agent is denoted by $N_i=\{j\in\mathcal{V}\mid (j,i)\in\mathcal{E}\}$.

$\mathcal{A}=(a_{ij})\in\mathbb{R}^{n\times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements and $a_{ij}>0\Leftrightarrow j\in N_i$. The in-degree and out-degree of vertex i are defined as $deg_{in}(i)=\sum_{j=1}^n a_{ij}$ and $deg_{out}(i)=\sum_{j=1}^n a_{ji}$, respectively. Then the Laplacian of the weighted digraph $\mathcal{G}$ is defined as $L_\mathcal{G}=\mathcal{D}-\mathcal{A}$, where $\mathcal{D}=\text{diag}(deg_{in}(1),\ deg_{in}(2),\ \cdots,\ deg_{in}(n))$. If $deg_{in}(i)=deg_{out}(i)$, $i=1,2,...,n$, then we call $\mathcal{G}$ a balanced digraph.

A sequence of edges $(i_1,i_2),(i_2,i_3),\cdots,(i_{k-1},i_k)$ is called a directed path from vertex i_1 to vertex i_k. If there is a directed path from i to j between any two distinct vertices $i,j\in\mathcal{V}$, then $\mathcal{G}$ is called strongly connected. A directed tree is a directed graph where every vertex except the root vertex, which has only children but no parent, has exactly one parent. A spanning tree of a digraph is a directed tree formed by graph edges that connected all the vertices of the graph.

[1] σ-algebra is defined in (Chow & Teicher, 1997).

Below is a theorem for the Laplacian matrix.

Theorem 1. (Ren & Beard, 2005) The Laplacian matrix $L_\mathcal{G}$ of a digraph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right half plane. Furthermore, $L_\mathcal{G}$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ has a spanning tree.

2.2 Consensus Protocols

Here we consider a system consisting of $N+1$ agents where an agent indexed by 0 acts as the leader and the other agents indexed by $1,2,...,N$, respectively, are referred to as the followers. The dynamics of the ith follower is described as follows:

$$\dot{x}_i(t)=u_i(t),\quad i=1,2,...,N, \tag{1}$$

where $x_i(t)\in\mathbb{R}$ and $u_i(t)\in\mathbb{R}$ are the state and control input of the ith follower, respectively. In general, the behavior of the leader is independent of the followers. x_0 denotes the state of the leader and keeps being a constant.

With regarding the $N+1$ agents as vertices, the topology relationships among them can be conveniently described by a digraph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ with $\mathcal{V}=\{0,1,2,...,N\}$ and

$$\mathcal{A}=\begin{bmatrix}0 & 0 & \cdots & 0\\ a_{10} & a_{11} & \cdots & a_{1N}\\ \vdots & \vdots & \ddots & \vdots\\ a_{N0} & a_{N1} & \cdots & a_{NN}\end{bmatrix}\in\mathbb{R}^{(N+1)\times(N+1)}.$$

For simplicity, let the digraph $\overline{\mathcal{G}}=(\overline{\mathcal{V}},\overline{\mathcal{E}},\overline{\mathcal{A}})$ represent the subgraph formed by the N followers and $B=\text{diag}(b_1,b_2,...,b_N)$ represent the leader adjacency matrix associated with $\mathcal{G}$ where

$$\overline{\mathcal{V}}=\mathcal{V}\setminus\{0\},\overline{\mathcal{A}}=\begin{bmatrix}a_{11} & \cdots & a_{1N}\\ \vdots & \ddots & \vdots\\ a_{N1} & \cdots & a_{NN}\end{bmatrix}\in\mathbb{R}^{N\times N}, b_i=a_{i0}\geq 0.$$

Obviously, $b_i>0\Leftrightarrow 0\in N_i$.

As it is well-known, real communication processes are often disturbed by random noises. In our models, the ith agent receives information from its neighbors with measurement noises. Let

$$y_{ji}(t)=x_j(t)+n_{ji}(t),\quad j\in N_i,$$

denote the measurement of the jth agent's state $x_j(t)$ by the ith agent, where $\{n_{ji}(t), j\in N_i,\ i=1,2,...,N\}$ are independent standard white noises.

A group of controls

$$\mathcal{U}=\{u_i,\ i=1,2,...,N\}$$

is called a measurement-based distributed protocol (Li & Zhang, 2007), if $u_i(t)\in\sigma(x_i(s),\bigcup_{j\in N_i}y_{ji}(s),\ 0\leq s\leq t),\forall\, t\geq 0,\ i=1,2,...,N$.

