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Possible Non-Integrability of Observable Space for Discrete-Time Nonlinear Control Systems ⋆

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Abstract: The purpose of this paper is to provide some explanation why for the continuous-time nonlinear control system the observable subspace of one-forms is always generically integrable and why this is not necessarily so in the discrete-time case. Moreover, a general subclass of discrete-time control systems is suggested for which the observable subspace is nonintegrable.

Keywords: Nonlinear control system; Discrete-time; Observable space; Integrability; Realization

1. INTRODUCTION

Observability is a fundamental property of control systems. For certain applications it will be useful to have system representations in which the observable and unobservable state variables can be clearly distinguished. For a continuous-time nonlinear system the decomposition into observable-unobservable subsystems has been carried out both via differential geometric, see e.g. Nijmeijer and van der Schaft [1990], Isidori [1995], and differential algebraic methods, see e.g. Conte et al. [2007].

In Conte et al. [2007] decomposition is carried out on the bases of tangent linearized system. It has been proved that in the case of continuous-time nonlinear control systems, the observable subspace of one-forms is always generically integrable, and therefore can be locally spanned by exact one-forms whose integrals define the observable state coordinates, see Conte et al. [2007]. As demonstrated by the examples in Kotta [2005, 2000], in the discrete-time case this is not necessarily so. The purpose of this paper is to provide some explanation why things are different for continuous- and discrete-time systems.

We show that the proof in Conte et al. [2007] cannot be extended to the discrete-time case. Moreover, a general subclass of discrete-time control systems is suggested for which the observable subspace is nonintegrable. The drawback of nonintegrablility of observable space is that the results on observable-unobservable decomposition of the state equations do not carry over to the discrete-time domain, in general, since the observable space cannot always be locally spanned by exact one-forms whose integrals would define the observable state coordinates.

Of course, in case one drops the requirement that the decomposed equations have to be in the form of the classical state equations and allows a generalized state equations that besides inputs may depend also on forward shifts of inputs, it becomes always possible to carry out the decomposition, see Rieger et al. [2008].

The paper is organized as follows. Section 2 recalls the algebraic formalism of differential one-forms. Secion 3 presents a motivating example. Section 4 shows how to construct systems with nonintegrable observable space. Finally, Section 5 demonstrates that one can always decompose system into observable-unobservable subsystems if one drops the requirement that the decomposed system has to be in the classical state space form.

2. ALGEBRAIC FORMALISM

Consider a discrete-time single-input single-output nonlinear system Σ described by the equations

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

where $u(t) \in U \subset I\!R$ is the input variable, $y(t) \in Y \subset I\!R$ is the output variable, $x(t) \in X$, an open subset of $I\!R^n$, is the state variable, $f : X \times U \to X$ and $h : X \to Y$ are the real analytic functions. In this paper we are, like in Conte et al. [2007], interested in the generic observability and realizability properties, i. e. in the properties that hold almost everywhere, except on a set of measure zero. That is, we look at dimensions (or ranks) over a field of functions, not over $I\!R$. Thus there is not argument either about the point where to evaluate dimensions or about constant dimensionality of codistributions. Integrability of codistributions is often characterized by conditions which require that specific functions on system variables vanish. Since there are smooth functions that are neither generically zero nor generically different from zero, the notion of generic property does not make sense, in general, for systems defined by smooth functions. The situation is different, if we restrict our attention to systems defined

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by means of analytic (or also meromorphic) functions and this motivates our choice.

In order to use the mathematical tools from the algebraic framework of differential one-forms, see Aranda-Bricaire et al. [1996], we assume that the following assumption holds for system (1).