The so-called leader-following consensus problem is to design a measurement-based distributed protocol such that as the system evolves, each follower's state will finally converge to the leader's. We propose a protocol for the ith agent as:

$$u_i(t) = a(t)\left[\sum_{j\in N_i} a_{ij}\left(y_{ji}(t) - x_i(t)\right) + b_i(y_{0i}(t) - x_i(t))\right], \qquad (2)$$

where $t \geq 0$, $i = 1,2,...,N$, $a(\cdot) : [0,\infty) \longrightarrow (0,\infty)$ is a piecewise continuous function, usually called a time-varying consensus gain (Li & Zhang, 2007) and the set of neighbors $N_i = N_i(\mathcal{G})$ of vertex i varies in the switching topology case.

Remark 1. (i) Different from the leader-following consensus protocols in (Jadbabaie *et al.*, 2003; Ren & Beard, 2005; Hu & Hong, 2005), the measurement noises are explicitly taken into account in the consensus protocol (2). (ii) From (2) it is obvious that the consensus protocol devised for the ith agent is indeed a measurement-based distributed protocol since it depends only on the state information of itself and its neighbors'.

Let α_i represent the ith row of the matrix $\overline{\mathcal{A}}$, $H = \text{diag}(\alpha_1, \alpha_2, ..., \alpha_N)$ which is an $N \times N^2$ dimensional block diagonal matrix, $n_0(t) = (n_{01},\ n_{02}, ..., n_{0N})^T$, $n_i(t) = (n_{1i},\ n_{2i}, ..., n_{Ni})^T$, $i = 1,2,...,N$, and $Q = (B,\ H)$ is an $N \times N(N+1)$ dimensional block matrix. Denote $X(t) = (x_1(t),\ x_2(t), \cdots, x_N(t))^T$. Substituting the consensus protocol (2) to the system (1), we have

$$\frac{dX(t)}{dt} = a(t)(-L_{\overline{\mathcal{G}}} - B)X(t) + a(t)B\cdot\mathbf{1}x_0 + a(t)QZ(t), \qquad (3)$$

where $Z(t) = (n_0^T(t),\ n_1^T(t),\ ..., n_N^T(t))^T$ is an $N(N+1)$ dimensional independent standard white noise sequence. We can construct an N dimensional standard Brownian motion $W(t) = (W_1(t),\ W_2(t),\ \cdots,\ W_N(t))^T$, and rewrite (3) as

$$dX(t) = a(t)(-L_{\overline{\mathcal{G}}} - B)X(t)dt + a(t)B\cdot\mathbf{1}x_0dt + a(t)GdW(t), \qquad (4)$$

where

$$G = \text{diag}\left(\sqrt{b_1^2 + \sum_{j\in N_1} a_{1j}^2},\ \cdots,\ \sqrt{b_N^2 + \sum_{j\in N_N} a_{Nj}^2}\right).$$

In the sequel, we will show that under the consensus protocol (2), each follower's state will converge to the leader's.

3. CONVERGENCE ANALYSIS

Prior to establishing the convergence properties of the consensus protocol (2), we first introduce a definition of consensus protocols for stochastic systems.

Definition 1. (Huang & Manton, 2006) A distributed protocol $\mathcal{U} = \{u_i,\ i = 1,2,...,N\}$ is called a strong mean square consensus protocol if $\mathcal{U} = \{u_i,\ i = 1,2,...,N\}$ renders the system (1) has the following properties: $\lim_{t\to\infty} E(x_i(t) - x^*)^2 = 0$, $i = 1,2,...,N$, where x^* is a random variable and $E(x^*)^2 < \infty$.

Secondly, we make the following assumptions:

$(\mathbf{A_1})$: $\mathcal{G}$ has a spanning tree.

$(\mathbf{A_2})$: $\int_0^\infty a(s)ds = \infty$.

$(\mathbf{A_3})$: $\int_0^\infty a^2(s)ds < \infty$.

Remark 2. Generally speaking, a digraph does not always have a spanning tree. However, a strongly connected digraph must have a spanning tree. Therefore, the spanning tree requirement of Assumption (A_1) is weaker than the strong connectivity condition. Assumptions (A_2)-(A_3) are standard assumptions often used in the stochastic approximation (Nevel'son & Has'minskii, 1976), which (especially (A_2)) in some cases happen to be the weakest conditions to ensure a consensus protocol (see Theorem 3 below).

Let $\delta(t) = X(t) - x_0 \cdot \mathbf{1}$. Then, by (4) we have

$$d\delta(t) = a(t)(-L_{\overline{\mathcal{G}}} - B)\delta(t)dt + a(t)GdW(t). \qquad (5)$$

Next, we will demonstrate that the protocol (2) is a strong mean square consensus protocol under Assumptions (A_1)-(A_3).

3.1 Fixed Topology

Theorem 2. For system (1) with the consensus protocol (2), if Assumptions (A_1)-(A_3) hold, then

$$\lim_{t\to\infty} E\|\delta(t)\|^2 = 0. \qquad (6)$$

That is, (2) is a strong mean square consensus protocol.

Proof. Noticing the definition of the matrix B and the fact that $L_{\overline{\mathcal{G}}}$ is the Laplacian matrix of $\overline{\mathcal{G}}$, we know

$$L_{\mathcal{G}} = \begin{bmatrix} 0 & 0 \\ -B\cdot\mathbf{1} & L_{\overline{\mathcal{G}}} + B \end{bmatrix}$$

is the Laplacian matrix of $\mathcal{G}$. Consequently, from Assumption (A_1) and Theorem 1 it follows that $-L_{\overline{\mathcal{G}}} - B$ is a stable matrix. Thus, the Lyapunov equation

$$(-L_{\overline{\mathcal{G}}} - B)P + P(-L_{\overline{\mathcal{G}}} - B)^T = -I_N \qquad (7)$$

has a unique positive definite solution P. Let

$$V(t) = \delta^T(t)P\delta(t).$$

Then, by (5) and Itô formula, we get

$$\begin{aligned} dV(t) = &- a(t)\delta^T(t)\delta(t)dt + a^2(t)tr(PGG^T)dt \\ &+ 2a(t)\delta^T(t)PGdW(t). \end{aligned}$$

Noticing that $P > 0$, we have

$$\begin{aligned} dV(t) \leq &- \frac{a(t)}{\lambda_{max}(P)}V(t)dt + a^2(t)tr(PGG^T)dt \\ &+ 2a(t)\delta^T(t)PGdW(t). \end{aligned} \qquad (8)$$

Now we prove that

$$E\int_{t_0}^t a(s)\delta^T(s)PGdW(s) = 0,\ \forall\ 0 \leq t_0 \leq t. \qquad (9)$$

For any given $t_0 \geq 0$ and $T \geq t_0$, let $\tau_K^{t_0} = \inf\{t \geq t_0 \mid \delta^T(t)P\delta(t) \geq K\}$, where K is a given positive integer. From (8) one can get

$$\begin{aligned} &E[V(t\wedge\tau_K^{t_0})\chi_{\{t\leq\tau_K^{t_0}\}}] - E[V(t_0)] \\ &\leq -\frac{1}{\lambda_{max}(P)}\int_{t_0}^t a(s)E[V(s\wedge\tau_K^{t_0})\chi_{\{s\leq\tau_K^{t_0}\}}]ds \\ &+tr(PGG^T)\int_{t_0}^t a^2(s)ds \end{aligned}$$

$$\leq tr(PGG^T)\int_{t_0}^{T} a^2(s)ds, \quad \forall\, t_0 \leq t \leq T,$$

which implies that there exists a constant $M_{t_0,T} > 0$ such that

$$E[V(t\wedge \tau_K^{t_0})\chi_{\{t\leq \tau_K^{t_0}\}}] \leq M_{t_0,T} < \infty, \quad \forall\, 0 \leq t_0 \leq T.$$

Noticing that $t\wedge\tau_K^{t_0} \xrightarrow[K\to\infty]{a.s.} t$, by the above inequality and the Fatou lemma, we have

$$\sup_{t_0\leq t\leq T} E[V(t)] \leq M_{t_0,T}.$$

Thus,

$$E\int_{t_0}^{t} a^2(s)V(s)ds \leq \sup_{t_0\leq t\leq T} E[V(t)]\int_{t_0}^{T} a^2(s)ds < \infty, \quad \forall\, t_0 \leq t \leq T.$$