Assumption 1. $f(x,u)$ is generically a submersion, i. e. generically

$$\text{rank}\,\frac{\partial f(x,u)}{\partial(x,u)} = n.$$

Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of variables $\{x(0), u(t),\ t \geq 0\}$. The forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by $\delta\zeta(x(t), u(t)) = \zeta(f(x(t), u(t)), u(t+1))$. Under Assumption 1, δ becomes injective, i. e. if $\delta(a) = \delta(b)$, then $a = b$ for all $a, b \in \mathcal{K}$, and the pair $(\mathcal{K}, \delta)$ is a difference field; up to an isomorphism, there exists a unique difference field $(\mathcal{K}^*, \delta^*)$, called the *inversive closure* of $(\mathcal{K}, \delta)$, such that $\mathcal{K} \subset \mathcal{K}^*$, $\delta^* : \mathcal{K}^* \to \mathcal{K}^*$ is an automorphism and the restriction of δ^* to $\mathcal{K}$ equals δ. Hereinafter we use the same symbol to denote the difference field $(\mathcal{K}, \delta)$ and its inversive closure. Over the field $\mathcal{K}$ one can define a difference vector space $\mathcal{E} := \text{sp}_{\mathcal{K}}\{\text{d}\varphi \mid \varphi \in \mathcal{K}\}$. The operator $\delta : \mathcal{K} \to \mathcal{K}$ induces a forward-shift operator $\delta : \mathcal{E} \to \mathcal{E}$ by

$$\sum_i a_i \text{d}\varphi_i \to \sum_i \delta a_i \text{d}(\delta\varphi_i), a_i, \varphi_i \in \mathcal{K}.$$

We will say that $\omega \in \mathcal{E}$ is an exact one-form if $\omega = \text{d}F$ for some $F \in \mathcal{K}$. A one-form ν for which $\text{d}\nu = 0$ is said to be closed. It is well-known that exact forms are closed, whereas closed forms are only locally exact.

Theorem 1. (Frobenius). Let $V = \text{sp}_{\mathcal{K}}\{\ \omega_1, \ldots, \omega_r\}$ be a subspace of $\mathcal{E}$. V is closed if and only if

$$\text{d}\omega_i \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0, \quad \text{for any } i = 1, \ldots, r. \tag{2}$$

In (2) "$\wedge$" denotes the wedge product. Under conditions (2) there exists locally a system of coordinates $\{\zeta_1, \ldots, \zeta_r\}$ such that V is generated by $\{\text{d}\zeta_1, \ldots, \text{d}\zeta_r\}$. In this case V is said to be completely integrable, see Choquet-Bruhat et al. [1989].

Define the spaces $\mathcal{Y}^k := \text{sp}_{\mathcal{K}}\{\text{d}y(t+j), 0 \leq j \leq k\}$, $\mathcal{Y} := \text{sp}_{\mathcal{K}}\{\text{d}y(t+j), j \geq 0\}$, $\mathcal{U} := \text{sp}_{\mathcal{K}}\{\text{d}u(t+j),\ j \geq 0\}$, $\mathcal{X} := \text{sp}_{\mathcal{K}}\{\text{d}x(t)\}$ and introduce the chain of subspaces

$$0 \subset \mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}_2 \ldots \subset \mathcal{O}_k \ldots \tag{3}$$

where $\mathcal{O}_k := \mathcal{X} \cap (\mathcal{Y}^k + \mathcal{U})$ is called the *observability filtration.*

Definition 1. The subspace $\mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$ is called the observable space of system (1).

The observable space can be computed as the limit of the observability filtration (3). This limit will be denoted by $\mathcal{O}_\infty$ and obviously we have

$$\mathcal{O}_\infty = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U}).$$

System (1) is said to be single-experiment observable if $\mathcal{O}_\infty = \mathcal{X}$, see Kotta [2005], or alternatively, if the observability matrix has generically full rank, see Sontag [1979],

$$\text{rank}_{\mathcal{K}} \frac{\partial(h(x), \delta h(x), \ldots, \delta^{n-1}h(x))}{\partial x} = n.$$

3. MOTIVATING EXAMPLE

We start with an example of a simple bilinear system.

Example 1. Consider system Σ, described by equations

$$\begin{aligned} x_1(t+1) &= x_2(t) \\ x_2(t+1) &= x_3(t) + x_1(t)u(t) \\ x_3(t+1) &= u(t) \\ y(t) &= x_2(t) \end{aligned} \tag{4}$$

The observable filtration for system (4) is as follows

$$\begin{aligned} \mathcal{O}_0 &= \text{sp}_{\mathcal{K}}\{\text{d}x_2(t)\} \\ \mathcal{O}_1 &= \mathcal{X} \cap (\mathcal{Y}^1 + \mathcal{U}) \\ &= \mathcal{X} \cap \text{sp}_{\mathcal{K}}\{\text{d}y(t), \text{d}y(t+1), \text{d}u(t+k), \quad k \geq 0\} \\ &= \mathcal{X} \cap \text{sp}_{\mathcal{K}}\{\text{d}x_2(t), \text{d}x_3(t) + u(t)\text{d}x_1(t) \\ &\quad + x_1(t)\text{d}u(t), \text{d}u(t+k), \quad k \geq 0\} \\ &= \mathcal{X} \cap \text{sp}_{\mathcal{K}}\{\text{d}x_2(t), \text{d}x_3(t) + u(t)\text{d}x_1(t), \text{d}u(t+k), \\ &\quad k \geq 0\} \\ &= \text{sp}_{\mathcal{K}}\{\text{d}x_2(t), \text{d}x_3(t) + u(t)\text{d}x_1(t)\} \\ \mathcal{O}_2 &= \mathcal{X} \cap (\mathcal{Y}^2 + \mathcal{U}) \\ &= \mathcal{X} \cap \text{sp}_{\mathcal{K}}\{\text{d}y(t), \text{d}y(t+1), \text{d}y(t+2), \text{d}u(t+k), \\ &\quad k \geq 0\} \\ &= \mathcal{O}_1 \end{aligned}$$

since $y(t+2) = u(t) + y(t)u(t+1)$. Therefore,

$$\begin{aligned} \mathcal{O}_1 &= \mathcal{O}_\infty = \text{sp}_{\mathcal{K}}\{\text{d}x_2(t), \text{d}x_3(t) + u(t)\text{d}x_1(t)\} \\ &:= \text{sp}_{\mathcal{K}}\{\omega_1, \omega_2\} \end{aligned}$$

Since dim $\mathcal{O}_\infty = 2 < n = 3$, system (4) is not observable. By applying the Frobenius theorem, one can easily check that $\mathcal{O}_\infty$ is not integrable since

$$\text{d}\omega_2 \wedge \omega_1 \wedge \omega_2 = \text{d}u(t) \wedge \text{d}x_1(t) \wedge \text{d}x_2(t) \wedge \text{d}x_3(t) \not\equiv 0.$$

Next we will demonstrate on the bases of Example 1 that the proof of integrability of the observable space (Theorem 4.13 in Conte et al. [2007]) does not carry over to the discrete-time case. If we try to adapt the proof to the discrete-time case, we have to compute $y(t)$, $y(t+1)$ and $y(t+2)$ as polynomials in $u(t)$ and $u(t+1)$:

$$\begin{aligned} y(t) &= x_2(t) \\ y(t+1) &= x_3(t) + x_1(t)u(t) \\ y(t+2) &= u(t) + x_2(t)u(t+1) \end{aligned} \tag{5}$$

There should be, according to Conte et al. [2007], at most $n = 3$ independent coefficients in these polynomials; in this example these coefficients are $c_1(x) = x_2$, $c_2(x) = x_3$ and $c_3(x) = x_1$. The proof then states that the system of equations (5) can be solved in $c_1(x)$, $c_2(x)$ and $c_3(x)$. This is certainly not true in this example.

4. A SUBCLASS OF SYSTEMS WITH NONINTEGRABLE OBSERVABLE SPACE

In this section we will show how to construct systems with nonintegrable observable space. Decomposition plays an important role, for example, in the realization problem. If the realization algorithm in Kotta et al. [2001] is applied to an input-output equation, the resulting state equations are observable.

However, it is well-known, that a nonlinear input-output (i/o) equation of order n has not in general a state space realization of the same order, both in continuous- and discrete-time cases, see Kotta and Mullari [2005], Kotta et al. [2001] and the references in Kotta and Mullari [2005]. Moreover, it has been proven that it is possible to construct a post-compensator in the form of the string of pure forward shifts which will always result in a realizable composite system, if the n-th order i/o difference equation has no state space realization of order n, see Nomm et al. [2004]. The order of the state equations corresponding to the composite system is, of course, higher than n. We will demonstrate that the state equations of the composite system, accompanied by the original (and not the new shifted) output, will always result in a non-observable system with non-integrable observable subspace. Though the composite system itself is observable, the states of the realization cannot be recovered from the *original outputs only,* but require information on past outputs. However, in the continuous-time case, one cannot overcome non-realizability of the i/o differential equation by using a post-compensator, see Mullari et al. [2006]. Finally, note that using the precompensator to guarantee the realizability does not yield the nonintegrable observable space.