By the arbitrariness of T, we obtain

$$E\int_{t_0}^{t} a^2(s)V(s)ds < \infty, \quad \forall\, 0 \leq t_0 \leq t.$$

This together with

$$E\int_{t_0}^{t} a^2(s)\|\delta^T(s)PG\|^2 ds \leq \|P\|\|G\|^2 E\int_{t_0}^{t} a^2(s)V(s)ds$$

gives (9). From (8) it follows that for any $t \geq 0$ and $h > 0$,

$$E[V(t+h)] - E[V(t)] \leq -\frac{1}{\lambda_{max}(P)}\int_t^{t+h} a(s)E[V(s)]ds + tr(PGG^T)\int_t^{t+h} a^2(s)ds,$$

Further,

$$\limsup_{h\to 0^+}\frac{E[V(t+h)]-E[V(t)]}{h} \leq -\frac{1}{\lambda_{max}(P)}a(t)E[V(t)] + tr(PGG^T)a^2(t).$$

Thus, by the comparison principle (Michel & Miller, 1977), we have that for any $t \in [0, t+h]$,

$$E[V(t)] \leq E[V(0)]\exp\{-\frac{1}{\lambda_{max}(P)}\int_0^t a(s)ds\} + tr(PGG^T)\int_0^t a^2(s)\exp\{-\frac{1}{\lambda_{max}(P)}\int_s^t a(\tau)d\tau\}ds. \quad (10)$$

By Assumption (A_3), for any given $\epsilon > 0$ there exists $s_0 > 0$ such that $\int_{s_0}^{\infty} a^2(s)ds < \epsilon$. Hence,

$$tr(PGG^T)\int_0^t a^2(s)\exp\{-\frac{1}{\lambda_{max}(P)}\int_s^t a(\tau)d\tau\}ds$$
$$\leq tr(PGG^T)\exp\{-\frac{1}{\lambda_{max}(P)}\int_{s_0}^t a(\tau)d\tau\}\int_0^{\infty} a^2(s)ds + tr(PGG^T)\int_{s_0}^{\infty} a^2(s)ds$$
$$\leq o(1) + tr(PGG^T)\epsilon, \quad t\to\infty.$$

Since ϵ is arbitrary,

$$\lim_{t\to\infty} tr(PGG^T)\int_0^t a^2(s)\exp\{-\frac{1}{\lambda_{max}(P)}\int_s^t a(\tau)d\tau\}ds = 0.$$

Noticing that $\|\delta(t)\|^2 \leq \frac{V(t)}{\lambda_{min}(P)}$, by Assumption (A_2) and (10), (6) holds. □

Remark 3. From Theorem 2 we can see that in the fixed topology case, under Assumptions (A_1)-(A_3), the designed protocol ensures that the state of each follower converges to that of the leader in mean square.

Remark 4. Different from (Li & Zhang, 2007), here we only require $\mathcal{G}$ has a spanning tree, and do not require it is balanced.

Remark 5. It is worth pointing out that, unlike (Jadbabaie *et al.*, 2003) and (Tsitsiklis *et al.*, 1986), here random measurement noises are considered. To reduce the influence of noises, time-varying consensus gains are adopted, which renders the closed-loop system (5) is a time-varying stochastic differential equation. Assumption (A_1) ensures the existence of a Lyapunov function, with which we complete the convergence analysis of the closed-loop system; while Assumptions (A_2)-(A_3) ensure that the mean square error $E[\|\delta(t)\|^2]$ converges to zero.

As stated in Theorem 2, Assumptions (A_1)-(A_3) are sufficient conditions to guarantee that the protocol (2) is a strong mean square consensus protocol. In what follows, we will prove that when the measurement noises are zeros, Assumptions (A_1)-(A_2) are necessary, too.

When $n_{ji}(t) \equiv 0$, the protocol (2) is reduced to

$$u_i(t) = a(t)\left[\sum_{j\in N_i} a_{ij}\,(x_j(t) - x_i(t)) + b_i(x_0 - x_i(t))\right], \quad t \geq 0,\ i = 1, 2, ..., N. \quad (11)$$

In this case, applying the protocol (11) to the system (1), we obtain a tracking error equation as follows:

$$d\delta(t) = a(t)(-L_{\overline{\mathcal{G}}} - B)\delta(t)dt. \quad (12)$$

For this tracking error, we state the following result.

Theorem 3. For the system (1), if the protocol (2) is applied and $n_{ji}(t) \equiv 0, j \in N_i, i = 1, 2, ..., N$, then for any initial value $X(0)$, $\lim_{t\to\infty}\|\delta(t)\| = 0$ if and only if Assumptions (A_1)-(A_2) hold.

Remark 6. When $n_{ji}(t) \equiv 0$, from Theorem 3 it can be seen that Assumptions (A_1)-(A_2) are the necessary and sufficient conditions ensuring the followers can eventually follow the leader, where Assumption (A_1) guarantees the connectivity of the network topology which makes the state of each follower eventually equal the leader's; and Assumption (A_2) guarantees the consensus error $\delta(t)$ converges to zero with a certain rate.

Remark 7. When $n_{ji}(t) \equiv 0$, in (Jadbabaie *et al.*, 2003; Tsitsiklis *et al.*, 1986) the theory of stochastic matrices and nonnegative matrices was used for the convergence analysis. However, here Assumption (A_2) cannot guarantee that there exists a positive constant $\alpha > 0$ such that $a(t) \geq \alpha$, $\forall\, t \geq 0$ (for example, $a(t) = \frac{1}{t+1}$, $\forall\, t \geq 0$), and so, the positive entries of those off-diagonal ones in the state matrix of the closed-loop system (12) are not uniformly bounded away from zero. Thus, the main condition Assumption 1(b) in (Tsitsiklis *et al.*, 1986) does not hold, and the stochastic matrix methods used in (Jadbabaie *et al.*, 2003; Tsitsiklis *et al.*, 1986) do not work here.

Remark 8. In Theorem 3 $\mathcal{G}$ is not required to be balanced, which makes the tools of symmetric digraphs used in (Li & Zhang, 2007) do not work. Here, we propose a Lyapunov-based approach to deal with the problem.

As shown above, we analyze the convergence of the consensus protocol (2) under the fixed topologies. However, in most real communication processes, the communication links among the agents often change in time. For example, in the flocking and vehicle formation control, the communication topology depends on the environment of the flocking and the relative positions of the vehicles, which are usually changing in time. Thus, it will be interesting to study the convergence of the consensus protocol under switching topologies.

3.2 Switching Topology

$\sigma(t) : [0, \infty) \longrightarrow \mathcal{I}_{\mathcal{T}^*}$ is a switching signal that determines the communication topology. The set $\mathcal{T}^*$ is a set of digraphs with a common vertex set $\mathcal{V}$. Since at most a digraph with vertex set $\mathcal{V}$ has $N(N+1)$ directed edges, the set $\mathcal{T}^*$ is finite and can be denoted as $\mathcal{T}^* = \{\mathcal{G}_1, ..., \mathcal{G}_{N^*}\}$, where N^* represents the total number of digraphs in $\mathcal{T}^*$ and $\mathcal{I}_{\mathcal{T}^*} = \{1, 2, ..., N^*\}$ is the index set associated with the elements of $\mathcal{T}^*$. We can rewrite consensus protocol (2) as

$$u_i(t) = a(t)\Bigg[\sum_{j\in N_i(\mathcal{G}_{\sigma(t)})} a_{ij}(\mathcal{G}_{\sigma(t)})(y_{ji}(t) - x_i(t)) + b_i(\mathcal{G}_{\sigma(t)})(y_{0i}(t) - x_i(t))\Bigg], \ t \geq 0, \tag{13}$$

where $i = 1, 2, ..., N$; $N_i(\mathcal{G}_{\sigma(t)})$ is the set of neighbors of agent i in the digraph $\mathcal{G}_{\sigma(t)}$; $a_{ij}(\mathcal{G}_{\sigma(t)})$ $(i, j = 1, 2, ..., N)$ is the element of the adjacency matrix of $\mathcal{G}_{\sigma(t)}$, and $B_{\mathcal{G}_{\sigma(t)}} = \mathrm{diag}(b_1(\mathcal{G}_{\sigma(t)}) ,..., b_N(\mathcal{G}_{\sigma(t)}))$ such that $b_i(\mathcal{G}_{\sigma(t)}) > 0$ if and only if $0 \in N_i(\mathcal{G}_{\sigma(t)})$.