To present the results of the paper we also need to recall the realizability conditions for higher order input-output (i/o) difference equation

$$y(t+n) = \phi(y(t), \ldots, y(t+n-1), u(t), \ldots, u(t+s)) \quad (6)$$

where $s < n$ and ϕ is a real analytic function defined on $I\!R^n \times I\!R^{s+1}$. These are formulated in terms of a sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$, defined by

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}y(t), \ldots, \mathrm{d}y(t+n-1), \mathrm{d}u(t), \ldots, \mathrm{d}u(t+s)\} \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \delta\omega \in \mathcal{H}_k\}, \quad k \geq 1. \end{aligned} \quad (7)$$

Theorem 2. Kotta et al. [2001] The nonlinear system, described by the i/o equation (6) of order n has a nth order state space realization iff for $1 \leq k \leq s+2$ the subspaces $\mathcal{H}_k$ defined by (7) are completely integrable.

We now study again Example 1. The input-output (i/o) equation, corresponding to the state equations (4) is

$$y(t+2) = u(t) + u(t+1)y(t). \quad (8)$$

According to realizability conditions in Theorem 2, the 2nd order i/o equation (8) has no (minimal) state space realization of order 2. Compute the sequence of subspaces

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}y(t), \mathrm{d}y(t+1), \mathrm{d}u(t), \mathrm{d}u(t+1)\} \\ \mathcal{H}_2 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}y(t), \mathrm{d}y(t+1), \mathrm{d}u(t)\} \\ \mathcal{H}_3 &= \cdots, \end{aligned}$$

then it is easy to see that the subspace $\mathcal{H}_3$,

$$\mathcal{H}_3 = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}y(t), \mathrm{d}y(t+1) - y(t-1)\mathrm{d}u(t)\} \quad (9)$$

is not integrable. However, using the postcompensator Σ_P, defined by

$$\tilde{y}(t+1) = y(t),$$

one can make a composite system $\Sigma_p \circ \Sigma$ realizable, see (Nomm et al. [2004]). Really, the i/o equation of the composite system is

$$\tilde{y}(t+3) = u(t) + u(t+1)\tilde{y}(t+1) \quad (10)$$

and for (10) the subspace

$$\mathcal{H}_3 = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}\tilde{y}(t), \mathrm{d}\tilde{y}(t+1), \mathrm{d}[\tilde{y}(t+2) - u(t)\tilde{y}(t)]\}$$

is integrable, leading the state coordinates $x_1(t) = \tilde{y}(t)$, $x_2(t) = \tilde{y}(t+1)$, $x_3(t) = \tilde{y}(t+2) - u(t)\tilde{y}(t)$, and the state equations (4), except the different output function that is now $\tilde{y}(t) = x_1(t)$.

Note that the subspaces $\mathcal{O}_\infty$ for system (4) and $\mathcal{H}_3$ for its i/o equation are closely related. The subspace $\mathcal{H}_3$ given by (9) can be rewritten as

$$\mathcal{H}_3 = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x_2(t), \mathrm{d}x_3(t) + u(t)\mathrm{d}x_1(t)\}$$

since

$$\mathrm{d}x_3(t) = \mathrm{d}\tilde{y}(t+2) - \tilde{y}(t)\mathrm{d}u(t) - u(t)\mathrm{d}\tilde{y}(t),$$

and

$$\mathrm{d}\tilde{y}(t) = \mathrm{d}x_1(t) = \mathrm{d}y(t-1)\ .$$

That is, in terms of the x variables, the same as $\mathcal{O}_\infty$.

We will show that the same applies to all non-realizable input-output equations that can be made realizable by using the postcompensator in the form of the string of pure time shifts. The state equations of the composite system, though observable themselves, yield a non-integrable $\mathcal{O}_\infty$, if the output function is chosen as the original output.

To make this paper self-sufficient we recall the algorithm to build a post-compensator that results in a realizable composition.