Let $\delta(t) = X(t) - x_0 \cdot \mathbf{1}$ as in Section 3.1. Then, by substituting the protocol (13) to the system (1), we get

$$d\delta(t) = a(t)(-L_{\overline{\mathcal{G}}_{\sigma(t)}} - B_{\mathcal{G}_{\sigma(t)}})\delta(t)dt + a(t)G_{\mathcal{G}_{\sigma(t)}}dW(t), \tag{14}$$

where $L_{\overline{\mathcal{G}}_{\sigma(t)}}$ is the Laplacian matrix of the digraph $\overline{\mathcal{G}}_{\sigma(t)}$ formed by N followers and

$$G_{\mathcal{G}_{\sigma(t)}} = \mathrm{diag}\Bigg(\sqrt{b_1^2(\mathcal{G}_{\sigma(t)}) + \sum_{j\in N_1(\mathcal{G}_{\sigma(t)})} a_{1j}^2(\mathcal{G}_{\sigma(t)})}, \cdots, \sqrt{b_N^2(\mathcal{G}_{\sigma(t)}) + \sum_{j\in N_N(\mathcal{G}_{\sigma(t)})} a_{Nj}^2(\mathcal{G}_{\sigma(t)})}\Bigg).$$

In the sequel, we will analyze the convergence of the consensus protocol (13) based on the closed-loop system (14). The main result of this section can be summarized as follows:

Theorem 4. For the system (1) with the protocol (13), if for any $t \geq 0$, $\overline{\mathcal{G}}_{\sigma(t)}$ is balanced, and $\mathcal{G}_{\sigma(t)}$ has a spanning tree, then under Assumptions (A_2)-(A_3),

$$\lim_{t\to\infty} E\|\delta(t)\|^2 = 0.$$

Remark 9. From Theorem 4 it can be seen that even under the switching topologies, the consensus protocol (2) can still guarantee the state of each follower converges to that of the leader, i.e., (2) is a strong mean square consensus protocol.

4. NUMERICAL EXAMPLE

In this section, we give a numerical simulation to illustrate our results.

We consider a system consisting of one leader indexed by 0 and two followers indexed by 1 and 2, respectively. $x_0 = 1$ is the state of the leader and the dynamics of ith follower is described as follows:

$$\dot{x}_i(t) = u_i(t), \ i = 1, 2. \tag{15}$$

Assume the initial states of the two followers indexed by 1 and 2 are $x_1(0) = 2$ and $x_2(0) = -2$, respectively. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be the communication topology graph as shown in Fig.1 with

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

$y_{ji}(t) = x_j(t) + n_{ji}(t)(j \in N_i, i = 1, 2)$ is the measurement of jth agent's state $x_j(t)$ by ith agent, where $\{n_{ji}(t), \ i, j = 0, 1, 2\}$ are independent standard white noises.

We choose the consensus gain function $a(t) = \frac{1}{t+1}, \ t \geq 0$, and apply the consensus protocol (2) to the system (15). The simulation result for the states of the leader and the two followers is shown in Fig.2, from which we can see that two followers can eventually follow the leader.

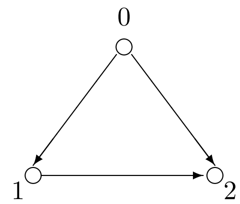

Fig.1 The communication topology between agents

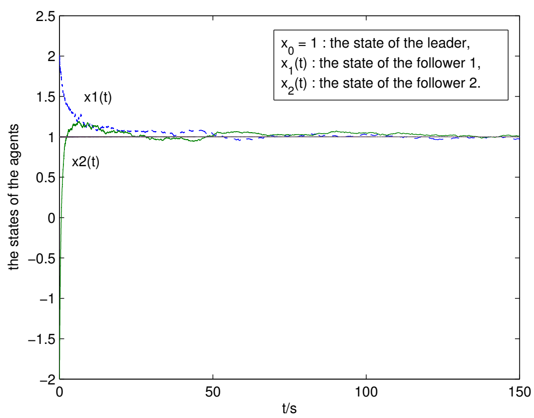

Fig.2 Curves of states of the agents

5. CONCLUSION

This paper is concerned with leader-following consensus control for multi-agent systems under measurement noises. Time-varying consensus gains are introduced into the network protocol designed. By using the stochastic analysis and algebraic graph theory, it is shown that the protocol designed is a strong mean square consensus protocol under the fixed topologies. In addition, if the subgraph formed by the followers is balanced, the protocol is still strong mean square consensus even under the switching topologies.

It is worth pointing out that this paper is only a preliminary step on leader-following consensus under measurement noises. When the state of the leader is not a constant, it will be more interesting to study consensus protocols to guarantee the convergence of the state of each follower to that of the time-varying leader under measurement noises. However, as Ren (2007) pointed out, the extension of consensus algorithms from a constant reference to a time-varying one is non-trivial. There are, of course, many other topics worth investigating, such as, how to design protocols for time-delay cases, and how to get almost sure consensus protocols, etc.