Algorithm

(1) $q := 0$. Define Σ_P as $\tilde{y}(t+q) = y(t)$.
(2) Calculate the subspaces for the composite system $\Sigma_P \circ \Sigma$ until one finds the first non-integrable subspace $\mathcal{H}_r$.
(3) Find the elements which cause non-integrability.
(4) Find the number of forward shifts N, required to overcome non-integrability of $\mathcal{H}_r$. The value N is determined by the highest order of negative shifts in the basis elements of (non-integrable part of) $\mathcal{H}_r$.
For example, if a basis contains a non-integrable element $\mathrm{d}y(t+k) - a(\zeta)\mathrm{d}u(t+j)$ where ζ represents the variable with negative shifts then by adding to this element $u(t+j)\mathrm{d}a(\zeta)$ and adding $\mathrm{d}\zeta$ to the basis will make the basis element integrable.
(5) $q := q + N$. Define Σ_P as $\tilde{y}(t+q) = y(t)$.
(6) Check realizability of $\Sigma_P \circ \Sigma$. If it is realizable, STOP. Otherwise return to Step 2.

Note that though the number of necessary shifts can be found step by step using the algorithm above, the alternative way is to examine only $\mathcal{H}_{s+2}$. The latter may be computationally much more involved, especially for complex systems.

Theorem 3. Assume that the i/o equation Σ is non-realizable and its first non-integrable subspace is $\mathcal{H}_r$. The observable subspace $\mathcal{O}_\infty$ for the state equations of the realizable composition $\Sigma_P \circ \Sigma$, accompanied with the output function of Σ, coincides with $\mathcal{H}_{s+2}$ if the latter is rewritten in terms of the state variables of $\Sigma_P \circ \Sigma$.

Proof. Follows directly from the algorithm. According to point (4) of the algorithm, by making $\mathcal{H}_r, \ldots, \mathcal{H}_{s+2}$ integrable, we add new *independent* variables into the span of $\mathcal{H}_r, \ldots, \mathcal{H}_{s+2}$, being the original outputs at negative time instances $x_1(t) = y(t-q), \ldots, x_q(t) = y(t-1)$. Therefore, though the composite system $\Sigma_P \circ \Sigma$ itself is observable with $\mathcal{O}_\infty = \mathcal{H}_{s+2}$, it's states cannot be recovered from outputs and their forward shifts only. ■

Example 2. Consider the non-realizable i/o equation Σ

$$y(t+3) = y(t+2)u(t+1) + y(t+1)u(t) + u(t+2)y(t) \quad (11)$$

with non-integrable $\mathcal{H}_3$ and $\mathcal{H}_4$:

$$\begin{aligned}\mathcal{H}_3 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}y(t), \mathrm{d}y(t+1), \mathrm{d}y(t+2)\\ &\quad -y(t-1)\mathrm{d}u(t+1), \mathrm{d}u(t)\}\\ \mathcal{H}_4 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}y(t), \mathrm{d}y(t+1) - y(t-2)\mathrm{d}u(t),\\ &\quad \mathrm{d}y(t+2) - y(t-1)\mathrm{d}u(t+1)\\ &\quad -\mathrm{d}[u(t)y(t+1)]\}.\end{aligned}$$

To make equations (11) realizable, one needs a second-order postcompensator Σ_P

$$\tilde{y}(t+2) = y(t). \tag{12}$$

The composition $\Sigma_P \circ \Sigma$ is realizable and observable with state equations

$$\begin{aligned}x_1(t+1) &= x_2(t)\\ x_2(t+1) &= x_3(t)\\ x_3(t+1) &= x_4(t) + x_1(t)u(t)\\ x_4(t+1) &= x_5(t) + u(t)[x_4(t) + x_1(t)u(t)]\\ x_5(t+1) &= u(t)[x_4(t) + x_1(t)u(t)]\end{aligned} \tag{13}$$

and output function

$$\tilde{y}(t) = x_1(t) \tag{14}$$

However, computing the observable space for system (13) with the original output function with $y(t) = x_3(t)$, one gets

$$\begin{aligned}\mathcal{O}_0 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x_3(t)\}\\ \mathcal{O}_1 &= \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x_3(t), \mathrm{d}x_4(t) + u(t)\mathrm{d}x_1(t)\}\\ \mathcal{O}_2 &= \mathcal{O}_\infty = \mathrm{sp}_{\mathcal{K}}\{\mathrm{d}x_3(t), \mathrm{d}x_4(t) + u(t)\mathrm{d}x_1(t),\\ &\quad \mathrm{d}x_5(t) + u(t+1)\mathrm{d}x_2(t)\}\ .\end{aligned}$$

Next, taking into account that by (13) and (12)

$$\begin{aligned}\mathrm{d}x_3(t) &= \mathrm{d}y(t)\\ \mathrm{d}x_4(t) + u(t)\mathrm{d}x_1(t) &= \mathrm{d}y(t+1) - y(t-2)\mathrm{d}u(t)\\ \mathrm{d}x_5(t) + u(t+1)\mathrm{d}x_2(t) &= \mathrm{d}y(t+2)\\ &\quad -y(t-1)\mathrm{d}u(t+1)\\ &\quad -\mathrm{d}[u(t)y(t+1)]\end{aligned}$$

we obtain exactly the basis elements of $\mathcal{O}_\infty$.

One may argue that the construction of this subclass is artificial and therefore of no value. Actually, if given this type of system it looks like as a typical control system. Moreover, it is possible to construct an infinite number of systems with nonintegrable observable space. It is not the case that we have just a few strange examples.

5. GENERALIZED STATE EQUATIONS

In several control problems it has become clear that it is necessary to consider more general dynamics that contain in addition to the input also a finite number of its time shifts. One well-known example is the inverse system that, in general, contains the shifts of its inputs. Another example is the generalized controller canonical form, see Fliess [1990b], and the extended observer canonical form, introduced in Lilge [1998] where, in contrast to output injection form, past measurements of the system output are used that allows extension of the class of systems for which an observer can be designed. A theoretical study of discrete-time control systems depending explicitly on input shifts was initiated by Fliess [1986, 1989, 1990a, 1992] and based on difference algebra.

In case we drop the requirement that the decomposed system has to be in the form (1) and allow the generalized state equations

$$\begin{aligned}z(t+1) &= F(z(t), u(t), \ldots, u(t+\alpha))\\ y(t) &= h(z(t))\end{aligned}$$

and generalized state transformations

$$z(t) = \psi(x(t), u(t), \ldots, u(t+\alpha-1))$$

like in Fliess [1992], both system (4) and (13) with $y(t) = x_3(t)$ allow the decomposition into an observable and unobservable subsystems, see Rieger et al. [2008].

Example 3. (Continuation of Example 1) Choosing the (generalized) state coordinates as follows

$$\begin{aligned}z_1(t) &= x_2(t)\\ z_2(t) &= x_3(t) + u(t)x_1(t)\\ z_3(t) &= x_1(t)\end{aligned}$$

yields the generalized state equations

$$\begin{aligned}z_1(t+1) &= z_2(t)\\ z_2(t+1) &= u(t) + u(t+1)z_1(t)\\ z_3(t+1) &= z_1(t)\\ y(t) &= z_1(t)\end{aligned}$$

Note that the first two equations define the observable subsystem and the last equation the unobservable subsystem.

Example 4. (Continuation of Example 2) Choosing the state coordinates as

$$\begin{aligned}z_1(t) &= x_3(t)\\ z_2(t) &= x_4(t) + x_1(t)u(t)\\ z_3(t) &= x_5(t) + x_2(t)u(t+1)\\ z_4(t) &= x_1(t)\\ z_5(t) &= x_2(t)\end{aligned}$$

yields the generalized state equations

$$\begin{aligned}z_1(t+1) &= z_2(t)\\ z_2(t+1) &= z_3(t) + z_2(t)u(t)\\ z_3(t+1) &= z_2(t)u(t) + z_1(t)u(t+2)\\ z_4(t+1) &= z_5(t)\\ z_5(t+1) &= z_1(t)\\ y(t) &= z_1(t)\end{aligned}$$

The first three equations define the observable subsystem and the last two equations the unobservable subsystem.

6. CONCLUSIONS

The paper demonstrates that in the case of discrete-time nonlinear control systems, the observable and unobservable state variables cannot be always distinguished, unlike in the case of continuous-time counterpart. A subclass, containing an infinite number of systems is introduced for which decomposition of state equations into observable-unobservable parts is impossible. This difficulty does not exist if one allows the decomposed system equations depend on future inputs. To conclude, the results of the paper point to the two aspects. First, that in the nonlinear domain, there exists a number of situations in which discrete- and continuous-time systems behave remarkably differently. Second, that in the nonlinear domain, perhaps the classical state equations are not the most natural choice, since in the framework of the generalized state equations, many difficulties cease to exist.

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