REFERENCES

Belta, C., & Kumar, V. (2002). Trajectory design for formation of robots by kinetic energy shaping, *Proceedings of the 2002 IEEE International Conference on Robotics and Automation*, Washington DC, May 11-15, 2002.

Chow, Y. S., & Teicher, H. (1997). *Probability Theory: Independence, Interchangeability, Martingales*, 3rd edition, New York: Springer-Verlag.

Cutts, C., & Speakman, J. (1994). Energy savings in formation flight of Pink-Footed geese, *Journal of Experimental Biology*, 189(1), 251-261.

Fax, J. A., & Murray, R. M. (2004). Information flow and cooperative control of vehicle formations, *IEEE Transactions on Automatic Control*, 49(9), 1465-1476.

Godsil, C., & Royle, G. (2001). *Algebraic Graph Theory*, New York: Springer.

Hu, J., & Hong, Y. (2005). Leader-following coordination of multi-agent systems with coupling time delays, *Physica A: Statistical Mechanics and its Applications*, 374(2), 853-863.

Huang, M., & Manton, J. H. (2006). Coordination and consensus of networked agents with noisy measurment: stochastic algorithms and asymptotic behavior, submitted to *SIAM Journal on Control and Optimization:* Special Issue on Control and Optimization in Cooperative Networks, 2006.

Jadbabaie, A., Lin, J., & Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Transactions on Automatic Control*, 48(6), 988-1001.

Kingston, D. B., Ren, W., & Beard, R. W. (2005). Consensus algorithms are input-to-state stable, *Proceedings of the 2005 American Control Conference*, Portland, OR, June 8-10, 2005.

Leonard, N. E., & Fiorelli, E. (2001). Virtual leaders, artificial potentials and coordinated control of groups, *Proceedings of the 40th IEEE Conference on Decision and Control* , Orlando, Florida, December 4-7, 2001.

Li, T., & Zhang, J. F. (2007). Mean square average consensus under measurement noises and fixed topologies: necessary and sufficient conditions, submitted for possible publication.

Lin, Z., Broucke, M., & Francis, B. (2004). Local control strategies for groups of mobile autonomous agents, *IEEE Transactions on Automatic Control*, 49(4), 622-629.

Michel, A. N., & Miller, R. K. (1977). *Qualitative Analysis of Large Scale Dynamical Systems*, New York: Academic Press.

Nevel'son, M. B., & Has'minskii, R. Z. (1976). *Stochastic Approximation and Recursive Estimation*, Providence, RI: American Mathematical Society.

Olfati-Saber, R. (2006). Flocking for multi-agent dynamics systems: algorithms and theory, *IEEE Transactions on Automatic Control*, 51(3), 401-420.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays, *IEEE Transactions on Automatic Control*, 49(9), 1520-1533.

Ren, W.(2007). Multi-vehicle consensus with a time-varying reference state, *Systems* & *Control Letters*, 56(7-8), 474-483.

Ren, W., & Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies, *IEEE Transactions on Automatic Control*, 50(5), 655-661.

Ren, W., Beard, R. W., & Atkins, E. M. (2005). A survey of consensus problems in multi-agent coordination, *Proceedings of the 2005 American Control Conference*, Portland, OR, June 8-10, 2005.

Ren, W., Beard, R. W., & Kingston, D. B. (2005). Multi-agent Kalman consensus with relative uncertainty, *Proceedings of the 2005 American Control Conference*, Portland, OR, June 8-10, 2005.

Ren, W., Beard, R. W., & Mclain, T. W. (2004). Coordination variables and consensus building in multiple vehicle systems, In V. Kumar, N. E. Leonard, & A. S. Morse (Eds.) *Cooperative control, Springer Series: Lecture notes in control and information sciences* (pp. 171-188), Berlin: Springer.

Tsitsiklis, J. N., Bertsekas, D. P. & Athans, M. (1986). Distributed asynchronous deterministic and stochastic gradient optimization algorithms, *IEEE Transactions on Automatic Control*, 31(9), 803-812.

Yu, H., Wang, Y. J., & Cheng, L. (2005). Control of stable flocking motion of multiply-agent with a leader, *Journal of Huazhong University of Science and Technology(Nature Science)*, 33(8), 56-58.